Lecturer: Jan Goetz	**Problem set 5**	PHYS-C0254 Quantum Circuits
TA: Hany Khalifa	Due April 5, 10:15 AM	March 29, 2021

1 *Density operator*

(1) Show that the density matrix $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$ is **Hermitian**, **positive semidefinite** (PSD), and has a unit trace. Where for a PSD matrix (operator) $A \in \mathcal{H}$ the following holds $\langle x|A|x\rangle \geq 0 \,\forall x \in \mathcal{H}$.

(2) Starting with the definition of a density operator, show that for a pure state the trace of ρ^2 is always equal to 1, whereas when the state is mixed the trace is always < 1.

(3) Suppose that the pure qubit state $|\varphi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ is undergoing a unitary evolution according to the Hamiltonian $H_0 = \hbar\omega_0\sigma_z/2$. Derive the time-dependent density operator of the state $\rho(t) = |\varphi(t)\rangle\langle\varphi(t)|$.

2 *The Hilbert space basis*

(1) Demonstrate the **ONB** property for both the computational and the Fourier basis in a Finite dimensional Hilbert space $\mathcal{H}^N$. **Hint:** The general formula for geometric progression is given by $\sum_{j=1}^{n} \gamma r^{j-1} = \frac{\gamma(1-r^n)}{1-r}$.

(2) A **qutrit** is a quantum mechanical information carrier that lives in the three dimensional Hilbert space $\mathcal{H}^3$.

- (a) Write down explicitly the qutrit computational basis, and verify that they constitute an **ONB**. What would an arbitrary qutrit state look like?
- (b) Similarly, write down the qutrit's Fourier basis and verify that they are indeed *orthonormal*.

3 *Qubit teleportation*

Let us assume that party A possesses an arbitrary quantum state $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$ and wants to transmit it to another party B. A doesn't know a priori the exact values of the state's probability amplitudes, moreover, any attempt to measure the qubit would only yield a partial information about the state[1]. Instead, when both parties share one of the Bell states, B will be able to reconstruct A's state without the need for any *physical* transmission of the state itself.

(1) Suppose that A and B share the Bell state $|\Phi^+\rangle$. At this point A possesses the qubit to be teleported and her share of the Bell state. Write down the the overall tensor product state of the three qubits.

(2) Now observe that the state $|00\rangle$ can be written in terms of the Bell states as $\frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Phi^-\rangle)$. Similarly, you can find expressions for $|01\rangle, |10\rangle, |11\rangle$. After some algebraic manipulations, rewrite the state derived in (1), such that A possesses one of the Bell states, whereas B possesses a single qubit with the same original probability amplitudes. You should get a three qubit state in a superposition of 4 possible combinations.

(3) Describe the action of the following set of qubit operations on the original qubit at A's lab, $\{X, Z, XZ\}$. Consider each operation separately.

(4) Assume now that A has the following set of Bell state projectors $\{|\Phi^+\rangle\langle\Phi^+|, |\Psi^+\rangle\langle\Psi^+|, |\Psi^-\rangle\langle\Psi^-|, |\Phi^-\rangle\langle\Phi^-|\}$, and she projects the two qubits in her possession onto one of these Basis. Write down the four possible qubit states that end up at B's lab.

(5) B equally likely to end up with one of the previous qubit states. Following (c) what kind of operations he has to perform on his qubit to retrieve the original one? Consider each case separately.

[1] Recall the Born rule.

4 *Qubit control*

Consider the qubit-field Hamiltonian given by $\mathcal{H}_{\text{q-f}} = \underbrace{\frac{\hbar\omega_0}{2}\sigma_z}_{\text{free}} + \underbrace{\hbar\gamma E_d \cos(\omega_d t + \phi)[\sigma^+ + \sigma^-]}_{\text{interaction}}$. Now define a frame rotating at the qubit's frequency as $U(t) = e^{-i\omega_0 \frac{\sigma_z}{2} t}$, such that $|\Psi\rangle_{\text{RF}} = U(t)\,|\Psi\rangle_{\text{Sch}}$, $H_{\text{RF}} = U(t) H_{\text{Sch}} U^\dagger(t) - i\hbar U(t)\dot{U}^\dagger(t)$. Show that in the rotating frame the qubit-field Hamiltonian is off-diagonal.
Hint: $e^{i\gamma A} B e^{-i\gamma A} = B + i\gamma[A,B] + \frac{(i\gamma)^2}{2!}[A,[A,B]] + \ldots$.

