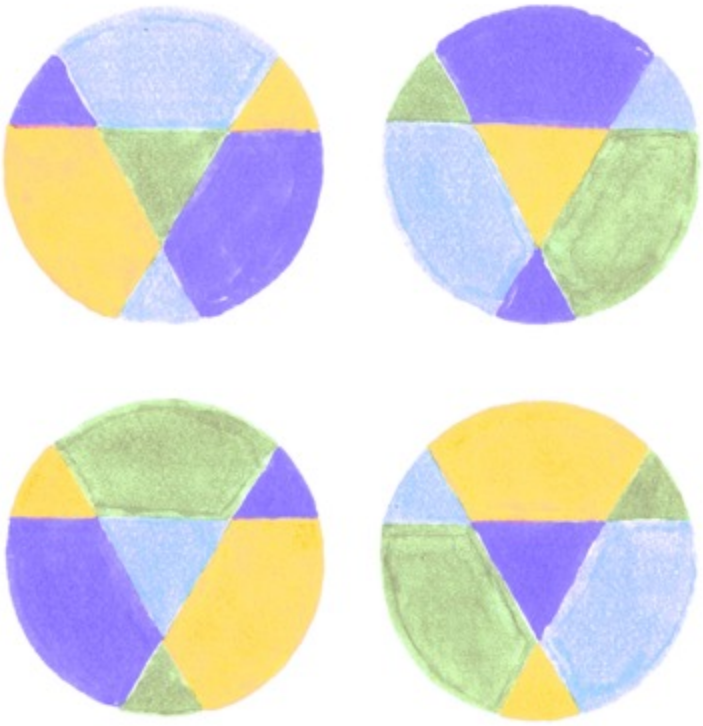
PROJECTIVE GEOMETRY
PART 6



Taneli Luotoniemi

CRYSTAL FLOWER IN HALLS OF MIRRORS 2021

Mathematics nor arts in their pure form are <u>not meant to be useful</u>

…but, let's list reasons:

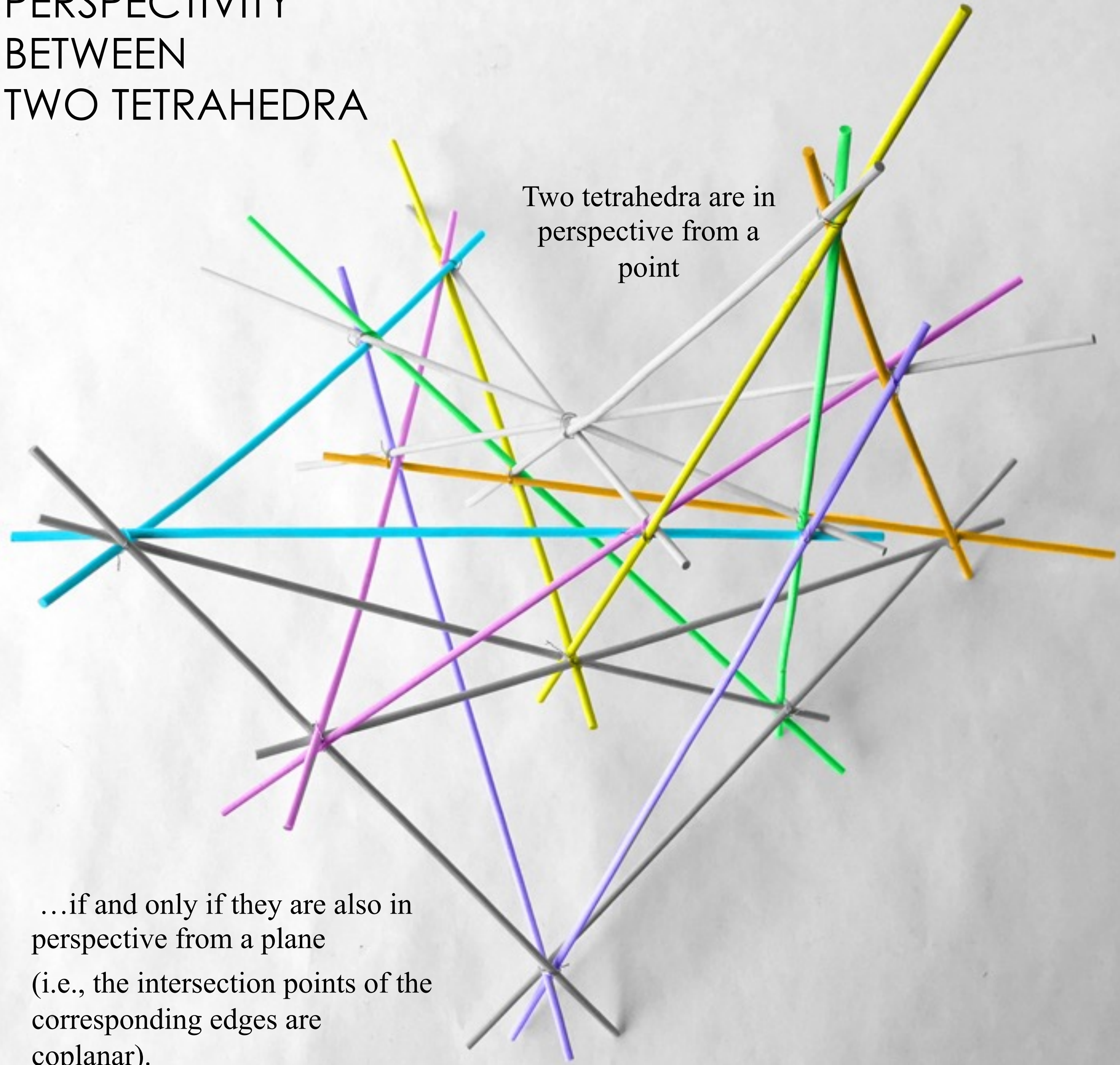
WHY LEARN PROJECTIVE GEOMETRY IN MATH&ARTS STUDIES

- to exercise your spatial reasoning
- to exercise logical thinking
- to learn visual composition
- to get a learn non-Euclidean geometry
- to learn linear perspective of visual arts
- to understand that geometry is not just measuring
- to challenge the point-biased approach of traditional geometry

COMPLETE HEXACHORON
(CONTINUED)

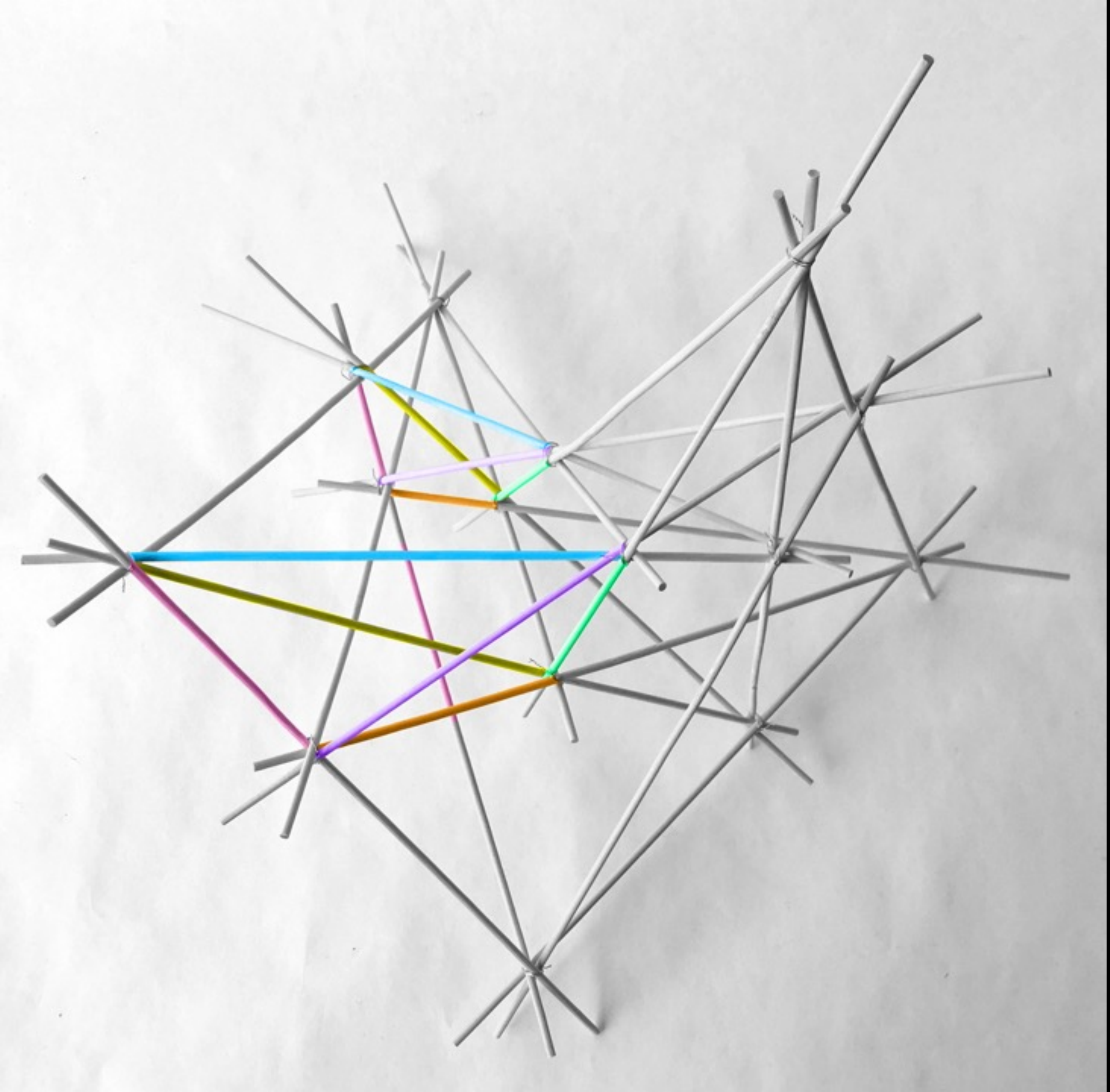
PERSPECTIVITY BETWEEN TWO TETRAHEDRA

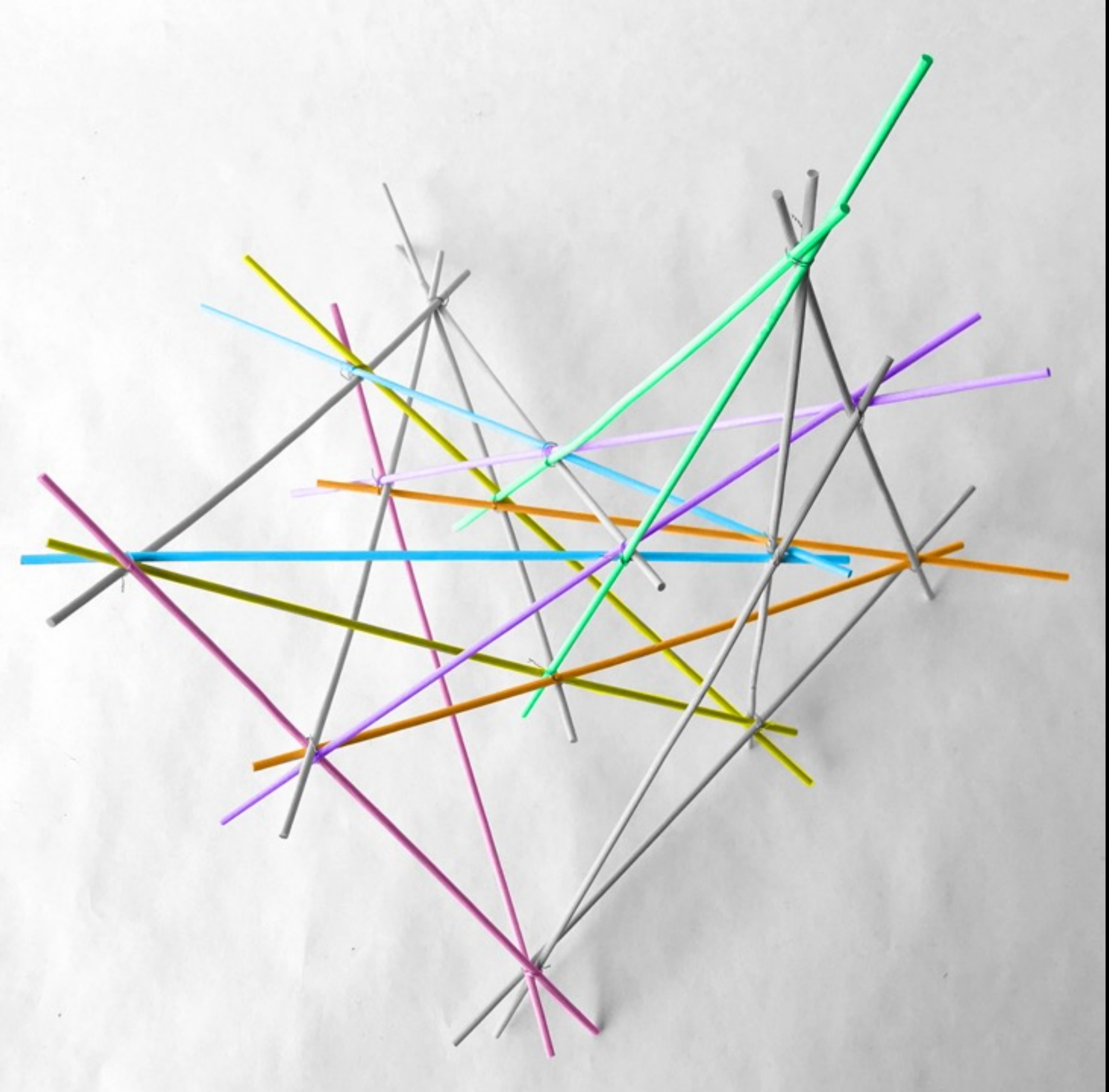
Two tetrahedra are in perspective from a point

