6 SHELL

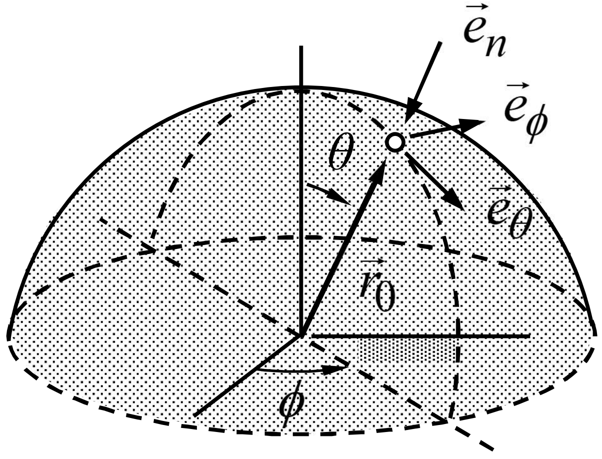
LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on the topics of week 14:

- □ Reissner-Mindlin and Kirchhoff shell models and Kirchhoff constraints.
- □ Shell equilibrium and constitutive equations in their tensor forms.
- □ Component representations of the membrane and shell equations for cylindrical and spherical geometries
- □ Derivation of shell equations by using the principle of virtual work, integration by parts, and the fundamental lemma of variation calculus.

EFFECT OF CURVATURE

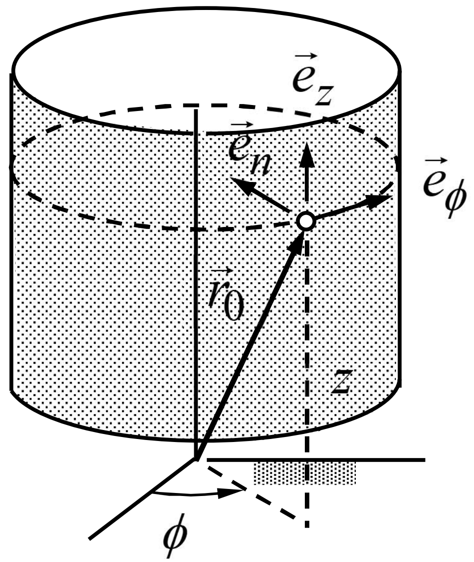
Sphere subjected to internal pressure:



$$N_{\phi\phi} = \frac{1}{2} pR \quad \text{and} \quad N_{\theta\theta} = \frac{1}{2} pR \;\Rightarrow$$

$$\overleftrightarrow{N} = \frac{1}{2} pR(\vec{e}_\phi \vec{e}_\phi + \vec{e}_\theta \vec{e}_\theta) = \frac{1}{2} pR\overleftrightarrow{I} \quad \text{(isotropic stress)}$$

Long cylinder subjected to internal pressure:

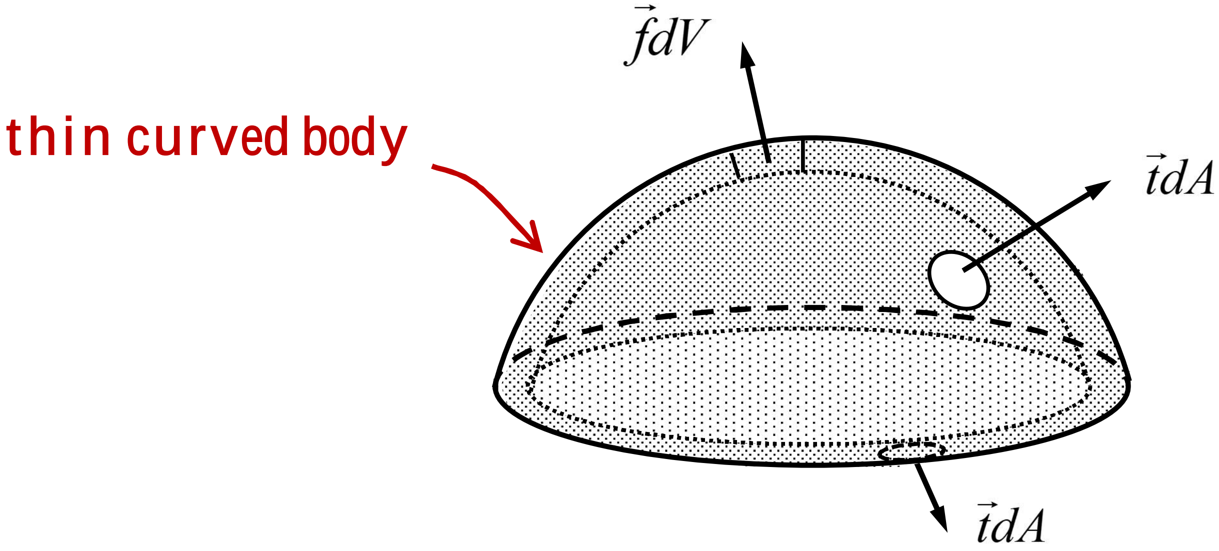


$$N_{zz} = \frac{1}{2} pR \quad \text{and} \quad N_{\phi\phi} = pR \;\Rightarrow$$

"curvature"

$$\overleftrightarrow{N} = \frac{1}{2} pR(\vec{e}_z \vec{e}_z + 2\vec{e}_\phi \vec{e}_\phi)$$

SHELL MODELS



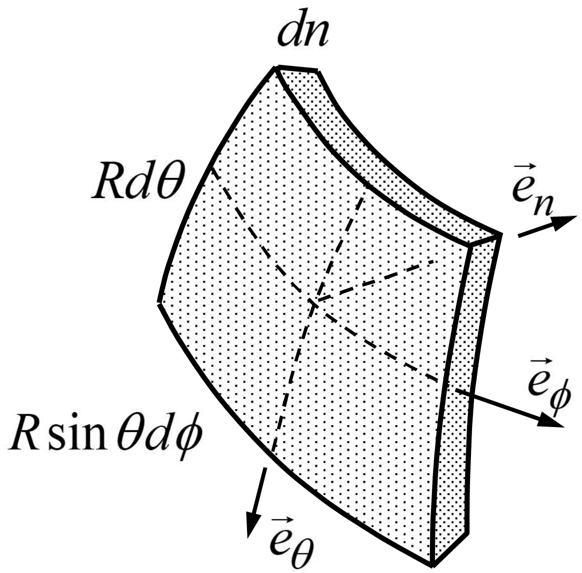
Kinematic assumption: Straight line segments perpendicular to the mid-surface remain straight in deformation (Reissner-Mindlin) or straight and perpendicular to the mid-surface (Kirchhoff) in deformation $\vec{u} = \vec{u}_0 + \vec{\theta}_0 \times n\vec{e}_n = \vec{u}_0 + n\vec{\omega}_0$.

Kinetic assumption: Stress component $\sigma_{nn} = 0$.

VOLUME AND AREA ELEMENT REPRESENTATIONS

The integrals of the virtual work expression are always over a body. Representations of the volume and area elements consist of the mid-surface elements and scaling factors taking into account the offset effect.

$$dV = \overset{\text{scaling factor}}{\left(\frac{R-n}{R}\right)^2} \underset{\text{small dimension}}{dn} \underbrace{(R^2\sin\theta)\overbrace{d\phi d\theta}^{\text{domain element}}}_{\text{mid-surface area element}}$$



In MEC-E8003, the region occupied by the body is (in most cases) a cuboid of the material coordinate space!

The scaling factors for the area elements depend on the direction of the boundary $\vec{n}$ (the unit outward normal vector). In terms of curvature of the mid-surface $\vec{\vec{\kappa}}$

$dV = JdndA$ and $dA = J(\vec{n})dnds$, where

$J(\vec{n}) = 1 - n\kappa(\vec{n})$ and $J = J(\vec{e}_\alpha)J(\vec{e}_\beta)$ and $\kappa(\vec{n}) = \vec{n}\cdot(\vec{e}_n \times \vec{\vec{\kappa}}_{\mathrm{c}} \times \vec{e}_n)\cdot\vec{n}$.

For example, in the cylindrical geometry and (z,ϕ,n) coordinates

$$\vec{\vec{\kappa}} = -\frac{1}{R}\vec{e}_\phi\vec{e}_\phi \quad\Rightarrow\quad \kappa(\vec{n}) = \vec{n}\cdot(\vec{e}_n \times \vec{\vec{\kappa}}_{\mathrm{c}} \times \vec{e}_n)\cdot\vec{n} = \frac{1}{R}(\vec{n}\cdot\vec{e}_{\mathrm{z}})(\vec{e}_{\mathrm{z}}\cdot\vec{n}) \quad\Rightarrow$$

$$J(\vec{e}_{\mathrm{z}}) = 1 - n\kappa(\vec{e}_{\mathrm{z}}) = 1 - \frac{n}{R}\,,\quad J(\vec{e}_\phi) = 1 - n\kappa(\vec{e}_\phi) = 1, \text{ and } J = J(\vec{e}_{\mathrm{z}})J(\vec{e}_\phi) = 1 - \frac{n}{R}.$$

GRADIENT REPRESENTATION

In derivation of equilibrium equations from virtual work expression of shell, gradient needs to be expressed in terms of the mid-surface gradient ∇_0, offset scaling $\overleftrightarrow{D}$, and the normal part:

scaling mid-surface part normal part

Generic: $$\nabla = \overleftrightarrow{D} \cdot (\nabla_0 + \vec{e}_n \frac{\partial}{\partial n})$$

Cylindrical: $$\nabla = (\vec{e}_z\vec{e}_z + \frac{R}{R-n}\vec{e}_\phi\vec{e}_\phi + \vec{e}_n\vec{e}_n)\cdot(\vec{e}_z \frac{\partial}{\partial z} + \frac{1}{R}\vec{e}_\phi \frac{\partial}{\partial \phi} + \vec{e}_n \frac{\partial}{\partial n})$$

Spherical: $$\nabla = (\frac{R}{R-n}\vec{e}_\phi\vec{e}_\phi + \frac{R}{R-n}\vec{e}_\theta\vec{e}_\theta + \vec{e}_n\vec{e}_n)\cdot(\frac{1}{R\sin\theta}\vec{e}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R}\vec{e}_\theta \frac{\partial}{\partial \theta} + \vec{e}_n \frac{\partial}{\partial n})$$