4.1 *Measuring* T_1

In order to measure the qubit's population relaxation time. First, a pulse is applied to the qubit to put it in the excited state, then the time taken by the qubit to decohere is observed. To see this in action, consider the on-resonance RWA Hamiltonian $H_{\text{RWA}} = \frac{\hbar\Omega}{2}(\sigma^+ + \sigma^-)$, where we have neglected the drive's phase off-set. Assume further that the qubit was initiated in the ground state $|\varphi(0)\rangle = |0\rangle$.

(1) Define a unitary $U = e^{-iH_{\text{RWA}}t/\hbar}$, then calculate the qubit's state at time t.
Hint: $e^{-i\theta U} = I - i\theta U + (i)^2 \frac{\theta^2}{2!} U^2 + \ldots$

(2) Let $t = \frac{\pi}{\Omega}$, what is the final state of the qubit?

4.2 *Ramsey interferometry*

Let us assume now that we have the same qubit in (4.1) undergoing the same unitary evolution. However, t is now set to $\frac{\pi}{2\Omega}$.

(1) What is the state of the qubit after the RWA interaction in this case?

(2) Perform the following phase transformation on the output in (1) $\begin{pmatrix} e^{i\delta\frac{\tau}{2}} & 0 \\ 0 & e^{-i\delta\frac{\tau}{2}} \end{pmatrix}$

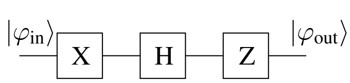
(3) Apply now another $\frac{\pi}{2}$ pulse similar to that in (1) and write down the state of the qubit accordingly. Describe the action of the $\pi/2$ pulse in the language of quantum gates.

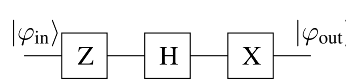
(4) Finally, calculate the probability of finding the qubit in the ground state.

5 *Quantum gates*

(1) Given the definitions of single qubit operations, show the following properties
$(a)\, XYX = -Y \quad (b)\, HXH = Z \quad (c)\, HYH = -Y \quad (d)\, HZH = X \quad (d)\, \Theta_{\frac{\pi}{2}} X \Theta^\dagger_{\frac{\pi}{2}} = Y$
$(e)\, \Theta_{\frac{\pi}{2}} Y \Theta^\dagger_{\frac{\pi}{2}} = -X \quad (f)\, \Theta_{\frac{\pi}{2}} Z \Theta^\dagger_{\frac{\pi}{2}} = Z$

(2) Suppose that $|\varphi_{\text{in}}\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. What is the final state $|\varphi_{\text{out}}\rangle$ in each of the following two cases?





(3) Consider a qubit in the $|0\rangle$ state entering a Hadamard gate. Then, the Hadamard output is fed into a CNOT gate as the *control* qubit while the *target* in initiated in the $|0\rangle$.

(a) Draw a diagram of the above circuit, and write down its final output state.

(b) Now perform separately the following 2-qubit operations on the circuit's output $I \otimes X$, $I \otimes Y$, $I \otimes Z$. What is the output state in each case?

(4) The Hamiltonian of the **iSWAP** gate is defined as $H = \frac{\hbar g}{2}(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)$. Define a unitary evolution as $U = e^{-iHt/\hbar}$.

(a) Derive the matrix representation of this transformation.
Hint: $e^{i\theta A} = \cos\theta I + i \sin\theta A$. $e^{A+B} = e^A e^B e^{-1/2[A,B]}$. where $A, B \in L(\mathcal{H})$.

(b) Describe the action of the $\sqrt{\textbf{iSWAP}}$ on the state $|0\rangle \otimes |1\rangle$.