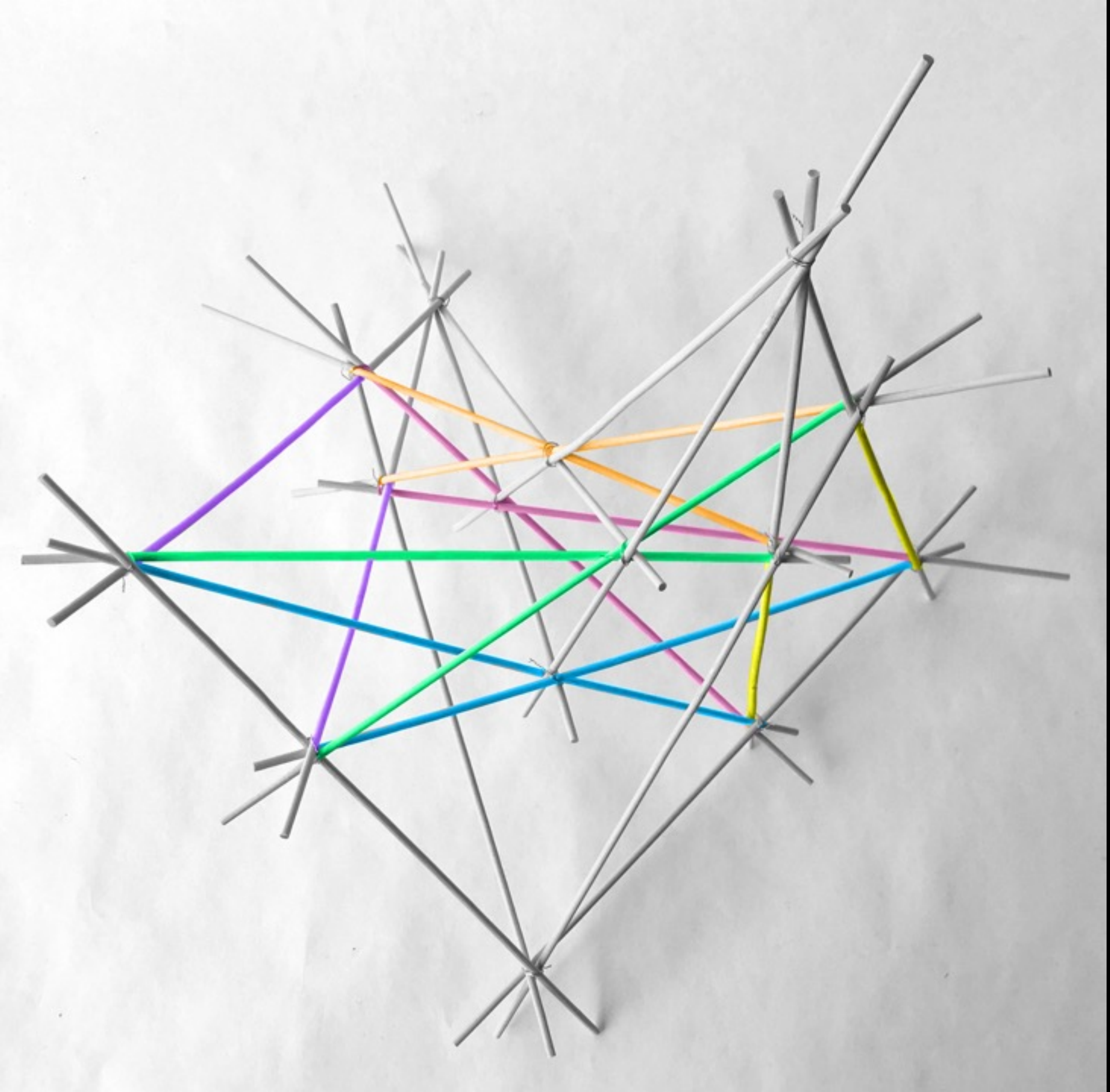


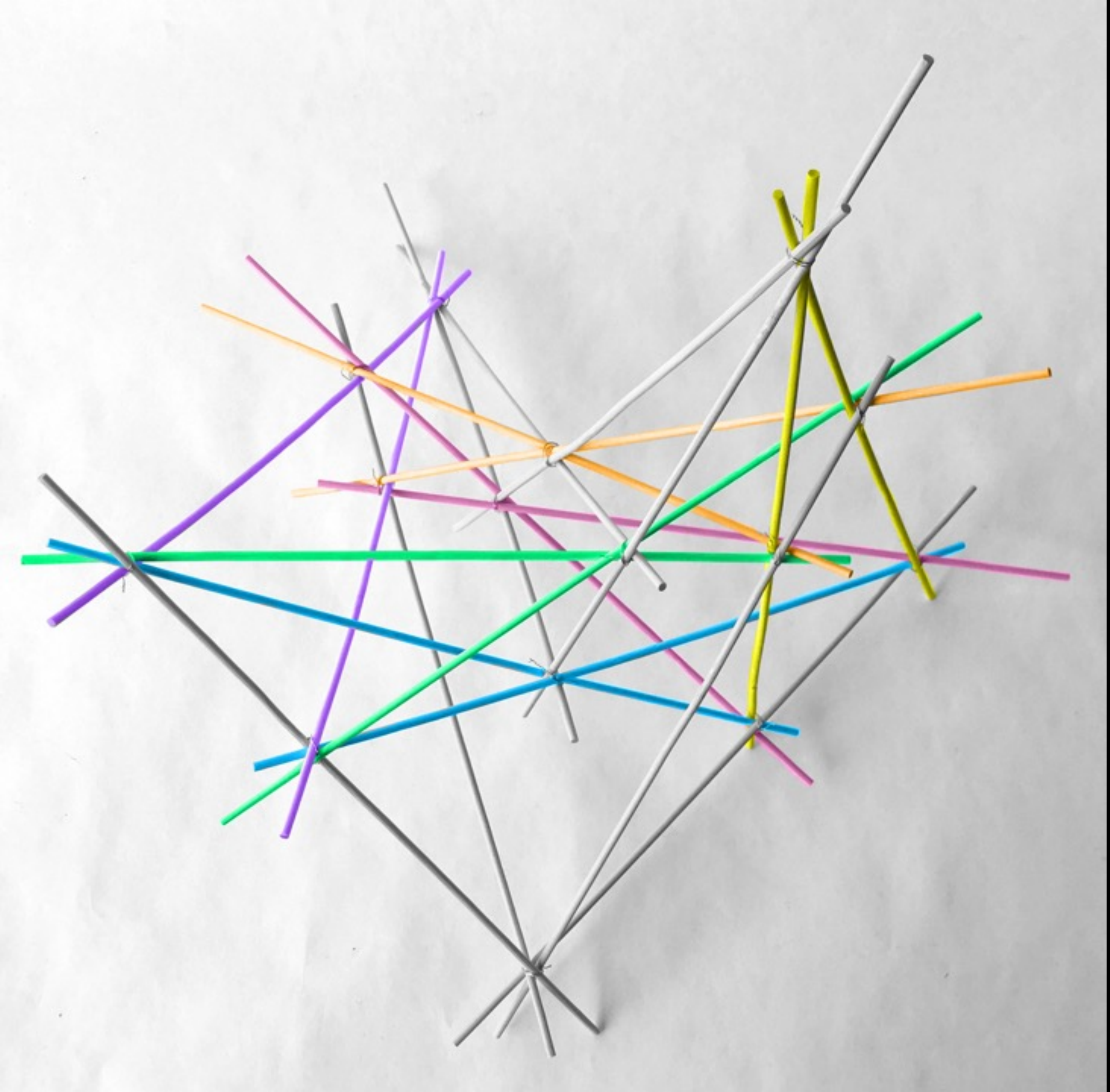
…if and only if they are also in perspective from a plane

(i.e., the intersection points of the corresponding edges are coplanar).









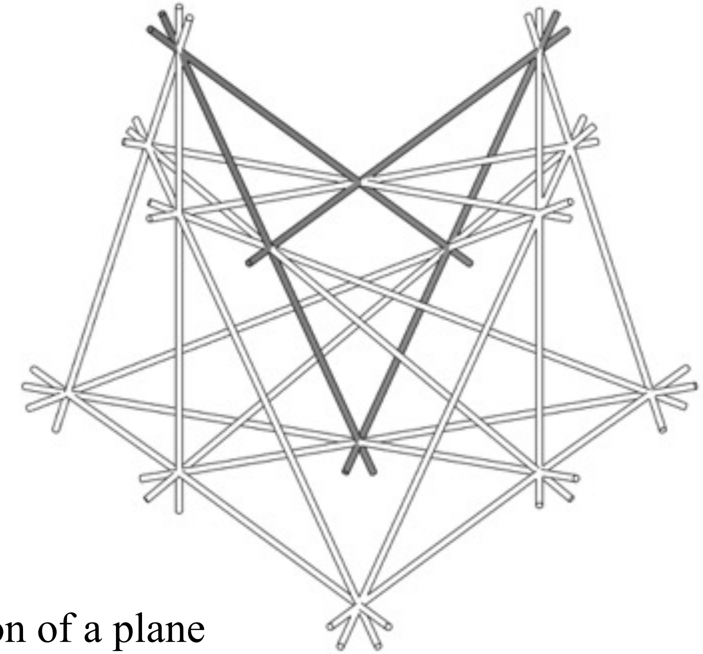
PHYSICAL MODEL

20 acrylic tubes & 38 painted wooden rods

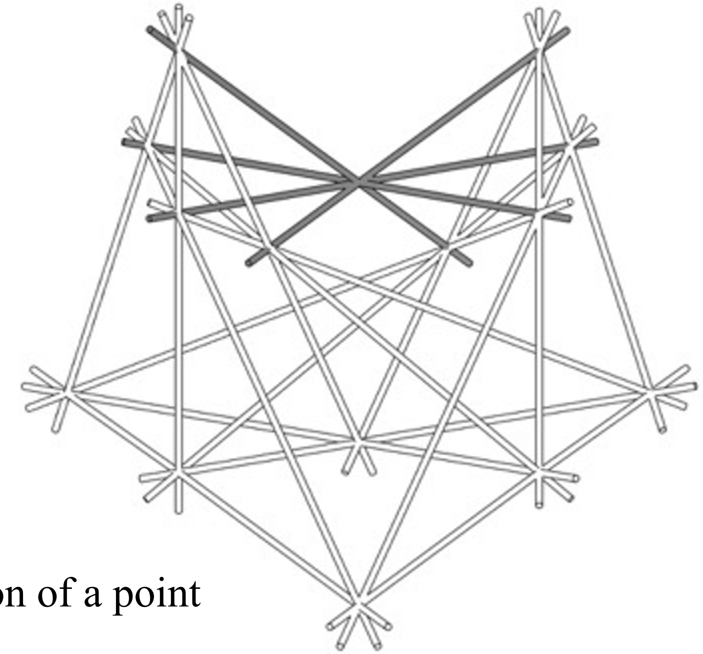
TIC-TAC-TOE

Game for two players (black & white) take turns inserting wooden rods into the tubes of the configuration.

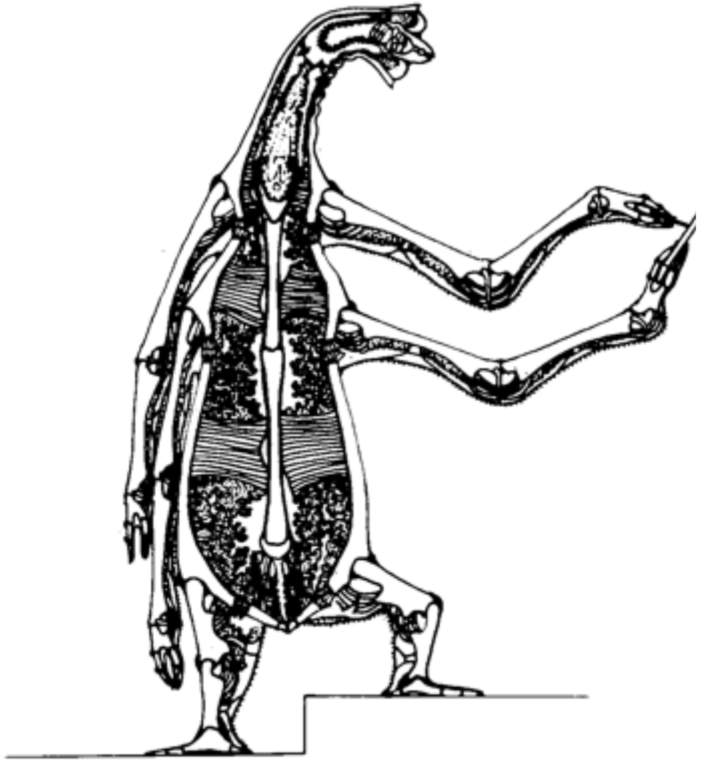
The first player to occupy an entire plane or an entire point wins.