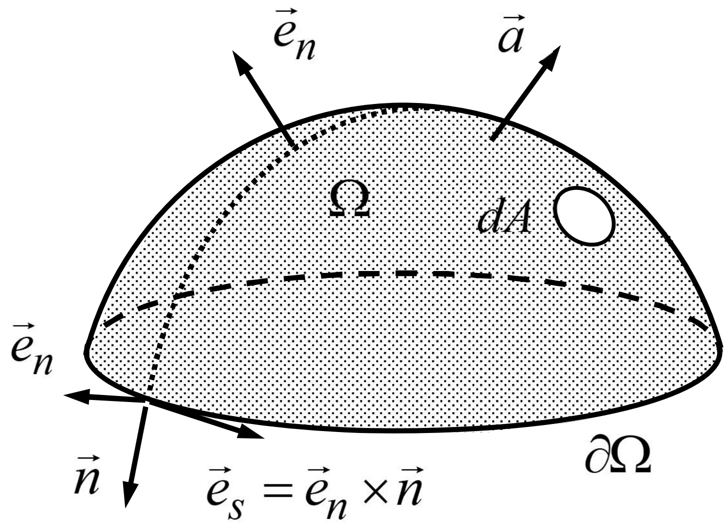
In flat geometry $\overleftrightarrow{D} = \overleftrightarrow{I}$ and in the thin body limit $(t / R \ll 1)$ $\overleftrightarrow{D} \approx \overleftrightarrow{I}$. Notice that integration by parts formula on curved surfaces is concerned with ∇_0.

GAUSS THEOREM ON CURVED SURFACES

As a generic vector identity, Gauss theorem is valid also when a thin body has curved mid-surface geometry. However, all parts of the boundary need to be accounted for correctly. Assuming that vector $\vec{a}$ does not depend on the transverse coordinate, it holds

$$\int_V \nabla \cdot \vec{a} dV = \int_{\partial V} \vec{n} \cdot \vec{a} dA \quad \text{and} \quad \frac{\partial \vec{a}}{\partial n} = 0 \quad \Leftrightarrow$$

$$\int_\Omega (\nabla_0 \cdot \vec{a} - \kappa \vec{e}_n \cdot \vec{a}) dA = \int_{\partial\Omega} (\vec{n} \cdot \vec{a}) ds .$$



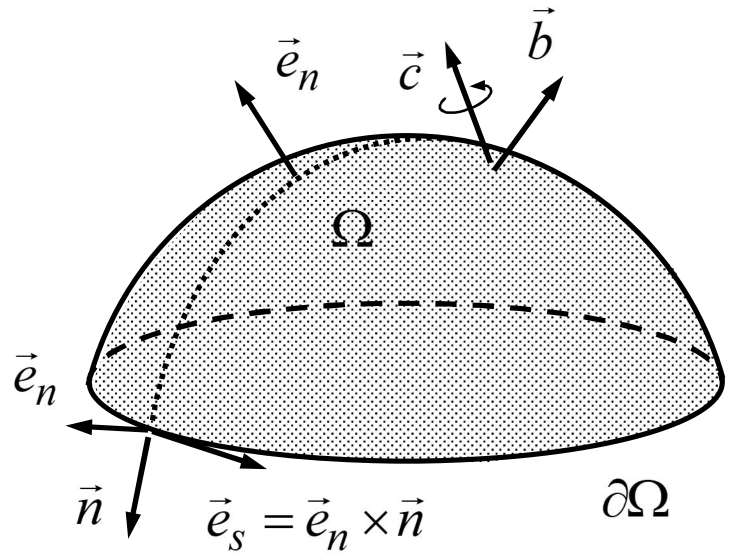
In the latter form, the area integral is over the mid-surface and the boundary integral over the boundary of the mid-surface. Term $\kappa = \nabla_0 \cdot \vec{e}_n$ is twice the mean curvature of the mid-surface or the trace of curvature tensor $\kappa = \vec{\vec{\kappa}} : \vec{\vec{I}}$.

6.1 SHELL EQUATIONS

Virtual work expression of shell, principle of virtual work, integration by parts on curved surfaces (Kelvin-Stokes), and the fundamental lemma of variation calculus give:

$$\left.\begin{aligned}&\nabla_0 \cdot \overleftrightarrow{F} - \kappa \vec{e}_n \cdot \overleftrightarrow{F} + \vec{b} = 0\\[2ex] &(\nabla_0 \cdot \overleftrightarrow{M} - \kappa \vec{e}_n \cdot \overleftrightarrow{M} - \vec{e}_n \cdot \overleftrightarrow{F} + \vec{c}) \times \vec{e}_n = 0\end{aligned}\right\}\quad \text{in } \Omega$$

$$\left.\begin{aligned}&\vec{n} \cdot \overleftrightarrow{F} - \underline{\vec{F}} = 0 \quad \text{or} \quad \vec{u}_0 - \underline{\vec{u}}_0 = 0\\[2ex] &(\vec{n} \cdot \overleftrightarrow{M} - \underline{\vec{M}}) \times \vec{e}_n = 0 \quad \text{or} \quad \vec{\theta}_0 - \underline{\vec{\theta}}_0 = 0\end{aligned}\right\}\quad \text{on } \partial\Omega$$



Conditions on $\partial\Omega$ need to be expressed finally in the boundary system with component representations of $\vec{n}$ and $\vec{e}_s = \vec{e}_n \times \vec{n}$.

RESULTANT DEFINITIONS

Stress and external force resultants are integrals over the thickness ($\vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n$). Stress resultant definition gives the constitutive equations:

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \begin{bmatrix} \overset{\leftrightarrow}{A} & \overset{\leftrightarrow}{C} \\ \overset{\leftrightarrow}{C} & \overset{\leftrightarrow}{B} \end{bmatrix} : \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix}, \quad \begin{bmatrix} \overset{\leftrightarrow}{A} & \overset{\leftrightarrow}{C} \\ \overset{\leftrightarrow}{C} & \overset{\leftrightarrow}{B} \end{bmatrix} = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} (\overset{\leftrightarrow}{D}_{\mathrm{c}} \cdot \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} \cdot \overset{\leftrightarrow}{D} J) dn, \quad \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix} = \begin{Bmatrix} \nabla_0 \vec{u}_0 + \vec{e}_n \vec{\omega}_0 \\ \nabla_0 \vec{\omega}_0 \end{Bmatrix}$$

$$\begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \vec{f} \begin{Bmatrix} 1 \\ n \end{Bmatrix} J dn + \sum \vec{t} \begin{Bmatrix} 1 \\ n \end{Bmatrix} J,$$

external force and moment per unit area

$$\begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \vec{t} \begin{Bmatrix} 1 \\ n \end{Bmatrix} J(\vec{n}) dn.$$

external force and moment per unit length

Elasticity tensor of plate $\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}$ is assumed to satisfy the minor and major symmetries and the kinetic assumption $\sigma_{nn} = 0$ ‘a priori’.

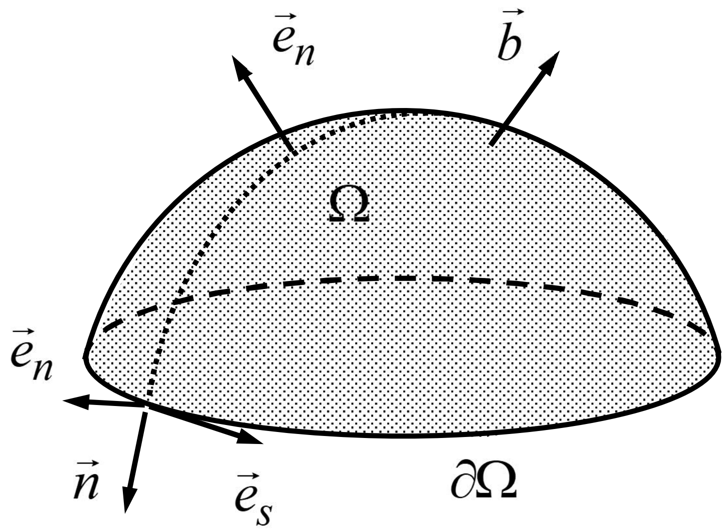
MEMBRANE EQUATIONS

Shell equations combine the thin-slab and bending modes of deformation that are always connected in curved geometry. The membrane model, i.e., thin-slab model in curved geometry, is useful in cases of non-negligible tension rigidity and negligible bending rigidity. Then

$$\nabla_0 \cdot \overleftrightarrow{N} + \vec{b} = 0 \quad \text{in} \quad \Omega \,,$$

$$\overleftrightarrow{N} = \overleftrightarrow{\overleftrightarrow{A}} : \nabla_0 \vec{u}_0 \quad \text{in} \quad \Omega,$$

$$\vec{n} \cdot \overleftrightarrow{N} - \underline{\vec{N}} = 0 \quad \text{or} \quad \vec{u}_0 - \underline{\vec{u}}_0 = 0 \quad \text{on} \quad \partial\Omega.$$



Equations follow from the shell equations with the kinetic assumptions $\vec{Q} = 0$ and $\overleftrightarrow{M} = 0$ for textile materials, skin of balloon, etc. of negligible bending rigidity.

CYLINDRICAL MEMBRANE (z,ϕ,n)

Equilibrium and constitutive equations of a cylindrical membrane follow from the coordinate system invariant forms of the membrane equations when gradient etc. are represented in (z,ϕ,n)-coordinate system:

$$\begin{Bmatrix} \dfrac{1}{R}\dfrac{\partial N_{\phi z}}{\partial \phi}+\dfrac{\partial N_{zz}}{\partial z}+b_z \\ \dfrac{\partial N_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial \phi}+b_\phi \\ \dfrac{1}{R}N_{\phi\phi}+b_n \end{Bmatrix}=0, \quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix}=t\left[E\right]_\sigma \begin{Bmatrix} \dfrac{\partial u_z}{\partial z} \\ \dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial \phi}-u_n) \\ \dfrac{1}{R}\dfrac{\partial u_z}{\partial \phi}+\dfrac{\partial u_\phi}{\partial z} \end{Bmatrix}, \text{ and } N_{\phi z}=N_{z\phi} \text{ in } \Omega.$$

Conditions on $\partial\Omega$ should be expressed in the boundary system with $\vec{n}=\vec{e}_z n_z+\vec{e}_\phi n_\phi$ and $\vec{e}_s=\vec{e}_n\times\vec{n}=\vec{e}_\phi n_r-\vec{e}_r n_\phi$.

In cylindrical geometry and (z,ϕ,n) coordinates, gradient operator takes the form

$$\nabla = \vec{e}_z \frac{\partial}{\partial z} + (\frac{R}{R-n})\frac{1}{R}\vec{e}_\phi \frac{\partial}{\partial \phi} + \vec{e}_n \frac{\partial}{\partial n} = \overleftrightarrow{D} \cdot (\nabla_0 + \vec{e}_n \frac{\partial}{\partial n}),$$

where $\nabla_0 = \vec{e}_z \frac{\partial}{\partial z} + \kappa \vec{e}_\phi \frac{\partial}{\partial \phi}$, $\overleftrightarrow{D} = \vec{e}_z \vec{e}_z + \frac{1}{1-\kappa n}\vec{e}_\phi \vec{e}_\phi + \vec{e}_n \vec{e}_n$, and $\kappa = \frac{1}{R}$.

Direct calculation with representations $\overleftrightarrow{N} = N_{zz}\vec{e}_z\vec{e}_z + N_{z\phi}\vec{e}_z\vec{e}_\phi + N_{\phi z}\vec{e}_\phi\vec{e}_z + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi$, $\vec{b} = b_z\vec{e}_z + b_\phi\vec{e}_\phi + b_n\vec{e}_n$ and the known derivatives of the basis vectors gives

$$\nabla_0 \cdot \overleftrightarrow{N} + \vec{b} = \vec{e}_z(\frac{1}{R}\frac{\partial N_{\phi z}}{\partial \phi} + \frac{\partial N_{zz}}{\partial z} + b_z) + \vec{e}_\phi(\frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial \phi} + b_\phi) + \vec{e}_n(\frac{1}{R}N_{\phi\phi} + b_n) = 0.$$

Elasticity tensor $\overset{\leftrightarrow}{\overset{\leftrightarrow}{A}}$ of shell depends on the isotropic plate elasticity $\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}$, scaling $\overset{\leftrightarrow}{D}$, and Jacobian $J = 1 - \kappa n$. Assuming a very thin membrane $t/R \ll 1$ for simplicity so that $J \approx 1$ and $\overset{\leftrightarrow}{D} \approx \overset{\leftrightarrow}{I}$ (the precise expressions will be discussed later)

$$\overset{\leftrightarrow}{\overset{\leftrightarrow}{A}} = \int (\overset{\leftrightarrow}{D}_{\mathrm{c}} \cdot \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} \cdot \overset{\leftrightarrow}{D} J) dn = \begin{Bmatrix} \vec{e}_\phi \vec{e}_\phi \\ \vec{e}_z \vec{e}_z \\ \vec{e}_\phi \vec{e}_z + \vec{e}_z \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} t \left[E \right]_\sigma \begin{Bmatrix} \vec{e}_\phi \vec{e}_\phi \\ \vec{e}_z \vec{e}_z \\ \vec{e}_\phi \vec{e}_z + \vec{e}_z \vec{e}_\phi \end{Bmatrix}.$$

Only the translation part $\vec{u} = u_z \vec{e}_z + u_\phi \vec{e}_\phi + u_n \vec{e}_n$ of the kinematic assumption matters. Direct calculation with the known derivatives of the basis vectors gives

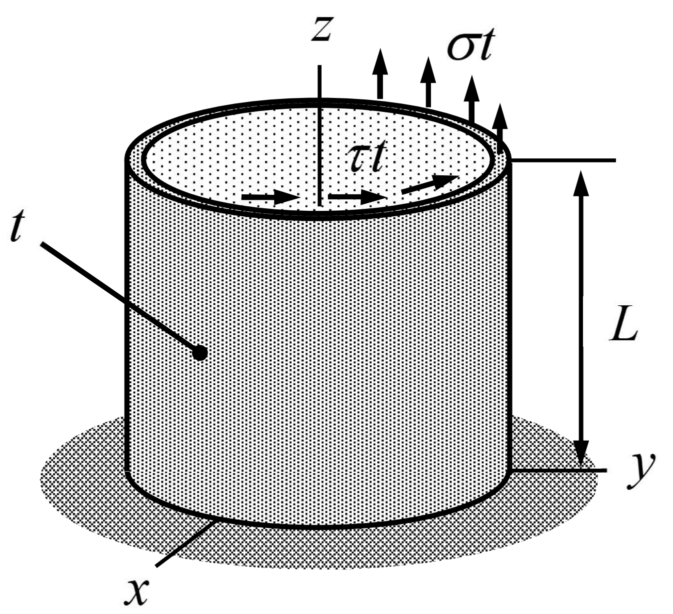
$$\nabla_0 \vec{u}_0 = \frac{\partial u_z}{\partial z} \vec{e}_z \vec{e}_z + \frac{\partial u_\phi}{\partial z} \vec{e}_z \vec{e}_\phi + \frac{1}{R} \frac{\partial u_z}{\partial \phi} \vec{e}_\phi \vec{e}_z + \frac{1}{R} \left(\frac{\partial u_\phi}{\partial \phi} - u_n \right) \vec{e}_\phi \vec{e}_\phi +$$

$$\frac{\partial u_n}{\partial z}\vec{e}_z\vec{e}_n + \frac{1}{R}(u_\phi + \frac{\partial u_n}{\partial \phi})\vec{e}_\phi\vec{e}_n.$$

Therefore, the constitutive equation $\overset{\leftrightarrow}{N} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{A}} : \nabla_0 \vec{u}_0$ takes the form

$$\overset{\leftrightarrow}{N} = \begin{Bmatrix} \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_z\vec{e}_z \\ \vec{e}_\phi\vec{e}_z + \vec{e}_z\vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} t\left[E\right]_\sigma \begin{Bmatrix} \frac{1}{R}(\frac{\partial u_\phi}{\partial \phi} - u_n) \\ \frac{\partial u_z}{\partial z} \\ \frac{\partial u_\phi}{\partial z} + \frac{1}{R}\frac{\partial u_z}{\partial \phi} \end{Bmatrix}, \text{ where } \left[E\right]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}. \Leftarrow$$

EXAMPLE 6.1 A thin walled cylindrical body of length L, (mid-surface) radius R, and thickness t is subjected to distributed loading $\vec{b} = b_z \vec{e}_z + b_\phi \vec{e}_\phi + b_n \vec{e}_n$ of constant components and boundary loading $\underline{\vec{F}} = t(\sigma \vec{e}_z + \tau \vec{e}_\phi)$ at the free end $z = L$. Assume rotational symmetry and use the membrane equations in (z, ϕ, n) coordinate system to solve for the mid-surface stress resultants.



Answer: $N_{zz} = \sigma t + b_z(L - z)$, $N_{z\phi} = \tau t + b_\phi(L - z)$, and $N_{\phi\phi} = -b_n R$

A rotational symmetric solution does not depend on ϕ. Then the equilibrium equations of the membrane model and the boundary conditions at the free edge simplify to

$$\frac{dN_{zz}}{dz} + b_z = 0, \quad \frac{dN_{z\phi}}{dz} + b_\phi = 0, \quad \frac{1}{R} N_{\phi\phi} + b_n = 0 \ \text{ in } (0, L)$$

$$N_{zz} - \sigma t = 0, \ \ N_{z\phi} - \tau t = 0 \ \text{ at } \ z = L.$$

Solution to the boundary value problem of two ordinary first order differential equations and one algebraic equation for the stress resultants is given by

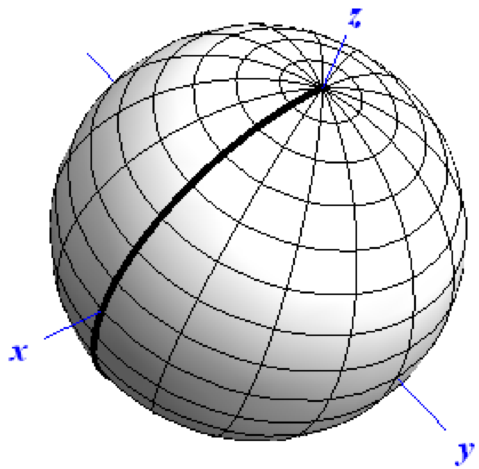
$$N_{zz} = \sigma t + b_z (L - z), \ \ N_{z\phi} = \tau t + b_\phi (L - z), \ \text{ and } \ N_{\phi\phi} = -b_n R.$$

SPHERICAL MEMBRANE (ϕ, θ, n)

$$\begin{Bmatrix} \dfrac{1}{R}[\csc\theta \dfrac{\partial N_{\phi\phi}}{\partial \phi} + \dfrac{\partial N_{\theta\phi}}{\partial \theta} + \cot\theta (N_{\theta\phi} + N_{\phi\theta})] + b_\phi \\ \dfrac{1}{R}[\csc\theta \dfrac{\partial N_{\phi\theta}}{\partial \phi} + \dfrac{\partial N_{\theta\theta}}{\partial \theta} + \cot\theta (N_{\theta\theta} - N_{\phi\phi})] + b_\theta \\ \dfrac{1}{R}(N_{\phi\phi} + N_{\theta\theta}) + b_n \end{Bmatrix} = 0$$

$$\begin{Bmatrix} N_{\phi\phi} \\ N_{\theta\theta} \\ N_{\phi\theta} \end{Bmatrix} = t\left[E\right]_\sigma \frac{1}{R} \begin{Bmatrix} \csc\theta(\cos\theta u_\theta + \dfrac{\partial u_\phi}{\partial \phi}) - u_n \\ \csc\theta \sin\theta \dfrac{\partial u_\theta}{\partial \theta} - u_n \\ \csc\theta \dfrac{\partial u_\theta}{\partial \phi} - \cot\theta u_\phi + \dfrac{\partial u_\phi}{\partial \theta} \end{Bmatrix} \text{ and } \quad N_{\theta\phi} = N_{\phi\theta}$$

EXAMPLE 6.2 Consider a balloon in (ϕ,θ,n) coordinates under positive pressure difference $\Delta p = p_{in} - p_{out}$. Assuming a rotational symmetric solution with respect to two axes, so that all stress resultants and displacement components are independent of ϕ and θ, find the membrane stress and displacement of the surface.