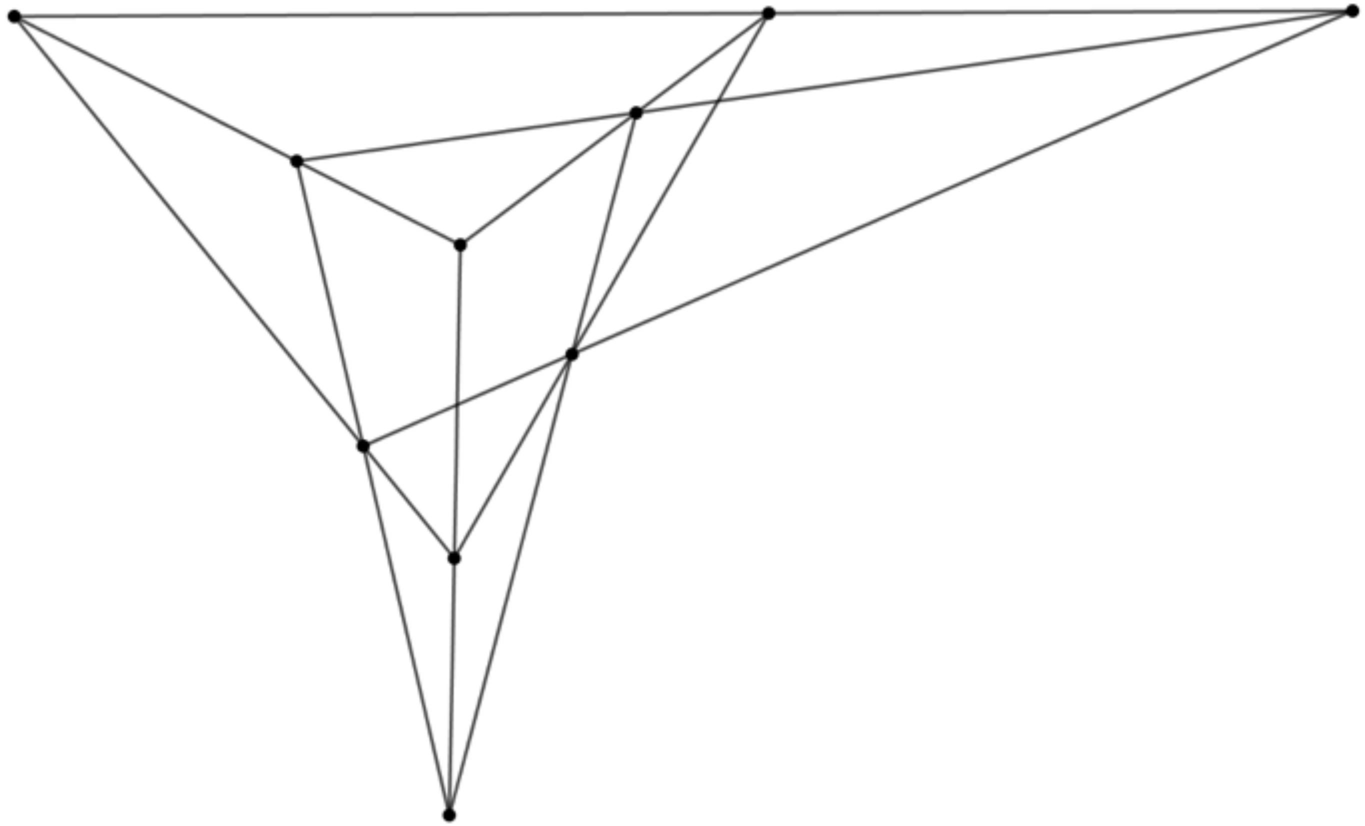
Win by an occupation of a plane



Win by an occupation of a point



A. K. Dewdney: “Planiverse” (1984)



THE DESARGUES CONFIGURATION AS A GNOMONIC PROJECTION

TETRAHEDRON
(3-SIMPLEX)

4 vertices
6 edges
4 faces (triangles)

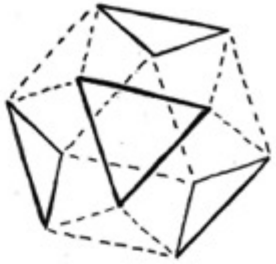
Tetrahedron: https://en.wikipedia.org/wiki/Tetrahedron

TETRAHEDRON (3-SIMPLEX)

4 vertices
6 edges
4 faces (triangles)

CUBOCTAHEDRON (EXPANDED TETRAHEDRON)

12 vertices
24 edges
14 faces (8 triangles & 6 squares)



Expansion: https://en.wikipedia.org/wiki/Expansion_(geometry)
Cuboctahedroni (i.e. expanded tetrahedron): https://en.wikipedia.org/wiki/Cuboctahedron

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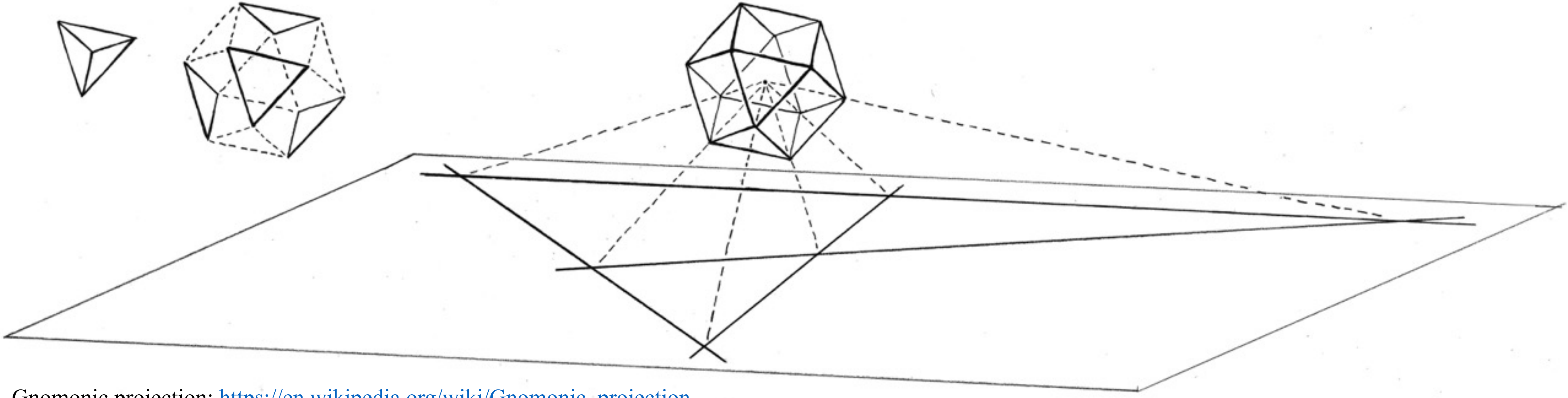
GNOMONIC PROJECTION



Gnomonic projection: https://en.wikipedia.org/wiki/Gnomonic_projection

TETRAHEDRON
(3-SIMPLEX)
4 vertices
6 edges
4 faces (triangles)

CUBOCTAHEDRON
(EXPANDED
TETRAHEDRON)
12 vertices
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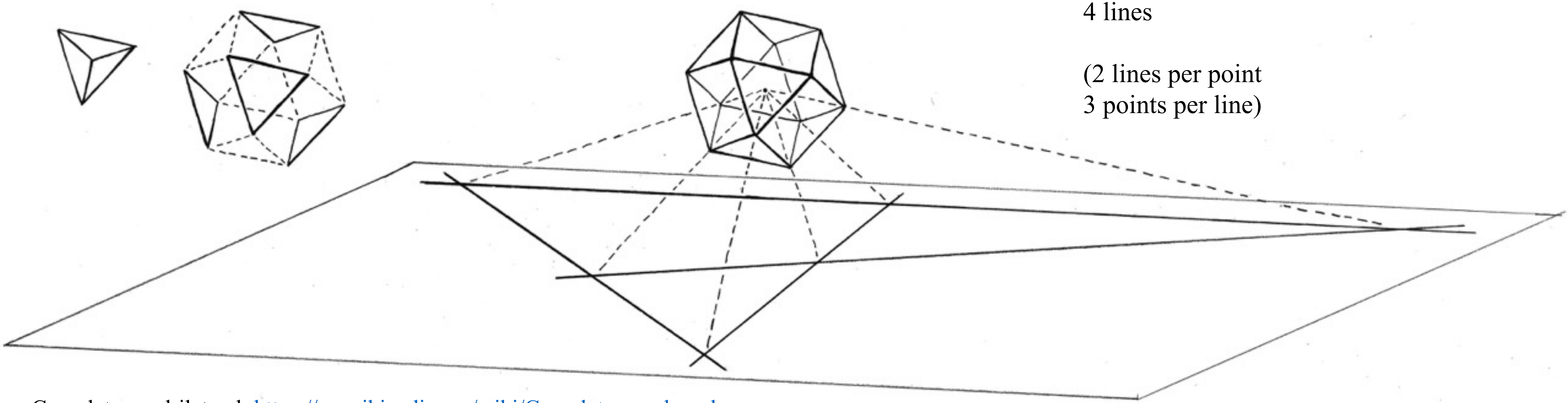
14 faces (8 triangles & 6 squares)

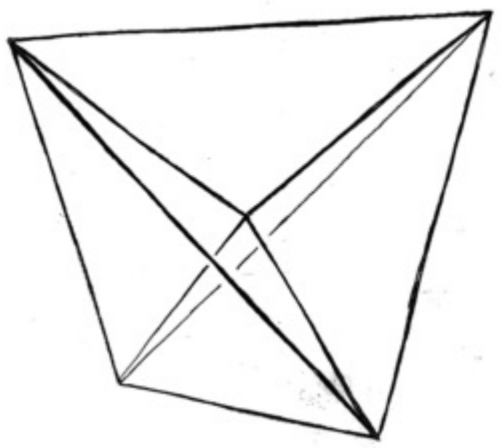
GNOMONIC
PROJECTION

THE COMPLETE
QUADRILATERAL
6 points
4 lines

(2 lines per point
3 points per line)



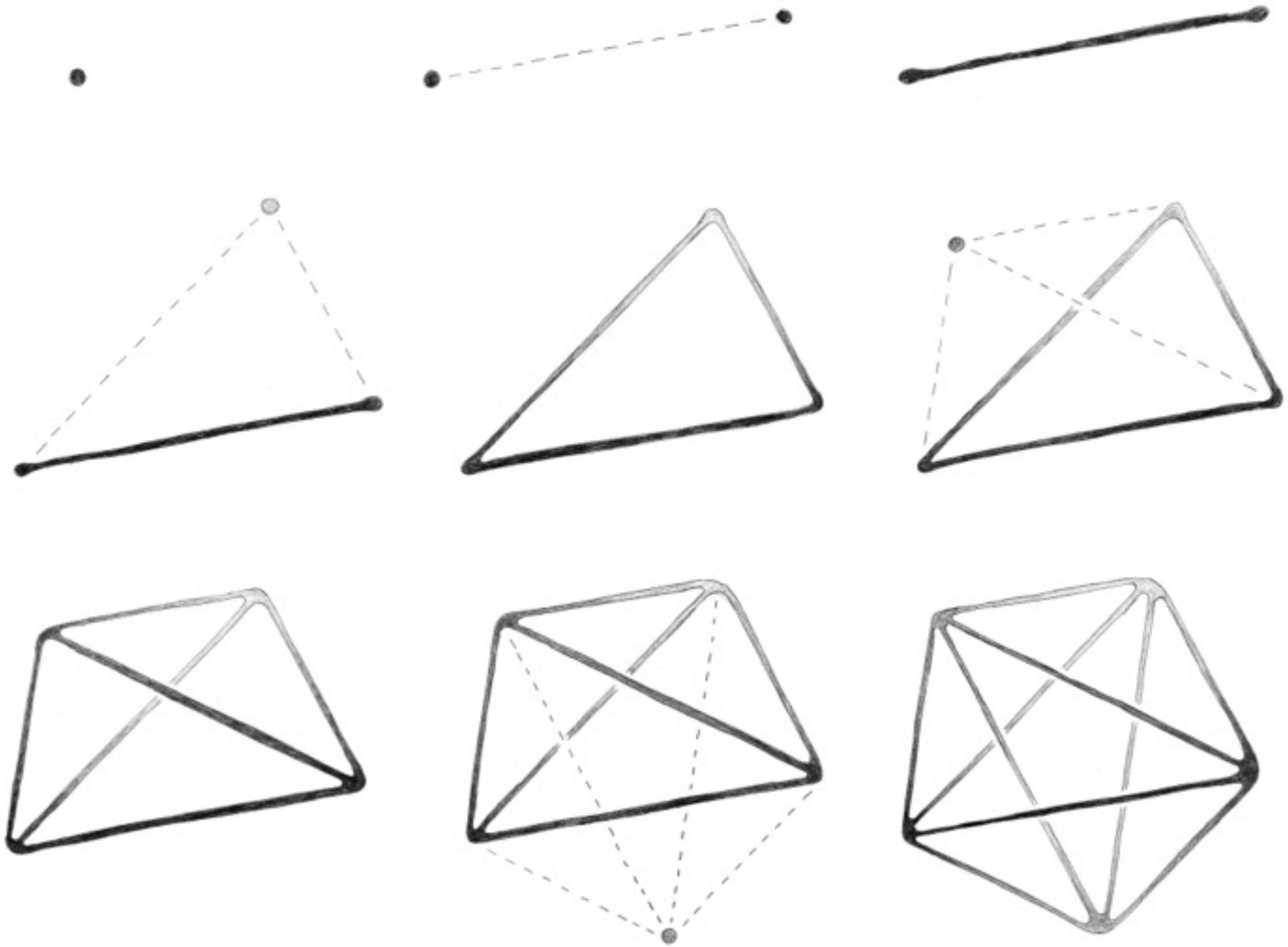
Complete quadrilateral: https://en.wikipedia.org/wiki/Complete_quadrangle