Answer: $\overleftrightarrow{N} = \dfrac{\Delta p R}{2}(\vec{e}_\phi\vec{e}_\phi + \vec{e}_\theta\vec{e}_\theta)$ and $\vec{u} = -\dfrac{\Delta p R^2(1-\nu)}{2tE}\vec{e}_n$

NOTICE. Linear elasticity theory assumes an equilibrium initial geometry with $\overleftrightarrow{N}_0$, Δp_0, and R_0. The aim is to find the new equilibrium $\overleftrightarrow{N}$, Δp, and R due to the change in pressure. Here, displacement gives the change in radius due to the increase in the pressure difference.

According to the assumption, derivatives with respect to ϕ and θ vanish. The components of distributed force are $b_\phi = b_\theta = 0$ and $b_n = -\Delta p$ (n is directed inwards). Equilibrium equations ($N_{\theta\phi} = N_{\phi\theta}$) simplify to

$$2\cot\theta N_{\phi\theta} = 0, \quad \cot\theta(N_{\theta\theta} - N_{\phi\phi}) = 0, \quad N_{\theta\theta} + N_{\phi\phi} - \Delta pR = 0 \quad \text{in } \Omega \quad \Rightarrow$$

$$N_{\phi\theta} = 0 \;\text{ and }\; N_{\theta\theta} = N_{\phi\phi} = \frac{\Delta pR}{2}.$$

Due to the rotational symmetry also $u_\theta = u_\phi = 0$. With the solution above, constitutive equations give

$$N_{\phi\phi} = N_{\theta\theta} = -\frac{tE}{1-\nu^2}\frac{1}{R}(1+\nu)u_n = \frac{\Delta pR}{2} \quad \Leftrightarrow \quad u_n = -\frac{pR^2(1-\nu)}{2tE}.$$

6.2 CYLINDRICAL SHELL (z,ϕ,n)

In curved geometry, the thin-slab and bending modes are always connected. In cylindrical geometry and (z,ϕ,n) coordinates, the equilibrium equations of shell take the forms to

$$\left\{\begin{matrix} \dfrac{1}{R}\dfrac{\partial N_{\phi z}}{\partial \phi}+\dfrac{\partial N_{zz}}{\partial z}+b_z \\ \dfrac{\partial N_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial \phi}-\dfrac{1}{R}Q_{\phi n}+b_\phi \end{matrix}\right\}=0, \quad \left\{\begin{matrix} \dfrac{1}{R}\dfrac{\partial Q_{\phi n}}{\partial \phi}+\dfrac{\partial Q_{zn}}{\partial z}+\dfrac{1}{R}N_{\phi\phi}+b_n \\ \dfrac{\partial M_{zz}}{\partial z}+\dfrac{1}{R}\dfrac{\partial M_{\phi z}}{\partial \phi}-Q_{nz}+c_z \\ \dfrac{\partial M_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial M_{\phi\phi}}{\partial \phi}-\dfrac{1}{R}M_{\phi n}-Q_{n\phi}+c_\phi \end{matrix}\right\}=0.$$

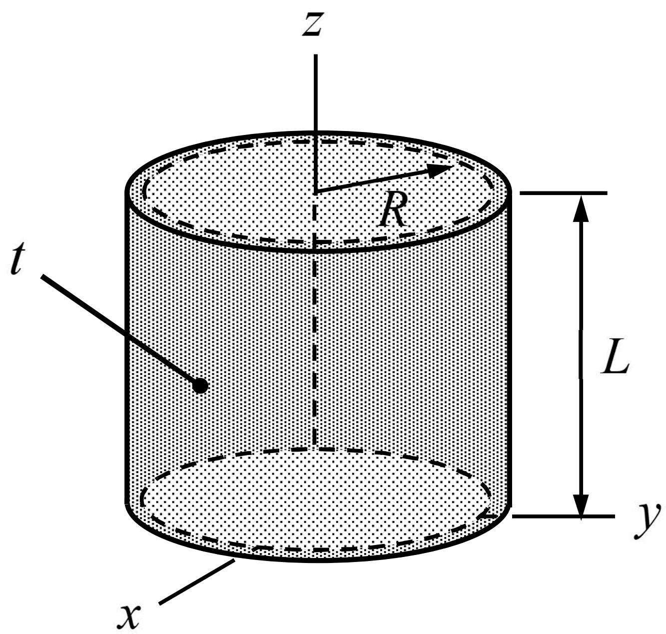
The boundary conditions on $\partial\Omega$ need to be deduced from the generic forms for the boundary system with $\vec{n}=\vec{e}_z n_z+\vec{e}_\phi n_\phi$ and $\vec{e}_s=\vec{e}_n\times\vec{n}=\vec{e}_\phi n_z-\vec{e}_z n_\phi$. The non-zero constitutive equations for a thin shell $(t/R)^2 \ll 1)$ take the forms

continues ...

$$\begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \\ N_{\phi z} \end{Bmatrix} = \begin{Bmatrix} \dfrac{tE}{1-\nu^2}[\dfrac{\partial u_z}{\partial z} + \nu \dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial \phi} - u_n)] - D\dfrac{1}{R}\dfrac{\partial \theta_\phi}{\partial z} \\ \dfrac{tE}{1-\nu^2}[\dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial \phi} - u_n) + \nu \dfrac{\partial u_z}{\partial z}] - D\dfrac{1}{R^2}\dfrac{\partial \theta_z}{\partial \phi} \\ Gt(\dfrac{1}{R}\dfrac{\partial u_z}{\partial \phi} + \dfrac{\partial u_\phi}{\partial z}) + \dfrac{1}{2}(1-\nu)D\dfrac{1}{R}\dfrac{\partial \theta_z}{\partial z} \\ Gt(\dfrac{1}{R}\dfrac{\partial u_z}{\partial \phi} + \dfrac{\partial u_\phi}{\partial z}) + \dfrac{1}{2}(1-\nu)D\dfrac{1}{R^2}\dfrac{\partial \theta_\phi}{\partial \phi} \end{Bmatrix}, \quad \begin{Bmatrix} Q_z \\ Q_\phi \end{Bmatrix} = Gt\begin{Bmatrix} \theta_\phi + \dfrac{\partial u_n}{\partial z} \\ \dfrac{1}{R}(\dfrac{\partial u_n}{\partial \phi} + u_\phi) - \theta_z \end{Bmatrix},$$

$$\begin{Bmatrix} M_{zz} \\ M_{\phi\phi} \\ M_{z\phi} \\ M_{\phi z} \end{Bmatrix} = D\begin{Bmatrix} \dfrac{\partial \theta_\phi}{\partial z} - \nu \dfrac{1}{R}\dfrac{\partial \theta_z}{\partial \phi} - \dfrac{1}{R}\dfrac{\partial u_z}{\partial z} \\ \nu \dfrac{\partial \theta_\phi}{\partial z} - \dfrac{1}{R}\dfrac{\partial \theta_z}{\partial \phi} + \dfrac{1}{R^2}(\dfrac{\partial u_\phi}{\partial \phi} - u_n) \\ \dfrac{1}{2}(1-\nu)[(\dfrac{1}{R}\dfrac{\partial \theta_\phi}{\partial \phi} - \dfrac{\partial \theta_z}{\partial z}) - \dfrac{1}{R}\dfrac{\partial u_\phi}{\partial z}] \\ \dfrac{1}{2}(1-\nu)[(\dfrac{1}{R}\dfrac{\partial \theta_\phi}{\partial \phi} - \dfrac{\partial \theta_z}{\partial z}) + \dfrac{1}{R^2}\dfrac{\partial u_z}{\partial \phi}] \end{Bmatrix}, \; M_{\phi n} = \frac{1}{2}(1-\nu)D\frac{1}{R}[\frac{1}{R}(\frac{\partial u_n}{\partial \phi} + u_\phi) - \theta_z]$$

EXAMPLE 6.3 Consider a cylindrical container of radius R subjected to distributed force b_n due to internal excess pressure p. Assuming rigid end plates and rotation symmetry (derivatives with respect to ϕ vanish and $u_\phi = \theta_z = 0$), derive the differential equation and the boundary conditions for the transverse deflection $w(z) = u_n(z)$ according to the Kirchhoff model. Material is linearly elastic with properties E and ν. Thickness of the container wall is t.



Answer: $\frac{d^4 w}{dz^4} + \frac{2\nu}{R^2}\frac{d^2 w}{dz^2} + \frac{Et}{DR^2} w - \frac{1}{D}\left(\frac{\nu N}{R} + b_n\right) = 0$

In the Kirchhoff model, constitutive equations for the shear forces are replaced by Kirchhoff constraints. With relationship $\theta_\phi = -du_n / dz$ and the assumptions of the problem, the non-zero constitutive equations for the stress resultants simplify to

$$N_{zz} = \frac{tE}{1-\nu^2}\left(\frac{du_z}{dz} - \nu\frac{1}{R}u_n\right) + D\frac{1}{R}\frac{d^2u_n}{dz^2},$$

$$N_{\phi\phi} = \frac{tE}{1-\nu^2}\left(\nu\frac{du_z}{dz} - \frac{1}{R}u_n\right),$$

$$M_{zz} = -D\left(\frac{d^2u_n}{dz^2} + \frac{1}{R}\frac{du_z}{dz}\right) .$$

Equilibrium equations simplify to

$$\frac{dN_{zz}}{dz} = 0, \quad Q_\phi = 0 \; , \quad \frac{dQ_z}{dz} + \frac{1}{R}N_{\phi\phi} + b_n = 0, \text{ and } \frac{dM_{zz}}{dz} - Q_z = 0 \; .$$

and after elimination of the shear force (using the moment equation)

$$\frac{dN_{zz}}{dz} = 0 \quad \text{and} \quad \frac{d^2 M_{zz}}{dz^2} + \frac{1}{R} N_{\phi\phi} + b_n = 0.$$

The constitutive equations for $N_{\phi\phi}$ and M_{zz} can be expressed in terms of u_n by using the equilibrium and constitutive equations for N_{zz}:

$$\frac{dN_{zz}}{dz} = 0 \quad \Rightarrow \quad N_{zz} = \frac{tE}{1-\nu^2}\left(\frac{du_z}{dz} - \nu \frac{1}{R} u_n\right) + D \frac{1}{R} \frac{d^2 u_n}{dz^2} = N = const. \quad \Rightarrow$$

$$\frac{du_z}{dz} = \nu \frac{1}{R} u_n + \frac{1-\nu^2}{tE}\left(N - D \frac{1}{R} \frac{d^2 u_n}{dz^2}\right).$$

Hence after elimination of du_z / dz and with $a = t / R$

$$N_{\phi\phi} = \frac{tE}{1-\nu^2}(\nu\frac{du_z}{dz} - \frac{1}{R}u_n) = -\frac{tE}{R}u_n + \nu(N - D\frac{1}{R}\frac{d^2u_n}{dz^2}),$$

$$M_{zz} = -D(\frac{d^2u_n}{dz^2} + \frac{1}{R}\frac{du_z}{dz}) = -D[(1-\frac{a^2}{12})\frac{d^2u_n}{dz^2} + \frac{1}{R}\frac{1-\nu^2}{tE}N + \nu\frac{1}{R^2}u_n].$$

Using notation $u_n \equiv w$, equilibrium equation in the transverse direction gives

$$\frac{d^2M_{zz}}{dz^2} + \frac{1}{R}N_{\phi\phi} + b_n = -D[(1-\frac{a^2}{12})\frac{d^4w}{dz^4} + \nu\frac{2}{R^2}\frac{d^2w}{dz^2}] - \frac{tE}{R^2}w + \nu\frac{N}{R} + b_n = 0\ . \quad \Leftarrow$$

Assuming that the end plates are rigid so that the displacement and rotation vanish at ends of the cylindrical container and $a^2 \ll 1$, the boundary value problem for the transverse displacement (positive inwards) takes the form

$$\frac{d^4 w}{dz^4} + \nu \frac{2}{R^2} \frac{d^2 w}{dz^2} + \frac{tE}{DR^2} w - \frac{1}{D} (\nu \frac{N}{R} + b_n) = 0 \quad \text{in} \quad (0, L),$$

$$w = \frac{dw}{dz} = 0 \quad \text{on} \quad \{0, L\}.$$

The fourth order differential equation can further be simplified by omitting the second derivative term as negligible compared to the fourth order derivative term.

6.3 SPHERICAL SHELL

In spherical geometry and (ϕ,θ,n) coordinate system, the equilibrium equations of shell simplify to

$$\begin{Bmatrix} \frac{1}{R}(\frac{\partial}{\partial\theta}N_{\theta\phi}+\csc\theta\frac{\partial}{\partial\phi}N_{\phi\phi}+2\cot\theta N_{\phi\theta}-Q_{\phi})+b_{\phi} \\ \frac{1}{R}(\frac{\partial}{\partial\theta}N_{\theta\theta}+\csc\theta\frac{\partial}{\partial\phi}N_{\phi\theta}+\cot\theta N_{\theta\theta}-\cot\theta N_{\phi\phi}-Q_{\theta})+b_{\theta} \\ \frac{1}{R}(\frac{\partial}{\partial\theta}Q_{\theta}+\csc\theta\frac{\partial}{\partial\phi}Q_{\phi}+\cot\theta Q_{\theta}+N_{\theta\theta}+N_{\phi\phi})+b_{n} \end{Bmatrix}=0,$$

$$\begin{Bmatrix} \frac{1}{R}(\frac{\partial}{\partial\theta}M_{\theta\phi}+\csc\theta\frac{\partial}{\partial\phi}M_{\phi\phi}+2\cot\theta M_{\phi\theta})-Q_{\phi}+c_{\phi} \\ \frac{1}{R}(\frac{\partial}{\partial\theta}M_{\theta\theta}+\csc\theta\frac{\partial}{\partial\phi}M_{\phi\theta}+\cot\theta M_{\theta\theta}-\cot\theta M_{\phi\phi})-Q_{\theta}+c_{\theta} \end{Bmatrix}=0,$$

continues...

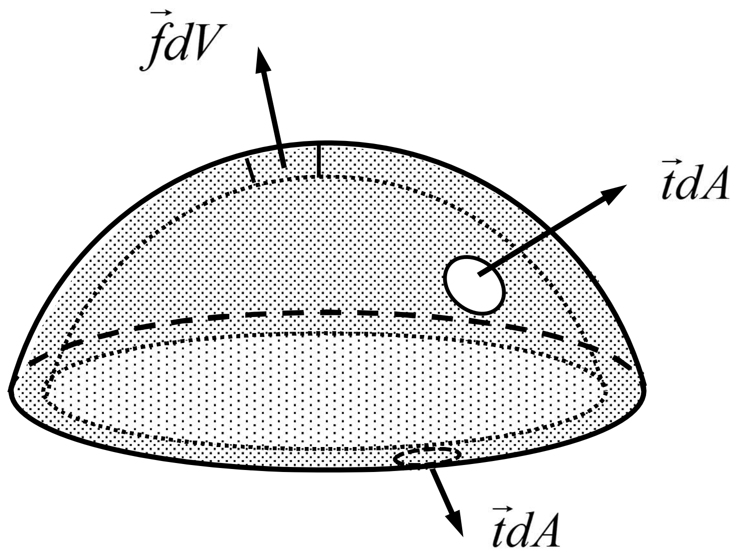
$$\begin{Bmatrix} N_{\phi\phi} \\ N_{\theta\theta} \\ N_{\phi\theta} \\ N_{\theta\phi} \end{Bmatrix} = \frac{Et}{1-\nu^2}\frac{1}{R}\begin{Bmatrix} (u_\theta \cot\theta + \frac{\partial u_\phi}{\partial \phi}\csc\theta - u_n) + \nu(\frac{\partial u_\theta}{\partial \theta} - u_n) \\ \nu(u_\theta \cot\theta + \frac{\partial u_\phi}{\partial \phi}\csc\theta - u_n) + (\frac{\partial u_\theta}{\partial \theta} - u_n) \\ \frac{1-\nu}{2}(-u_\phi \cot\theta + \frac{\partial u_\theta}{\partial \phi}\csc\theta + \frac{\partial u_\phi}{\partial \theta}) \\ \frac{1-\nu}{2}(-u_\phi \cot\theta + \frac{\partial u_\theta}{\partial \phi}\csc\theta + \frac{\partial u_\phi}{\partial \theta}) \end{Bmatrix}, \quad (\csc\theta = \frac{1}{\sin\theta})$$

$$\begin{Bmatrix} M_{\phi\phi} \\ M_{\theta\theta} \\ M_{\phi\theta} \\ M_{\theta\phi} \end{Bmatrix} = D\frac{1}{R}\begin{Bmatrix} -\theta_\phi \cot\theta + \frac{\partial \theta_\theta}{\partial \phi}\csc\theta - \nu\frac{\partial \theta_\phi}{\partial \theta} \\ \nu(-\theta_\phi \cot\theta + \frac{\partial \theta_\theta}{\partial \phi}\csc\theta) - \frac{\partial \theta_\phi}{\partial \theta} \\ \frac{1-\nu}{2}(\frac{\partial \theta_\theta}{\partial \theta} - \theta_\theta \cot\theta - \frac{\partial \theta_\phi}{\partial \phi}\csc\theta) \\ \frac{1-\nu}{2}(\frac{\partial \theta_\theta}{\partial \theta} - \theta_\theta \cot\theta - \frac{\partial \theta_\phi}{\partial \phi}\csc\theta) \end{Bmatrix}, \quad \begin{Bmatrix} Q_\phi \\ Q_\theta \end{Bmatrix} = tG\begin{Bmatrix} \theta_\theta + \frac{1}{R}(u_\phi + \frac{\partial u_n}{\partial \phi}\csc\theta) \\ -\theta_\phi + \frac{1}{R}(u_\theta + \frac{\partial u_n}{\partial \theta}) \end{Bmatrix}.$$

6.4 VIRTUAL WORK DENSITIES

Virtual work densities of the plate and shell models coincide

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta\overleftrightarrow{\varepsilon} \\ \delta\overleftrightarrow{\eta} \end{Bmatrix}_{\mathrm{c}}^{\mathrm{T}} : \begin{Bmatrix} \overleftrightarrow{F} \\ \overleftrightarrow{M} \end{Bmatrix}, \text{ where } \begin{Bmatrix} \overleftrightarrow{\varepsilon} \\ \overleftrightarrow{\eta} \end{Bmatrix} = \begin{Bmatrix} \nabla_0 \vec{u}_0 + \vec{e}_n \vec{\omega}_0 \\ \nabla_0 \vec{\omega}_0 \end{Bmatrix}$$



$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u} \\ \delta\vec{\omega} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix}, \quad \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u} \\ \delta\vec{\omega} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix}$$

in which

$$\begin{Bmatrix} \overleftrightarrow{F} \\ \overleftrightarrow{M} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ n \end{Bmatrix} J \overleftrightarrow{D}_{\mathrm{c}} \cdot \overleftrightarrow{\sigma} dn, \quad \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{f} J dn + \sum \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{t} J, \text{ and } \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ n \end{Bmatrix} J(\vec{n}) \vec{t} dn$$

All the kinematical quantities need to be expressed in terms of the kinematical quantities of the mid-surface $\vec{u}_0$, $\vec{\theta}_0$, ∇_0 etc. With $\vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n$, displacement gradient

$$\nabla\vec{u} = \overleftrightarrow{D} \cdot (\nabla_0 + \vec{e}_n \frac{\partial}{\partial n})(\vec{u}_0 + n\vec{\omega}_0) = \overleftrightarrow{D} \cdot (\overleftrightarrow{\varepsilon} + n\overleftrightarrow{\eta}),$$

where $\overleftrightarrow{\varepsilon} = \nabla_0\vec{u}_0 + \vec{e}_n\vec{\omega}_0$ and $\overleftrightarrow{\eta} = \nabla_0\vec{\omega}_0$ are the strain measures.