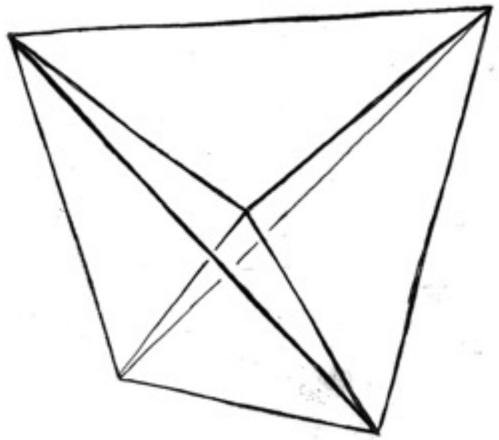
PENTACHORON
5 vertices
10 edges
10 faces (triangles)
5 cells (tetrahedra)

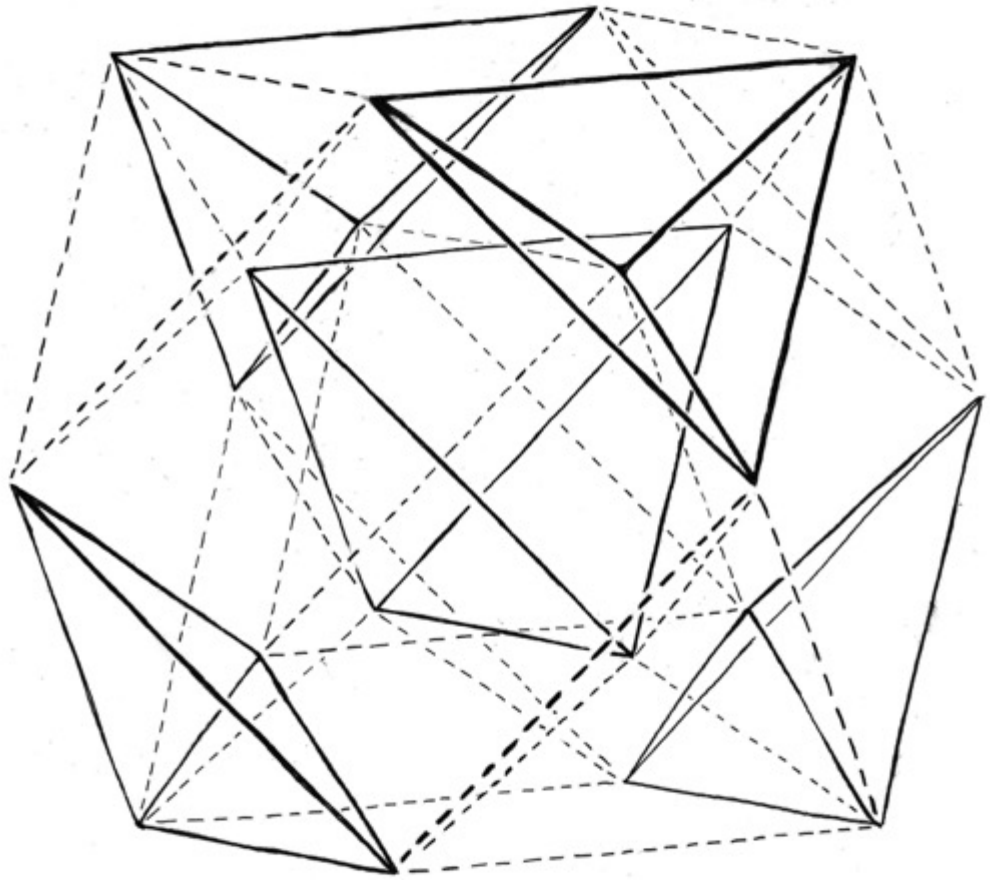
Pentachoron: https://en.wikipedia.org/wiki/5-cell

CONSTRUCTING THE PENTACHORON



Simplex: https://en.wikipedia.org/wiki/Simplex





PENTACHORON → EXPANDED PENTACHORON

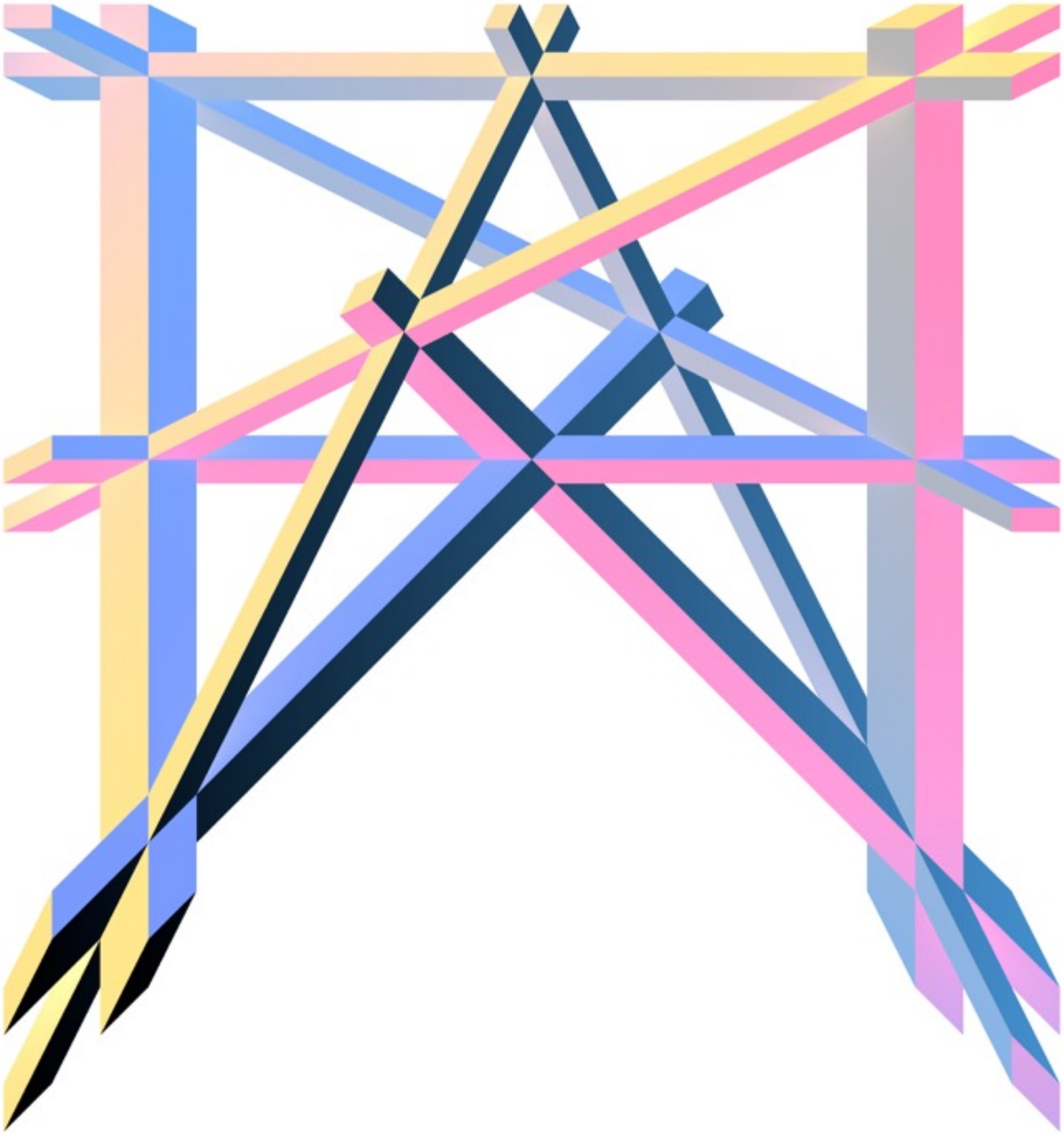
PENTACHORON	EXPANDED PENTACHORON
5 vertices	20 vertices
10 edges	60 edges
10 faces (triangles)	70 faces (40 triangles + 30 squares)
5 cells (tetrahedra)	30 cells (10 tetrahedra + 20 triangular prisms)

Expanded pentachoron:
https://en.wikipedia.org/wiki/Runcinated_5-cell

GNOMONIC PROJECTION
OF THE EXPANDED PENTACHORON
=
THE DESARGUES
CONFIGURATION

10 points
10 lines
5 planes

(3 lines and 3 planes per point
3 points and 2 planes per line
6 points and 4 lines per plane)



5-SIMPLEX

6 vertices
15 edges
20 faces (triangles)
15 cells (tetrahedra)
6 hypercells (pentachora)

5-simplex: https://en.wikipedia.org/wiki/5-simplex

5-SIMPLEX → EXPANDED 5-SIMPLEX

5-SIMPLEX	EXPANDED 5-SIMPLEX
6 vertices	30 vertices
15 edges	120 edges
20 faces (triangles)	210 faces (120 triangles + 90 squares)
15 cells (tetrahedra)	180 cells (60 tetrahedra +120 triangular prisms)
6 hypercells (pentachora)	62 hypercells (12 pentachora + 30 tetrahedral prisms + 20 three-by-three duoprisms)

Expanded 5-simplex: https://en.wikipedia.org/wiki/Expanded_5-simplex#Stericated_5-simplex

GNOMONIC PROJECTION OF THE EXPANDED 5-SIMPLEX = COMPLETE HEXACHORON CONFIGURATION

15 points
20 lines
15 planes
6 hyperplanes

(4 lines, 6 planes, and 4 hyperplanes per point
3 points, 3 planes, and 3 hyperplanes per line
6 points, 4 lines, and 2 hyperplanes per plane
10 points, 10 lines, and 5 planes per hyperplane)

Hyperplane: https://en.wikipedia.org/wiki/Hyperplane

COMPLETE HEXACHORON

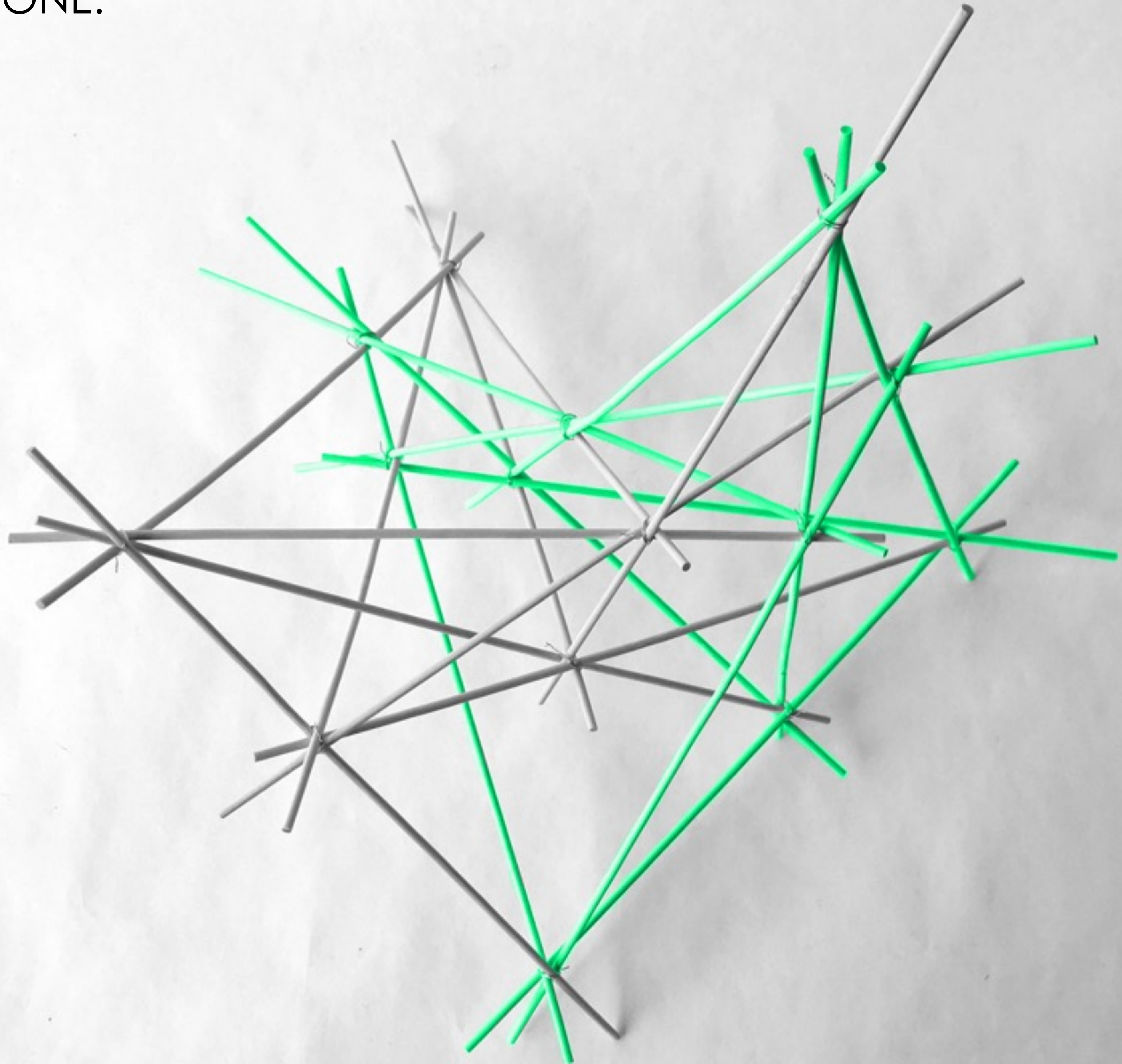
(SIX HYPERPLANES IN GENERAL POSITION INTERSECTING EACH OTHER IN PROJECTIVE 4-SPACE)