With the vector identities $\vec{a} : (\overleftrightarrow{b} \cdot \vec{c}) = (\vec{a} \cdot \overleftrightarrow{b}) : \vec{c}$ and $(\vec{a} \cdot \overleftrightarrow{b})_{\mathrm{c}} = \overleftrightarrow{b}_{\mathrm{c}} \cdot \vec{a}_{\mathrm{c}}$, the virtual work density of internal forces takes the form

$$\delta w_V^{\mathrm{int}} = -(\nabla\delta\vec{u})_{\mathrm{c}} : \overleftrightarrow{\sigma} = -(\delta\overleftrightarrow{\varepsilon}_{\mathrm{c}} + n\delta\overleftrightarrow{\eta}_{\mathrm{c}}) : (\overleftrightarrow{D}_{\mathrm{c}} \cdot \overleftrightarrow{\sigma}) = -\begin{Bmatrix} \delta\overleftrightarrow{\varepsilon} \\ \delta\overleftrightarrow{\eta} \end{Bmatrix}_{\mathrm{c}}^{\mathrm{T}} : \begin{Bmatrix} \overleftrightarrow{D}_{\mathrm{c}} \cdot \overleftrightarrow{\sigma} \\ n\overleftrightarrow{D}_{\mathrm{c}} \cdot \overleftrightarrow{\sigma} \end{Bmatrix}.$$

Integration over the domain occupied by the body gives writing the volume element in the form $dV = JdndA$, in which dA is the mid-surface area element,

$$\delta W^{\text{int}} = \int_{\Omega} [-\begin{Bmatrix} \delta\vec{\vec{\varepsilon}} \\ \delta\vec{\vec{\eta}} \end{Bmatrix}_{\text{c}}^{\text{T}} : (\int \begin{Bmatrix} 1 \\ n \end{Bmatrix} J\vec{\vec{D}}_{\text{c}} \cdot \vec{\vec{\sigma}} dn)] dA = \int_{\Omega} (-\begin{Bmatrix} \delta\vec{\vec{\varepsilon}} \\ \delta\vec{\vec{\eta}} \end{Bmatrix}_{\text{c}}^{\text{T}} : \begin{Bmatrix} \vec{\vec{F}} \\ \vec{\vec{M}} \end{Bmatrix}) dA \quad \Leftarrow$$

in which the stress resultants

$$\begin{Bmatrix} \vec{\vec{F}} \\ \vec{\vec{M}} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ n \end{Bmatrix} J\vec{\vec{D}}_{\text{c}} \cdot \vec{\vec{\sigma}} dn$$

are work conjugates to the strain measures. It is noteworthy that $\vec{\vec{F}}$ and/or $\vec{\vec{M}}$ of shell theory need not to be symmetric although the balance law of moment of momentum requires that $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\text{c}}$.

Volume and area forces contribute to the virtual work of external forces. The surface contribution needs to be divided into parts coming from the outer and inner surfaces and from the edge $\delta W_{\Omega}^{\text{ext}}$ and $\delta W_{\partial\Omega}^{\text{ext}}$, respectively:

$$\delta W^{\text{ext}} = \int_{\Omega} \vec{f} \cdot \delta\vec{u}\,dV + \int_{\partial\Omega} \vec{t} \cdot \delta\vec{u}\,dA \quad \Rightarrow$$

$$\delta W_{\Omega}^{\text{ext}} = \int_{\Omega} [\begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot (\int \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{f}Jdn) + \sum \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{t}J)]dA \quad \Rightarrow$$

$$\delta W_{\Omega}^{\text{ext}} = \int_{\Omega} \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} dA, \quad \text{where} \quad \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{f}Jdn) + \sum \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{t}J\,. \quad \Leftarrow$$

$$\delta W_{\partial\Omega}^{\text{ext}} = \int_{\partial\Omega} [\begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot (\int \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{t}J(\vec{n})dn)]ds \quad \Rightarrow$$

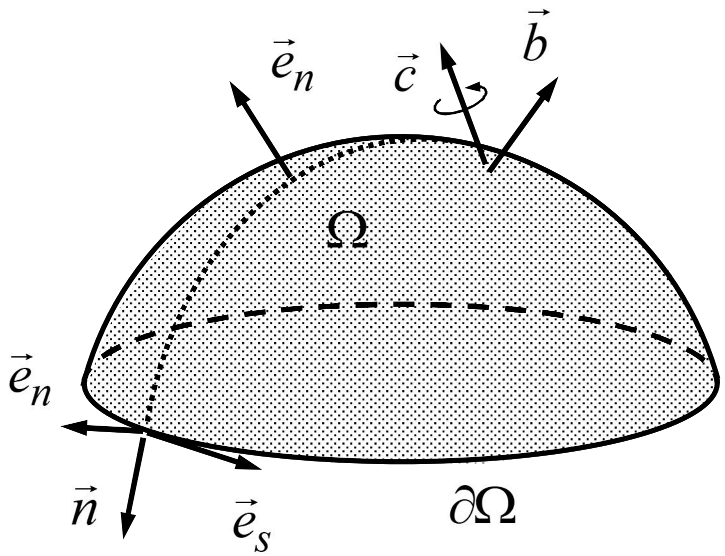
$$\delta W_{\partial\Omega}^{\text{ext}} = \int_{\partial\Omega} \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} ds, \quad \text{where} \quad \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{t}J(\vec{n})dn\,. \quad \Leftarrow$$

6.5 EQUILIBRIUM EQUATIONS

Virtual work expression of shell, principle of virtual work, integration by parts on curved surfaces (Kelvin-Stokes), and the fundamental lemma of variation calculus give:

$$\left.\begin{aligned}
&\nabla_0 \cdot \overset{\leftrightarrow}{F} - \kappa \vec{e}_n \cdot \overset{\leftrightarrow}{F} + \vec{b} = 0 \\
&(\nabla_0 \cdot \overset{\leftrightarrow}{M} - \kappa \vec{e}_n \cdot \overset{\leftrightarrow}{M} - \vec{e}_n \cdot \overset{\leftrightarrow}{F} + \vec{c}) \times \vec{e}_n = 0
\end{aligned}\right\} \quad \text{in } \Omega$$

$$\left.\begin{aligned}
&\vec{n} \cdot \overset{\leftrightarrow}{F} - \underline{\vec{F}} = 0 \quad \text{or} \quad \vec{u}_0 - \underline{\vec{u}}_0 = 0 \\
&(\vec{n} \cdot \overset{\leftrightarrow}{M} - \underline{\vec{M}}) \times \vec{e}_n = 0 \quad \text{or} \quad \vec{\theta}_0 - \underline{\vec{\theta}}_0 = 0
\end{aligned}\right\} \quad \text{on } \partial\Omega$$



Conditions on $\partial\Omega$ need to be expressed finally in the boundary system with component representations of $\vec{n}$ and $\vec{e}_s = \vec{e}_n \times \vec{n}$.

Virtual work expression of the shell model coincides with the plate model. However, as the mid-surface is not flat, the simple Gauss theorem is replaced by a version valid on curved surfaces

$$\delta W = \int_{\Omega} \left(-\begin{Bmatrix} \delta\overleftrightarrow{\varepsilon} \\ \delta\overleftrightarrow{\eta} \end{Bmatrix}_{\mathrm{c}}^{\mathrm{T}} : \begin{Bmatrix} \overleftrightarrow{F} \\ \overleftrightarrow{M} \end{Bmatrix}\right) dA + \int_{\Omega} \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} dA + \int_{\partial\Omega} \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} ds \, .$$

Integration by parts in terms containing derivatives of the variations gives (mean curvature $\kappa = \nabla_0 \cdot \vec{e}_n$) with the version of the Gauss theorem and the tensor identity $\nabla \cdot (\overleftrightarrow{b} \cdot \vec{a}) = (\nabla \cdot \overleftrightarrow{b}) \cdot \vec{a} + \overleftrightarrow{b}_{\mathrm{c}} : \nabla\vec{a}$ gives

$$\int_{\Omega} \overleftrightarrow{F} : (\nabla_0 \delta\vec{u}_0)_{\mathrm{c}} dA = -\int_{\Omega} (\nabla_0 \cdot \overleftrightarrow{F} - \kappa\vec{e}_n \cdot \overleftrightarrow{F}) \cdot \delta\vec{u}_0 dA + \int_{\partial\Omega} (\vec{n} \cdot \overleftrightarrow{F} \cdot \delta\vec{u}_0) ds,$$

$$\int_{\Omega} \overleftrightarrow{M} : (\nabla_0 \delta\vec{\omega}_0)_{\mathrm{c}} dA = -\int_{\Omega} (\nabla_0 \cdot \overleftrightarrow{M} - \kappa\vec{e}_n \cdot \overleftrightarrow{M}) \cdot \delta\vec{\omega}_0 dA + \int_{\partial\Omega} (\vec{n} \cdot \overleftrightarrow{M} \cdot \delta\vec{\omega}_0) ds$$

and thereby an equivalent but a more useful form of the virtual work expression

$$\delta W = \int_{\Omega} \begin{Bmatrix} \nabla \cdot \overleftrightarrow{F} - \kappa \vec{e}_n \cdot \overleftrightarrow{F} + \vec{b} \\ \nabla \cdot \overleftrightarrow{M} - \kappa \vec{e}_n \cdot \overleftrightarrow{M} - \vec{e}_n \cdot \overleftrightarrow{F} + \vec{c} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \delta \vec{u}_0 \\ \delta \vec{\omega}_0 \end{Bmatrix} dA - \int_{\partial\Omega} \begin{Bmatrix} \vec{n} \cdot \overleftrightarrow{F} - \underline{\vec{F}} \\ \vec{n} \cdot \overleftrightarrow{M} - \underline{\vec{M}} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \delta \vec{u}_0 \\ \delta \vec{\omega}_0 \end{Bmatrix} ds.$$

When definition $\delta\vec{\omega}_0 = \delta\vec{\theta}_0 \times \vec{e}_n$ and the vector identity $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ are used there (to recover the original rotation variable), the principle of virtual work and the fundamental lemma of variation calculus imply that

$$\nabla_0 \cdot \overleftrightarrow{F} - \kappa \vec{e}_n \cdot \overleftrightarrow{F} + \vec{b} = 0 \quad \text{in } \Omega$$

equilibrium eqs.

$$(\nabla_0 \cdot \overleftrightarrow{M} - \kappa \vec{e}_n \cdot \overleftrightarrow{M} - \vec{e}_n \cdot \overleftrightarrow{F} + \vec{c}) \times \vec{e}_n = 0 \quad \text{in } \Omega$$

$$\vec{n} \cdot \overleftrightarrow{F} - \underline{\vec{F}} = 0 \quad \text{or} \quad \vec{u}_0 - \underline{\vec{u}}_0 = 0 \quad \text{on } \partial\Omega$$

boundary conditions

$$(\vec{n} \cdot \overleftrightarrow{M} - \underline{\vec{M}}) \times \vec{e}_n = 0 \quad \text{or} \quad \vec{\theta}_0 - \underline{\vec{\theta}}_0 = 0 \quad \text{on } \partial\Omega$$

6.6 CONSTITUTIVE EQUATIONS

Constitutive equations $\vec{F} = \vec{F}(\vec{u}_0, \vec{\theta}_0)$, $\vec{M} = \vec{M}(\vec{u}_0, \vec{\theta}_0)$ follow from the generalized Hooke's law, the definition of small strain, and the kinetic and kinematic assumptions of the model:

notice this!

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \begin{bmatrix} \vec{A} & \vec{C} \\ \vec{C} & \vec{B} \end{bmatrix} : \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\eta} \end{Bmatrix}, \quad \text{where} \quad \begin{bmatrix} \vec{A} & \vec{C} \\ \vec{C} & \vec{B} \end{bmatrix} = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} (\vec{D}_{\mathrm{c}} \cdot \vec{E} \cdot \vec{D} J)\, dn$$

Elasticity tensor $\vec{E}$ is assumed to satisfy the minor and major symmetries and condition $\vec{e}_n \vec{e}_n : \vec{E} = 0$. Elasticity tensors $\vec{A}$, $\vec{B}$ and $\vec{C}$ of shell depend on the material, positioning of the mid-surface (actually the reference surface), thickness of the shell, and curvature of the mid-surface. Assuming a thin shell so that $\vec{D} \approx \vec{I}$ and $J = 1$, the expressions boil down to the plate expressions.

Constitutive equations follow from the stress resultant definitions when the stress expression is substituted there

$$\begin{Bmatrix} \vec{\vec{F}} \\ \vec{\vec{M}} \end{Bmatrix} = \int (\begin{Bmatrix} 1 \\ n \end{Bmatrix} J\vec{\vec{D}}_{\mathrm{c}} \cdot \vec{\vec{\sigma}})dn.$$

Notice that the stress resultant tensors may not be symmetric even though the stress tensor always is. The gradient expression was earlier found to be

$$\nabla\vec{u} = \vec{\vec{D}} \cdot (\nabla_0 + \vec{e}_n \frac{\partial}{\partial n})(\vec{u}_0 + n\vec{\omega}_0) = \begin{Bmatrix} 1 \\ n \end{Bmatrix}^{\mathrm{T}} \vec{\vec{D}} \cdot \begin{Bmatrix} \vec{\vec{\varepsilon}} \\ \vec{\vec{\eta}} \end{Bmatrix}, \text{ where } \begin{Bmatrix} \vec{\vec{\varepsilon}} \\ \vec{\vec{\eta}} \end{Bmatrix} = \begin{Bmatrix} \nabla_0\vec{u}_0 + \vec{e}_n\vec{\omega}_0 \\ \nabla_0\vec{\omega}_0 \end{Bmatrix}.$$

Let us assume a linearly elastic material and an elasticity tensor satisfying the minor and major symmetries and condition $\vec{e}_n\vec{e}_n : \vec{\vec{\vec{\vec{E}}}} = 0$. Stress-strain relationship gives (tensor identity $\vec{\vec{a}} : (\vec{b} \cdot \vec{c}) = (\vec{\vec{a}} \cdot \vec{b}) : \vec{c}$)

$$\vec{\sigma} = \vec{\vec{E}} : \nabla\vec{u} = \begin{Bmatrix} 1 \\ n \end{Bmatrix}^{\mathrm{T}} (\vec{\vec{E}} \cdot \vec{D}) : \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\eta} \end{Bmatrix}.$$

The stress-resultant definition gives now expression

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int (\begin{Bmatrix} 1 \\ n \end{Bmatrix} J\vec{D}_{\mathrm{c}} \cdot \vec{\sigma}) dn = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} (J\vec{D}_{\mathrm{c}} \cdot \vec{\vec{E}} \cdot \vec{D}) dn : \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\eta} \end{Bmatrix} \quad \Rightarrow$$

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \begin{bmatrix} \vec{\vec{A}} & \vec{\vec{C}} \\ \vec{\vec{C}} & \vec{\vec{B}} \end{bmatrix} : \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\eta} \end{Bmatrix}, \text{ where } \begin{bmatrix} \vec{\vec{A}} & \vec{\vec{C}} \\ \vec{\vec{C}} & \vec{\vec{B}} \end{bmatrix} = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} (J\vec{D}_{\mathrm{c}} \cdot \vec{\vec{E}} \cdot \vec{D}) dn. \quad \Leftarrow$$

which depends on the material properties, position of the mid-surface (actually the reference surface), thickness of the shell, and curvature of the reference surface. Without simplifications the membrane and bending modes are always connected.

CYLINDRICAL SHELL CONSTITUTIVE EQUATIONS

Derivation of the constitutive equations is a straightforward but somewhat tedious task. If the origin of the $n-$axis is placed at the mid-surface, constitutive equations take the forms $(F_{nn} = M_{nn} = 0)$

$$F_{zz} = \frac{tE}{1-\nu^2}(\varepsilon_{zz} + \nu\varepsilon_{\phi\phi}) - D\frac{1}{R}\kappa_{zz} = \frac{tE}{1-\nu^2}[\frac{\partial u_z}{\partial z} + \nu\frac{1}{R}(\frac{\partial u_\phi}{\partial \phi} - u_n)] - D\frac{1}{R}\frac{\partial \theta_\phi}{\partial z},$$

$$F_{\phi\phi} = \frac{tE}{1-\nu^2}[g\varepsilon_{\phi\phi} + \nu\varepsilon_{zz} + (g-1)R\kappa_{\phi\phi}] = \frac{tE}{1-\nu^2}[g\frac{1}{R}(\frac{\partial u_\phi}{\partial \phi} - u_n) + \nu\frac{\partial u_z}{\partial z} - (g-1)\frac{\partial \theta_z}{\partial \phi}],$$

$$F_{z\phi} = Gt(\varepsilon_{z\phi} + \varepsilon_{\phi z}) - \frac{1}{2}(1-\nu)D\frac{1}{R}\kappa_{z\phi} = Gt(\frac{\partial u_\phi}{\partial z} + \frac{1}{R}\frac{\partial u_z}{\partial \phi}) + \frac{1}{2}(1-\nu)D\frac{1}{R}\frac{\partial \theta_z}{\partial z},$$

$$F_{\phi z} = Gt[g\varepsilon_{\phi z} + \varepsilon_{z\phi} + (g-1)R\kappa_{\phi z}] = Gt[g\frac{1}{R}\frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z} + (g-1)\frac{\partial \theta_\phi}{\partial \phi}],$$

continues...

$$F_{zn} = Gt(\varepsilon_{nz} + \varepsilon_{zn}) - \frac{1}{2}(1-\nu)D\frac{1}{R}(\kappa_{nz} + \kappa_{zn}) = Gt(\theta_{\phi} + \frac{\partial u_n}{\partial z}),$$

$$F_{nz} = Gt(\varepsilon_{nz} + \varepsilon_{zn}) - \frac{1}{2}(1-\nu)D\frac{1}{R}(\kappa_{nz} + \kappa_{zn}) = Gt(\theta_{\phi} + \frac{\partial u_n}{\partial z}),$$

$$F_{\phi n} = Gt[g\varepsilon_{\phi n} + \varepsilon_{n\phi} + (g-1)R\kappa_{\phi n}] = Gtg[\frac{1}{R}(\frac{\partial u_n}{\partial \phi} + u_{\phi}) - \theta_z],$$

$$F_{n\phi} = Gt(\varepsilon_{n\phi} + \varepsilon_{\phi n}) - \frac{1}{2}(1-\nu)D\kappa_{n\phi} = Gt[-\theta_z + \frac{1}{R}(\frac{\partial u_n}{\partial \phi} + u_{\phi})],$$

$$M_{zz} = D(\kappa_{zz} + \nu\kappa_{\phi\phi} - \frac{1}{R}\varepsilon_{zz}) = D(\frac{\partial \theta_{\phi}}{\partial z} - \nu\frac{1}{R}\frac{\partial \theta_z}{\partial \phi} - \frac{1}{R}\frac{\partial u_z}{\partial z}),$$

$$M_{\phi\phi} = D(f\kappa_{\phi\phi} + \nu\kappa_{zz} + f\frac{1}{R}\varepsilon_{\phi\phi}) = D[-f\frac{1}{R}\frac{\partial \theta_z}{\partial \phi} + \nu\frac{\partial \theta_{\phi}}{\partial z} + f\frac{1}{R^2}(\frac{\partial u_{\phi}}{\partial \phi} - u_n)],$$

continues...