CONSTRUCTION OF THE DESARGUES CONFIGURATION:

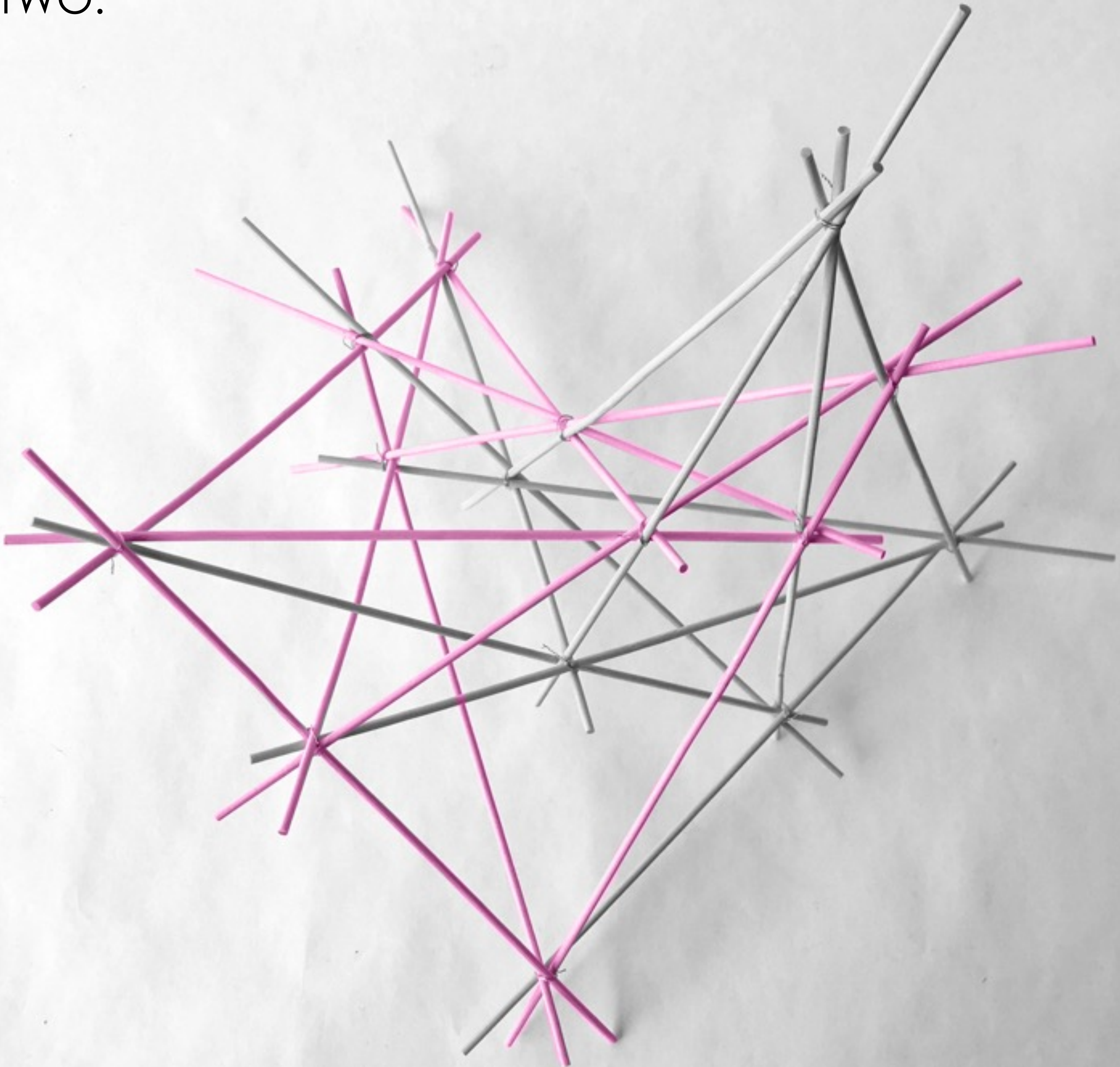
Take 5 random points floating in 4-dimensional space, and join them to get 10 lines and 10 planes.

If these lines and planes are cut with a hyperplane that neither contains nor is parallel to any of them, the intersections of the lines and planes will be 10 points and 10 lines forming an instance of Desargues configuration (green).

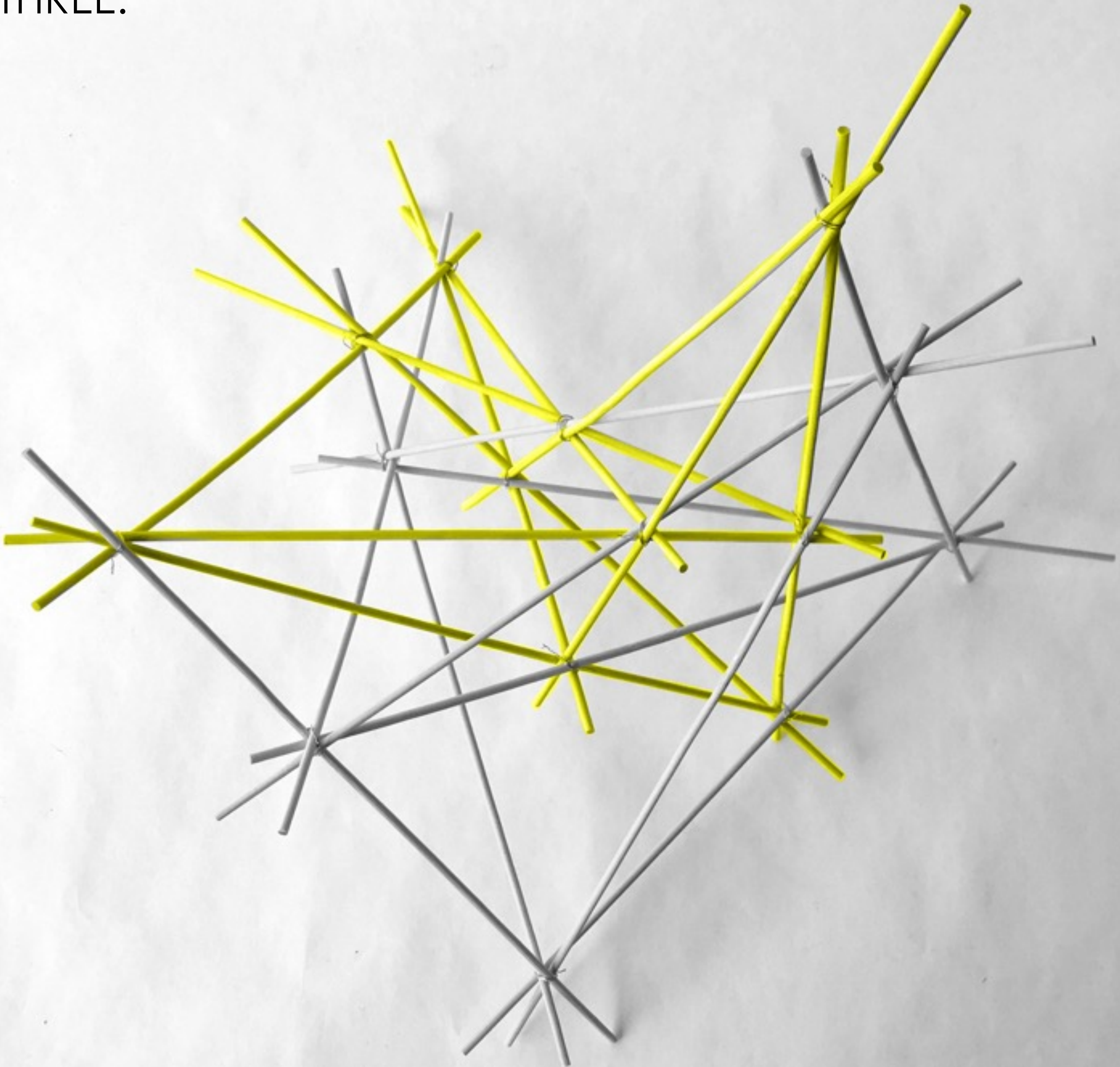
ONE:



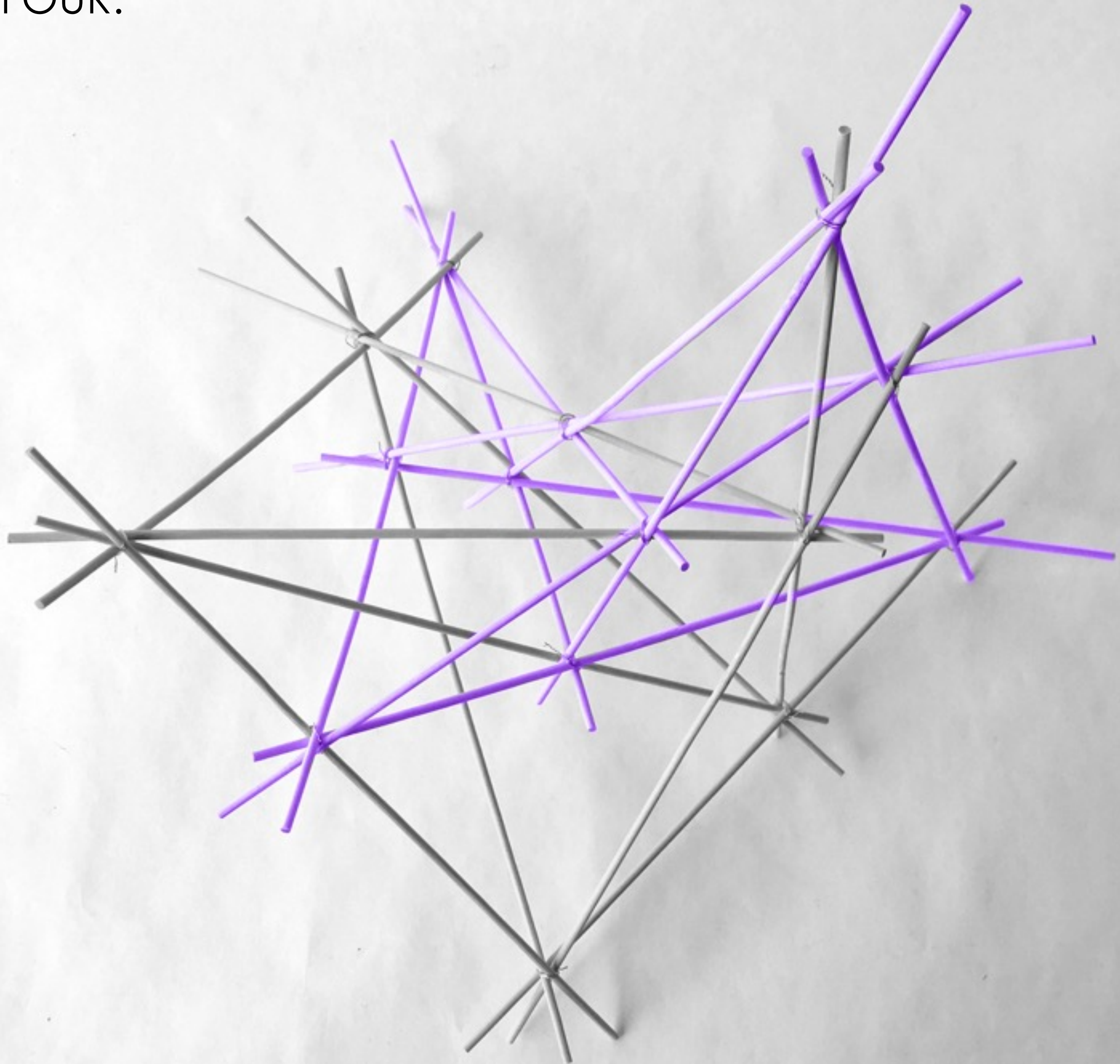
TWO:



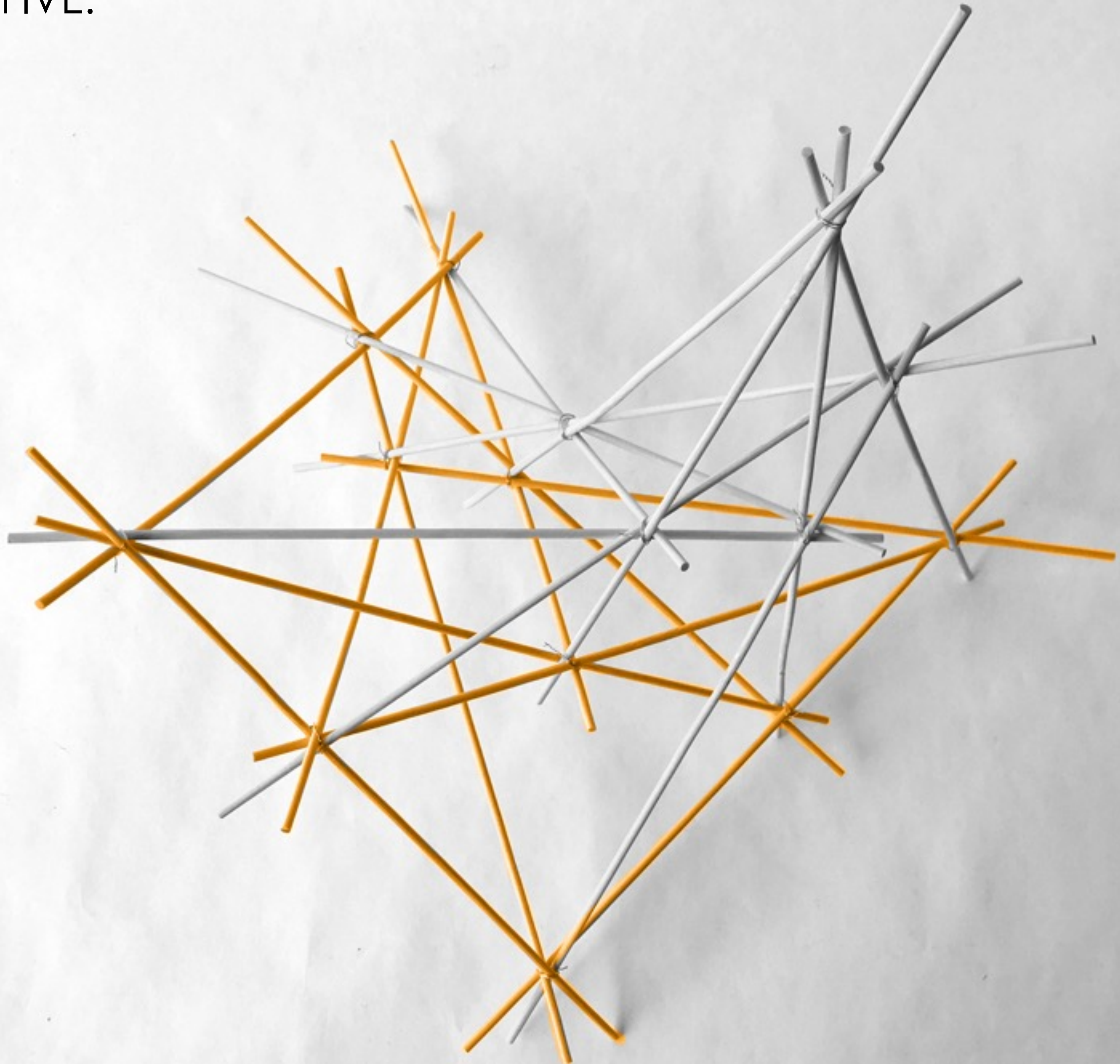
THREE:



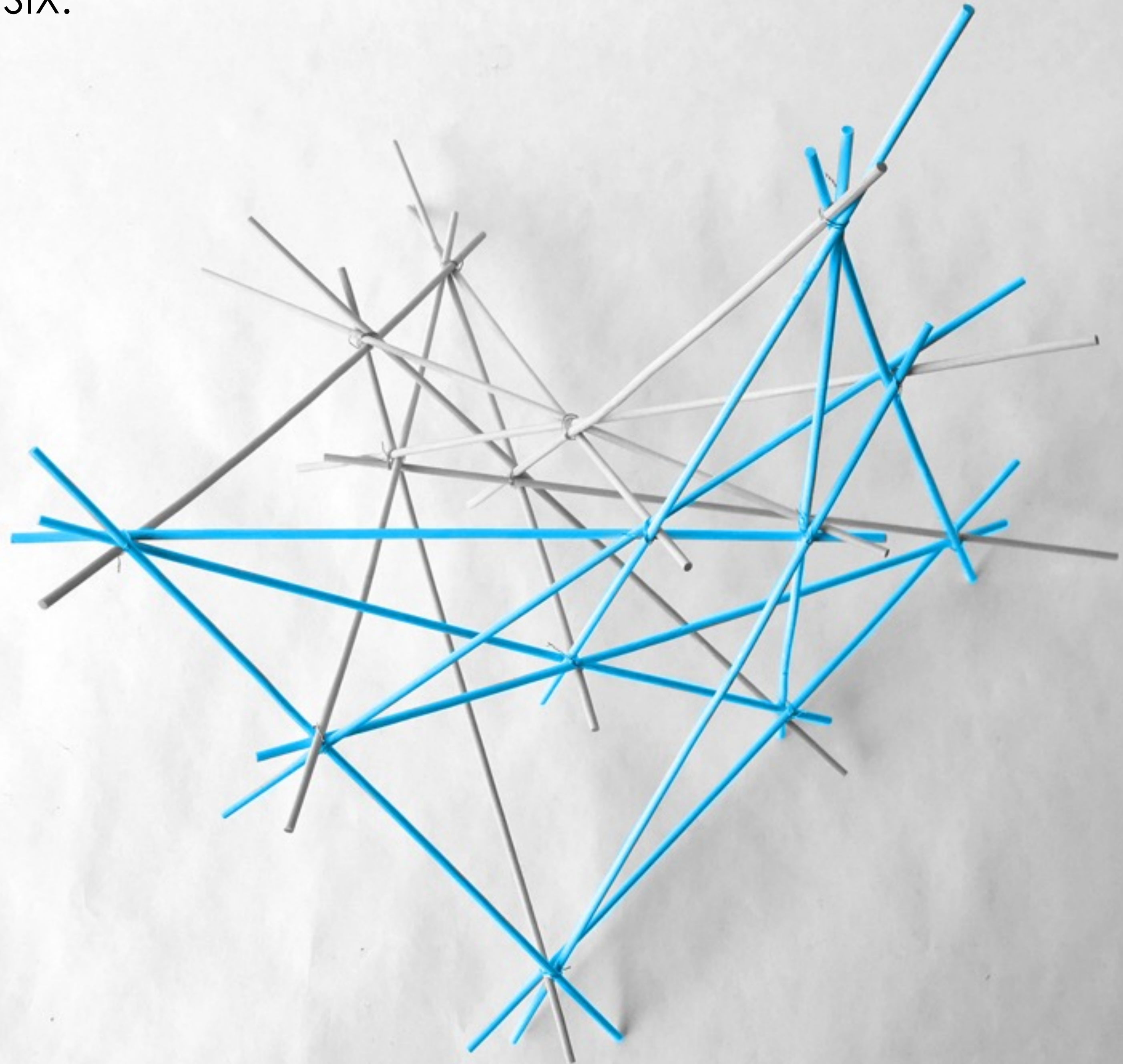
FOUR:



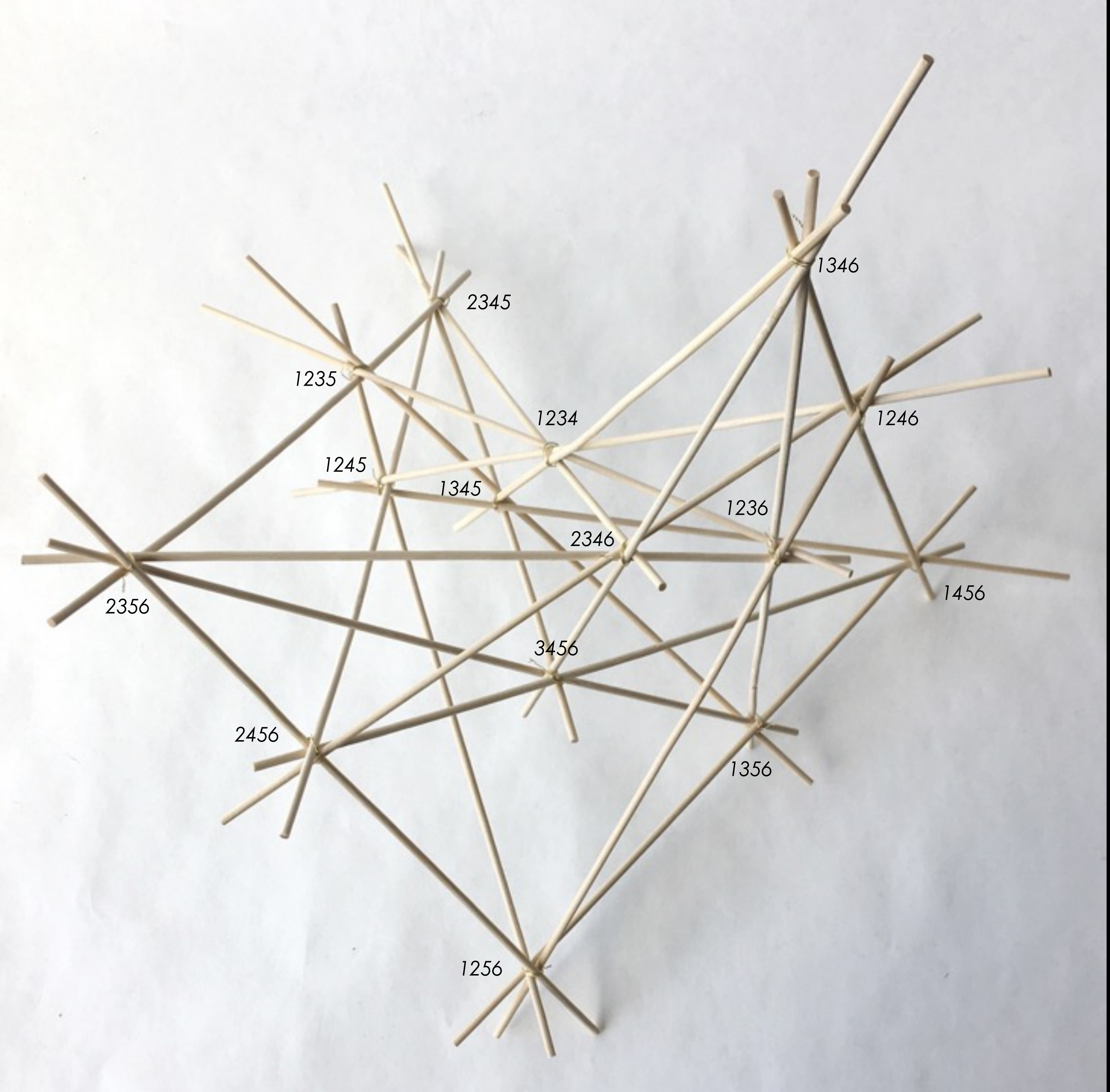
FIVE:



SIX:

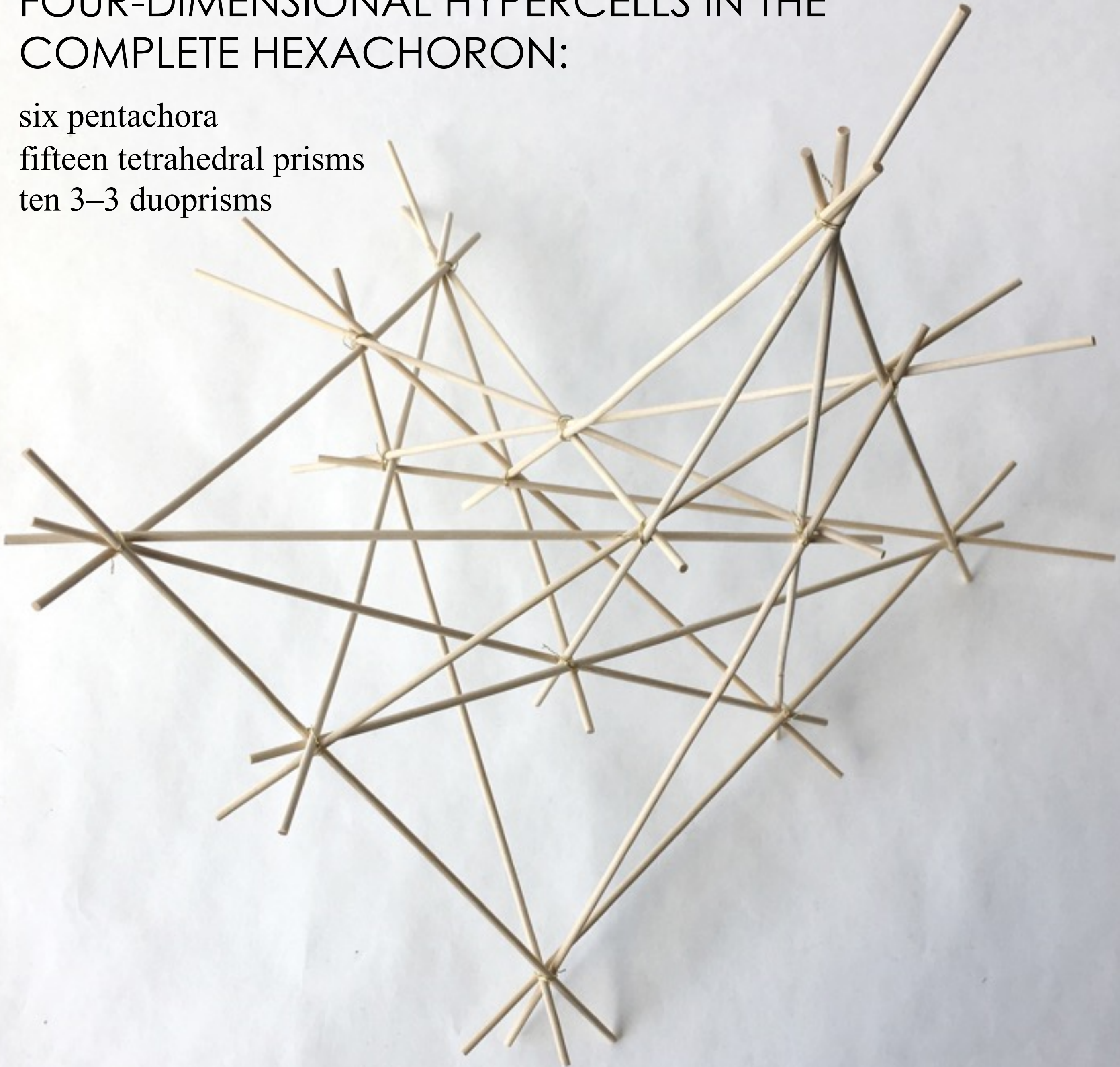


Each 3-space (hyperplane) of the complete hexachoron is a Desargues configuration, and the planes, the lines, and the points are realized as all possible combinations of two, three, and four members, respectively, from the set of these six 3-spaces.



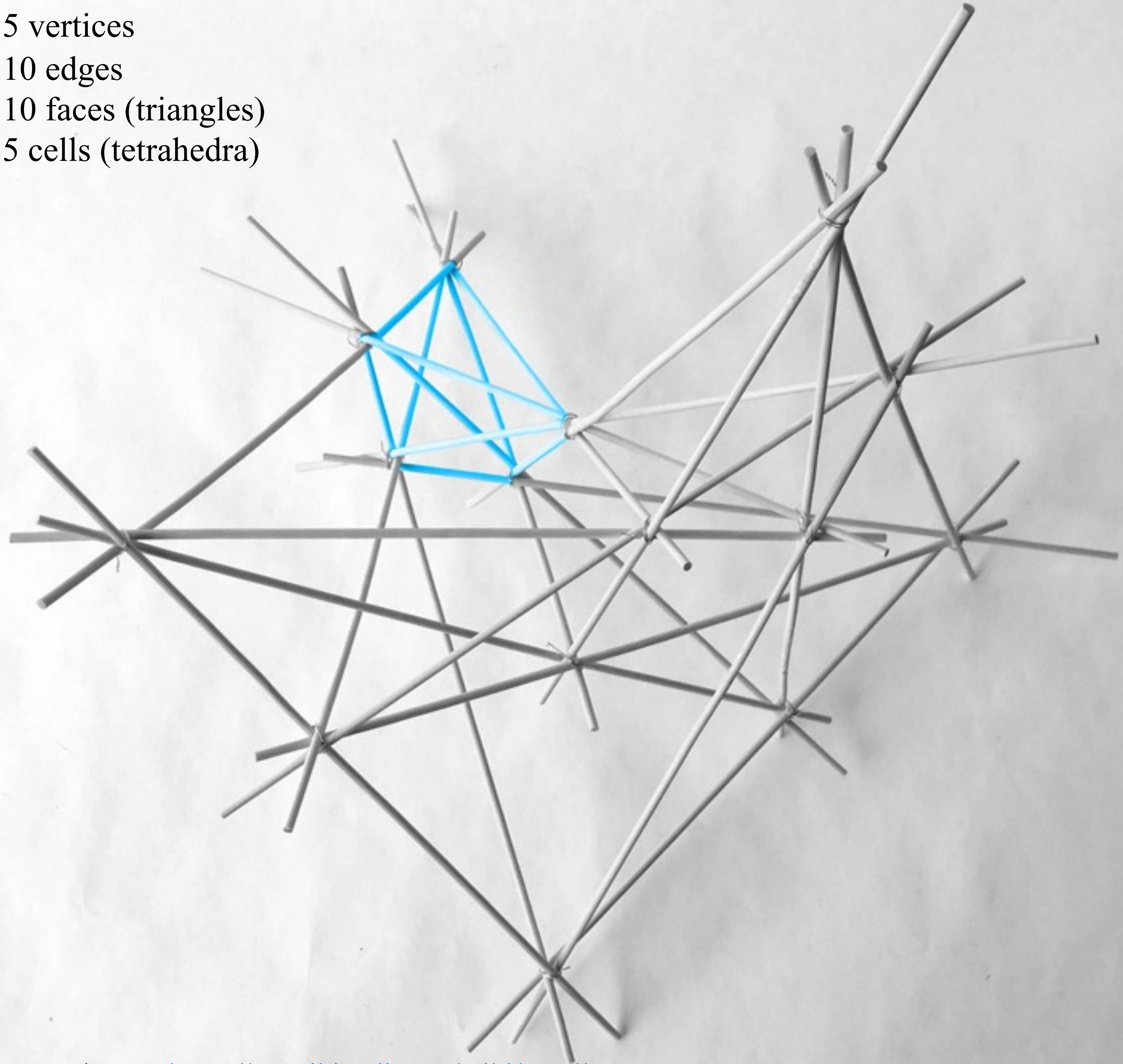
FOUR-DIMENSIONAL HYPERCELLS IN THE COMPLETE HEXACHORON:

six pentachora
fifteen tetrahedral prisms
ten 3–3 duoprisms



PENTACHORON

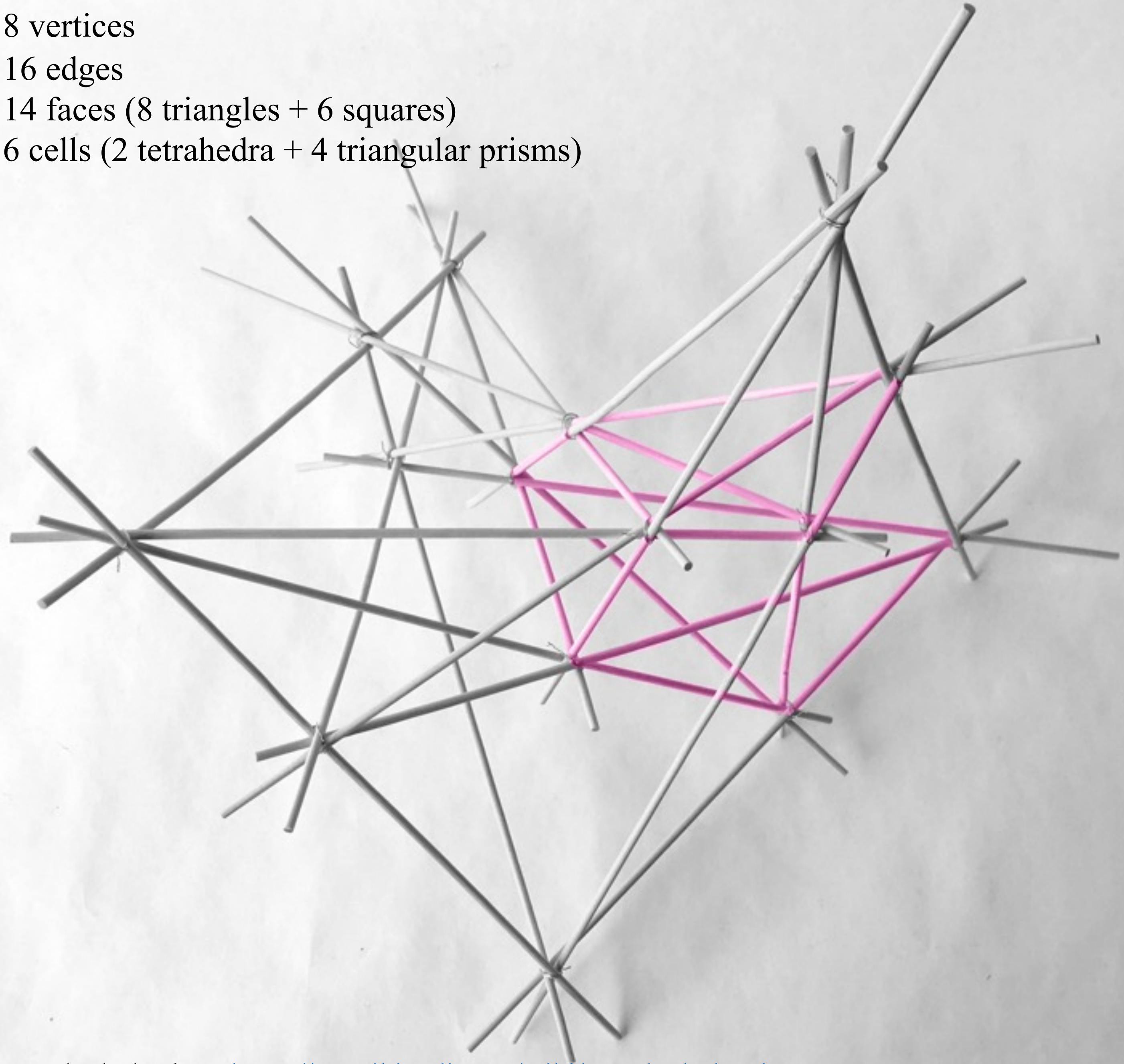
5 vertices
10 edges
10 faces (triangles)
5 cells (tetrahedra)



Pentachoron: https://en.wikipedia.org/wiki/5-cell

TETRAHEDRAL PRISM

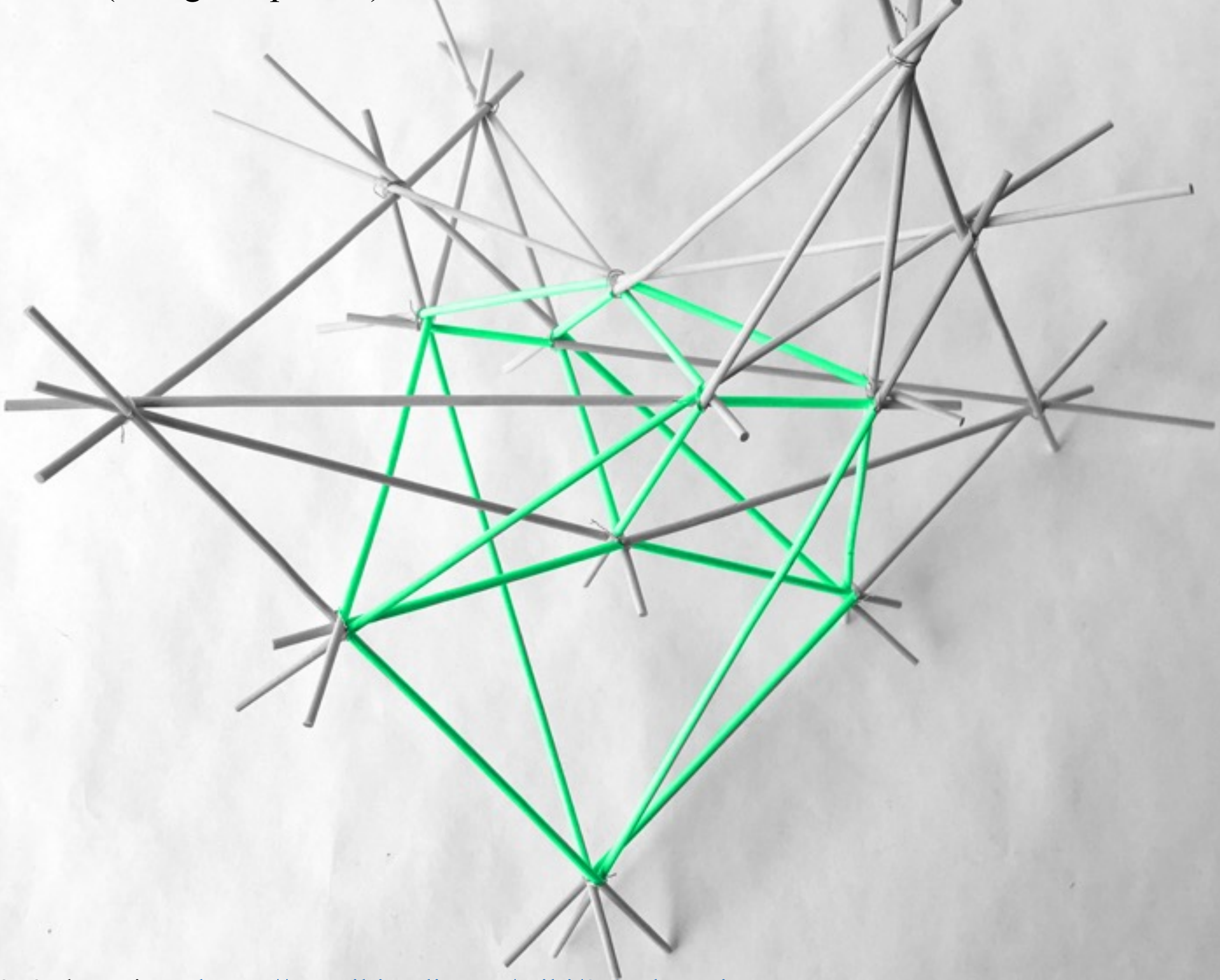
8 vertices
16 edges
14 faces (8 triangles + 6 squares)
6 cells (2 tetrahedra + 4 triangular prisms)