$$M_{z\phi} = \frac{1}{2}(1-\nu)D(\kappa_{z\phi} + \kappa_{\phi z} - \frac{1}{R}\varepsilon_{z\phi}) = \frac{1}{2}(1-\nu)D[(-\frac{\partial\theta_z}{\partial z} + \frac{1}{R}\frac{\partial\theta_\phi}{\partial\phi}) - \frac{1}{R}\frac{\partial u_\phi}{\partial z}],$$

$$M_{\phi z} = \frac{1}{2}(1-\nu)D(f\kappa_{\phi z} + \kappa_{z\phi} + f\frac{1}{R}\varepsilon_{\phi z}) = \frac{1}{2}(1-\nu)D(f\frac{1}{R}\frac{\partial\theta_\phi}{\partial\phi} - \frac{\partial\theta_z}{\partial z} + f\frac{1}{R^2}\frac{\partial u_z}{\partial\phi}),$$

$$M_{zn} = \frac{1}{2}(1-\nu)D[\kappa_{zn} + \kappa_{nz} - \frac{1}{R}(\varepsilon_{zn} + \varepsilon_{nz})] = -\frac{1}{2}(1-\nu)D\frac{1}{R}(\frac{\partial u_n}{\partial z} + \theta_\phi),$$

$$M_{nz} = \frac{1}{2}(1-\nu)D[\kappa_{nz} + \kappa_{zn} - \frac{1}{R}(\varepsilon_{nz} + \varepsilon_{zn})] = -\frac{1}{2}(1-\nu)D\frac{1}{R}(\frac{\partial u_n}{\partial z} + \theta_\phi),$$

$$M_{n\phi} = \frac{1}{2}(1-\nu)D(\kappa_{n\phi} + \kappa_{\phi n} - \frac{1}{R}\varepsilon_{n\phi}) = 0,$$

$$M_{\phi n} = \frac{1}{2}(1-\nu)D(f\frac{1}{R}\varepsilon_{\phi n} + \kappa_{n\phi} + f\kappa_{\phi n}) = \frac{1}{2}(1-\nu)Df\frac{1}{R}[-\theta_z + \frac{1}{R}(\frac{\partial u_n}{\partial\phi} + u_\phi)],$$

where the functions depending on the relative thickness $a = t/R$

continues...

$$g = \frac{1}{a}\log\left(\frac{2+a}{2-a}\right) \approx 1 + \frac{a^2}{12} + \frac{a^4}{80} + \dots, \quad \text{and} \quad f = 12\frac{1}{a^2}(g-1).$$

In the simplified constitutive equations, shell is assumed to be thin in the sense that $a = t/R \ll 1$ so that the first ($g \approx 1,\ f = 0$) or the first two terms ($g \approx 1 + a^2/12,\ \ f = 1$) of g give an accurate enough representation. No matter the number of terms used, constitutive equations satisfy the moment balance of the domain element

$$F_{nz} - F_{zn} = 0\ ,\ \ F_{z\phi} - F_{\phi z} + \frac{1}{R}M_{\phi z} = 0,\ \text{and}\ F_{n\phi} - F_{\phi n} + \frac{1}{R}M_{\phi n} = 0$$

‘a priori’. Also, stress resultants vanish in the rigid body motion of the shell

$$\vec{u}(z,\phi,n) = \vec{U}_0 + \vec{\Omega}_0 \times \vec{r}_0 \quad \text{and} \quad \vec{\theta}(z,\phi,n) = \vec{\Omega}_0$$

in which $\vec{U}_0$ and $\vec{\Omega}_0$ are constant vectors.

SIMPLIFIED CONSTITUTIVE EXPRESSIONS

The practical expressions of constitutive equations are often simplified by omitting the "small terms". The simplified expressions of the stress resultants should

(1) vanish in rigid body motion of the shell $\vec{u} = \vec{U} + \vec{\Omega} \times \vec{r}_0$ and $\vec{\theta} = \vec{\Omega}$ in which $\vec{U}$ and $\vec{\Omega}$ are constant vectors in the Cartesian (x, y, z) coordinate system

(2) satisfy the moment equilibrium $\vec{e}_n \cdot \underline{\overleftrightarrow{F}} \times \vec{e}_n + \nabla_0 \cdot (\underline{\overleftrightarrow{F}} \times \vec{\rho}_0) + \nabla_0 \cdot (\underline{\overleftrightarrow{M}} \times \vec{e}_n) = 0$, in which the underbars denote constants with respect to the gradient operator.

Both conditions are satisfied by the constitutive equations of spherical shell and by the cylindrical no matter the number of terms used for g_α (not all simplifications of the constitutive equations satisfy conditions (1) and (2)).

The latter requirement means that a material element should be in equilibrium under constant stress resultants and vanishing external loading (symmetry of stress $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}}$ of classical elasticity is one of the outcomes of the requirement). In curved geometry with $\vec{t} = \vec{n} \cdot \vec{\vec{\sigma}}$ and $\vec{r} = \vec{r}_0 + n\vec{e}_n$

$$\vec{F} = \int_{\partial\Omega} \int \vec{t} dnds = \int_{\partial\Omega} \vec{n} \cdot \vec{\vec{F}} ds = \int_{\Omega} (\nabla \cdot \vec{\vec{F}} - \kappa \vec{e}_n \cdot \vec{\vec{F}}) dA = 0,$$

$$\vec{M} = \int_{\partial\Omega} \int \vec{r} \times \vec{t} dnds = -\int_{\Omega} [\nabla_0 \cdot (\vec{\vec{F}} \times \vec{r}_0 + \vec{\vec{M}} \times \vec{e}_n) - \kappa \vec{e}_n \cdot (\vec{\vec{F}} \times \vec{r}_0 + \vec{\vec{M}} \times \vec{e}_n)] dA = 0.$$

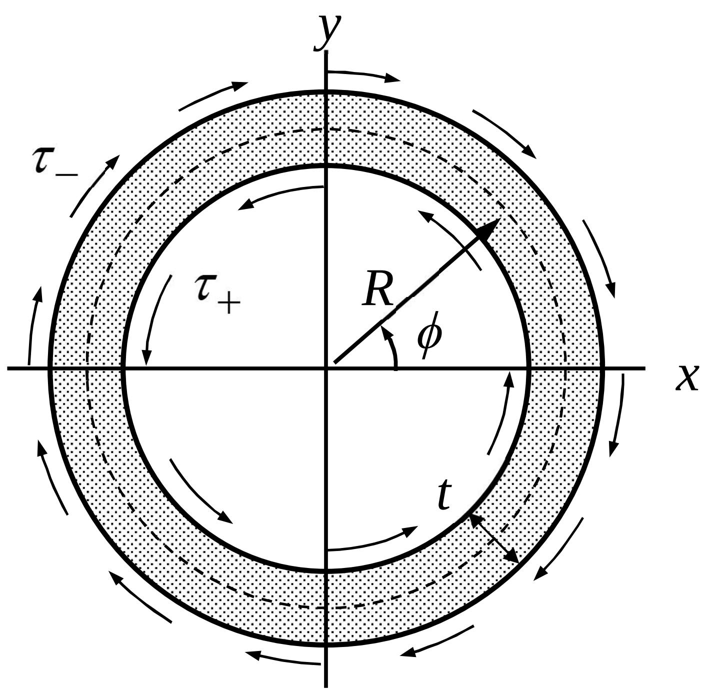
The generic equilibrium equations of plate show that the first condition is satisfied and the second implies (as Ω is arbitrary)

$$\vec{e}_n \cdot \vec{\vec{F}} \times \vec{e}_n + \nabla_0 \cdot (\underline{\vec{\vec{F}}} \times \vec{r}_0) + \nabla_0 \cdot (\underline{\vec{\vec{M}}} \times \vec{e}_n) = 0.$$

in which the underbars denote constants with respect to the gradient operator.

EXAMPLE 6.4 Consider a cylinder subjected to shear forces acting on the inner and outer surfaces as shown. Use the Reissner-Mindlin type shell model in $(z,\phi,n)-$coordinate system to derive the expression of displacement $\vec{u}(n)$. Assume that the only non-zero displacement/rotation component θ_z is constant and that the cylinder is in equilibrium so that the shear forces per unit area satisfy $\tau = \tau_-(1+a/2)^2 = \tau_+(1-a/2)^2$ where $a = t/r$.

Answer $\vec{u} = \dfrac{\tau}{G} n\vec{e}_\phi$ when $a = \dfrac{t}{R} \ll 1$



As all other displacement/rotation components except θ_z are assumed to vanish, the equilibrium and constitutive equations ($g_\alpha \approx 1 + a^2/12$ and $f_\alpha = 1$) take the forms

$$\frac{1}{R}M_{\phi n} + Q_\phi - c_\phi = 0,\ \ Q_\phi = -Gt\theta_z,\ \text{and}\ M_{\phi n} = -Gt\frac{a^2}{12}R\theta_z .$$

The distributed force and moment follow from definition

$$\begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \vec{f} \begin{Bmatrix} 1 \\ n \end{Bmatrix} J dn + \sum \vec{t} \begin{Bmatrix} 1 \\ n \end{Bmatrix} J$$

in which $\vec{f}$ is the external volume force (due to gravity for example) and $\vec{t}$ is the given area force acting on the outer and inner surfaces. The sum is over the coordinates $\{n_-, n_+\}$ of surfaces. Notice that – side is the outer surface and + the inner surface since

n is directed inwards in (z,ϕ,n) coordinates. Here $\vec{f}=0$ and scaling coefficient expression $J=1-n/R$ for the cylindrical shell

$$\vec{c}=\sum \vec{t}nJ=(1+\frac{a}{2})(-\frac{t}{2})(-\tau_{-})\vec{e}_{\phi}+(1-\frac{a}{2})(\frac{t}{2})(\tau_{+})\vec{e}_{\phi}=R\frac{a}{1-(a/2)^{2}}\tau\vec{e}_{\phi}\ .$$

When the constitutive equations are substituted there, equilibrium equation simplifies to (assuming that $a^{2}\ll 1$)

$$-Gt\theta_{z}-Ra\tau=0\ \Rightarrow\ \theta_{z}=-\frac{\tau}{G}.$$

Finally, using the kinematic assumption of the shell-model $\vec{u}=\theta_{z}\vec{e}_{z}\times n\vec{e}_{n}=-n\theta_{z}\vec{e}_{\phi}$ and therefore

$$\vec{u}=\frac{\tau}{G}n\vec{e}_{\phi}.$$