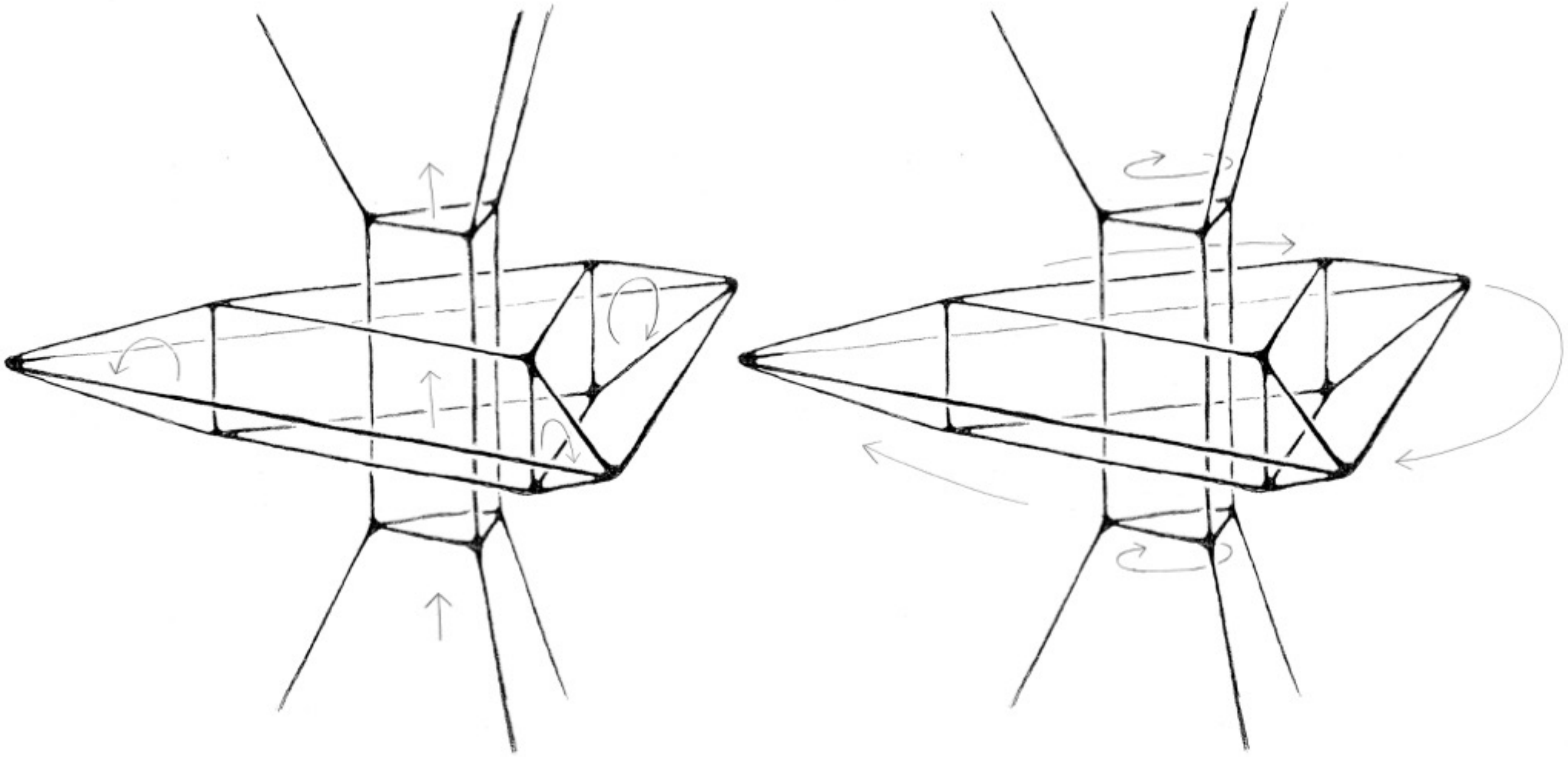
Tetrahedral prism: https://en.wikipedia.org/wiki/Tetrahedral_prism

3-3 DUOPRISM

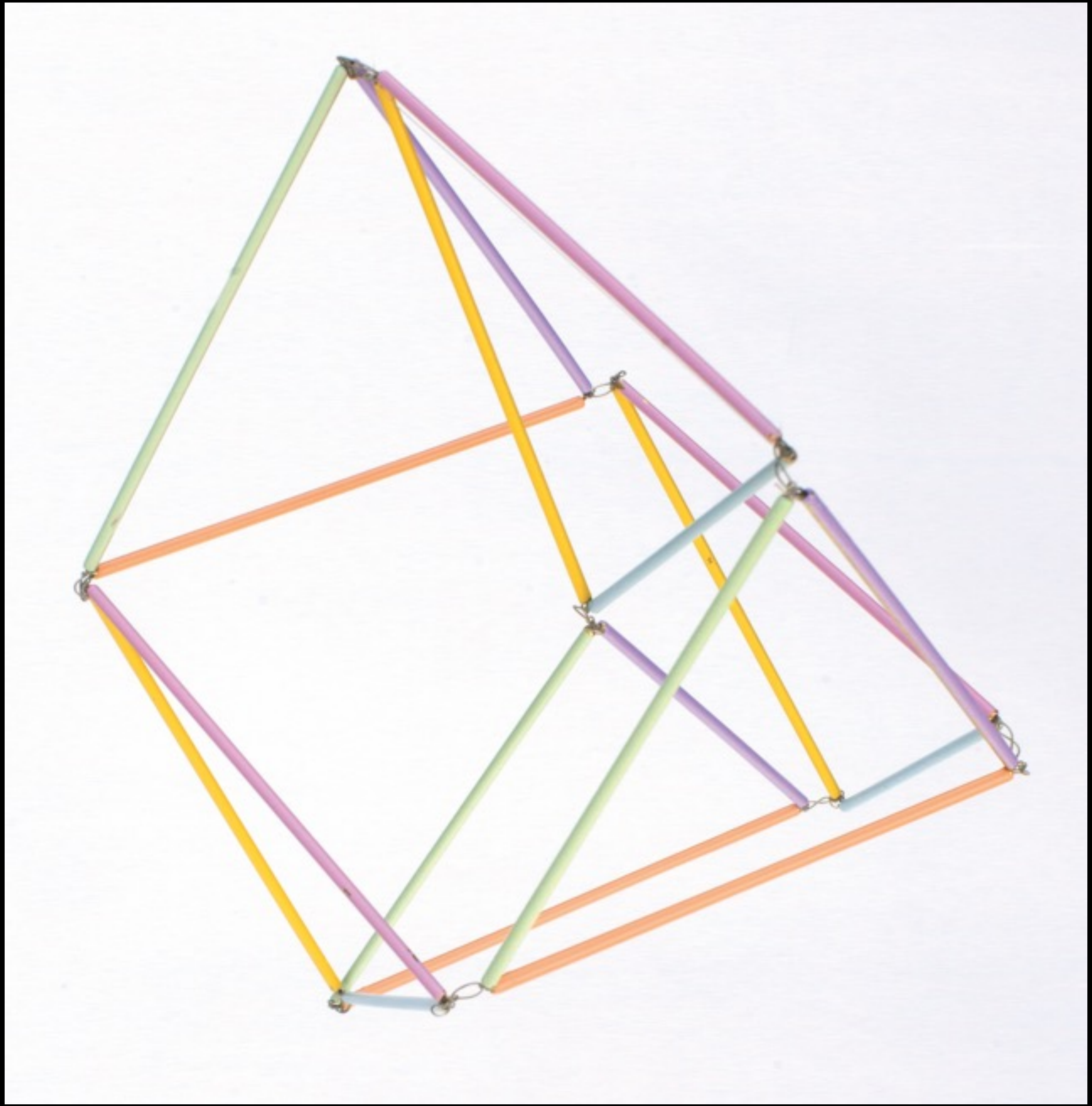
9 vertices
18 edges
15 faces (9 triangles + 6 squares)
6 cells (triangular prisms)



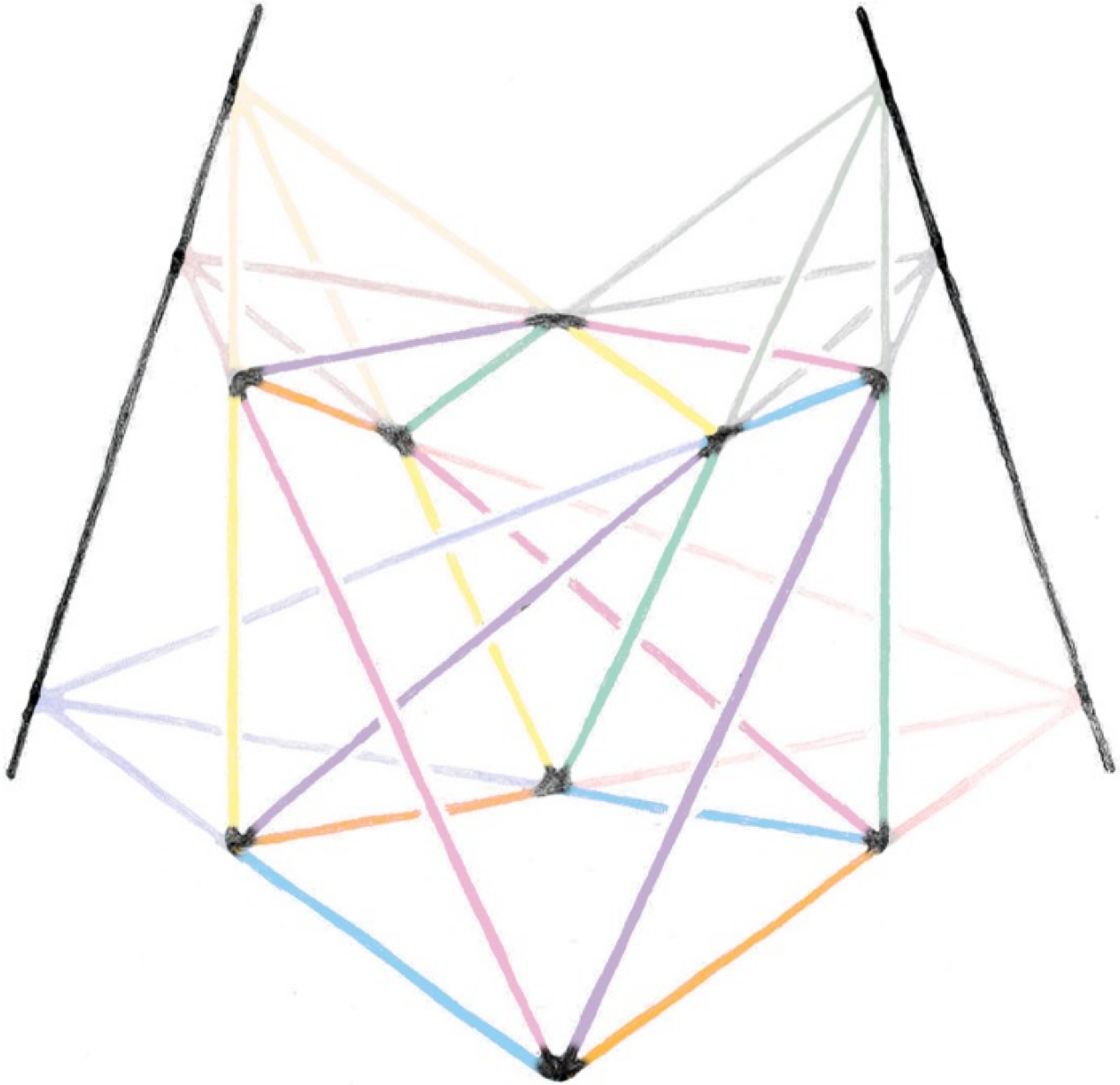
3–3-duoprism: https://en.wikipedia.org/wiki/3-3_duoprism



The 'Prismary'



The parallel edges of the 3-3 duoprism converge towards the vanishing points on the horizon lines, of which there are two – one for both of the cycles of prisms



REYE’S CONFIGURATION

REYE'S CONFIGURATION

12 points
16 lines
12 planes

4 lines and 6 planes per point
3 points and 3 planes per line
6 points and 4 lines per plane (a complete quadrilateral)

Stick model building video (slightly blurry)
https://youtu.be/fby53U_n4o8