Problem set 6

Due April 13, 10:15 AM

1 *Quantum Fourier transform(QFT)*

The quantum Fourier transform (QFT) is the quantum analogue of the classical discrete Fourier transform(DFT). Nonetheless, their roles in computational algorithms are totally different.

(1) Show that for any k-qubit state $|n\rangle = |n_1, n_2, \ldots, n_k\rangle$, $n_i \in \{0, 1\}$, and the tensor product Hadamard operator $H^{\otimes k}$, the following holds

$$H^{\otimes k} |n\rangle = \frac{1}{\sqrt{2^k}} \sum_{m \in \{0,1\}^k} (-1)^{n \cdot m} |m\rangle$$

where $\{0, 1\}^k$ is the set of all binary strings of length k, $n \cdot m = \langle n|m\rangle$ modulo 2, and $|m\rangle = |m_1, m_2, \ldots, m_k\rangle$.
Hint: Recall that $H^{\otimes k} |n\rangle = H |n_1\rangle \otimes H |n_2\rangle \otimes \ldots \otimes H |n_k\rangle$.

(2) Show that the QFT is unitary.

(3) Derive the matrix representation of the QFT operator.

(4) Consider the following normalized qutrit state $|\Psi\rangle = \frac{\mathrm{i}}{\sqrt{19}} |0\rangle - \frac{4}{\sqrt{19}} |1\rangle + \frac{\sqrt{2}}{\sqrt{19}} |2\rangle$. Calculate its QFT. First write down the corresponding matrix representation of the QFT operator, then it is straight forward to transform the state vector.

2 Grover's search algorithm

In this exercise we develop a basic understanding of Grover's algorithm by going through its implementation process in details.

(1) Given a database of size $N = 2^n$. Define an observable $X = \sum_{i=0}^{N-1} x_i |x_i\rangle \langle x_i|$, where the set of vectors $\{|x_i\rangle\}_i$ constitute an ONB, and $\{x_i\}$ are their associated eigenvalues . Each $|x_i\rangle$ is an n-qubit representation of the corresponding n-bit binary string. The database is prepared in a general superposition state $|s\rangle \in \mathcal{H}^{\otimes n}$, this can be achieved by means of an n-tensor product Hadamard gate $H^{\otimes n}$

  (a) Let $U_w = \mathcal{I} - 2 |x_w\rangle \langle x_w|$ defines a unitary linear operator denoted as the *oracle* [1], and let $|s\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |x_i\rangle$ be the initial database state. Show that $U_w |s\rangle = |s\rangle - \frac{2}{\sqrt{N}} |x_w\rangle$.
  Hint: The database state can be written as $|s\rangle = \frac{1}{\sqrt{N}} \left(\sum_{\substack{i=0 \\ i \neq w}}^{N-1} |x_i\rangle + |x_w\rangle \right)$.

(2) Next we introduce another unitary operator as $U_s = 2 |s\rangle \langle s| - \mathcal{I}$, and define the Grover operator as the composition $G = U_s U_w$

  (a) Show that when G acts on $|s\rangle$ the resulting state is $|\Psi\rangle = \left(1 - \frac{4}{N}\right) |s\rangle + \frac{2}{\sqrt{N}} |x_w\rangle$

  (b) Using the Born rule calculate the probability of finding the correct state $|x_w\rangle$ for a very large N, and compare your result with the original probability before applying the Grover operator $|\langle x_w|s\rangle|^2$.

(3) We now redefine the database state $|s\rangle$ according to the orthogonal complement property of the Hilbert space $\mathcal{H}^{\otimes n} = \mathcal{H}_u^{\otimes n} \bigoplus \mathcal{H}_w^{\otimes n}$, where $\mathcal{H}_u^{\otimes n} \perp \mathcal{H}_w^{\otimes n}$, and $\langle u|x_w\rangle = 0$.
Thus, we can write the normalized database state $|s\rangle$ as $\sqrt{1 - \frac{1}{N}} |u\rangle + \frac{1}{\sqrt{N}} |x_w\rangle$.

[1] In computer science, oracles are operators that perform specific task, without giving much attention to their exact implementations.

Then, interpreting the oracle operator U_w as a small rotation through an angle θ, such that, $\cos\theta = \langle s|x_w\rangle = 1 - \frac{2}{N}$, yields,

$$|s\rangle = \cos\frac{\theta}{2}\,|u\rangle + \sin\frac{\theta}{2}\,|x_w\rangle$$

(a) Show that k-Grover iterations give the state

$$\cos\left[(2k+1)\frac{\theta}{2}\right]|u\rangle + \sin\left[(2k+1)\frac{\theta}{2}\right]|x_w\rangle$$

(b) Calculate the probability of finding the state $|x_w\rangle$.

(c) Find k_{max} that maximizes the above probability. Assume very large N, such that $\theta \approx \frac{2}{\sqrt{N}}$.

3 The variational method

Suppose that the Hamiltonian of a 1-dimensional quantum system is described as

$$H = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

Use the Gaussian ansatz $\Psi(x) = e^{-\alpha x^2}$ to obtain the lowest upper bound on the ground state in the two cases $(a)\, V(x) = \gamma|x| \qquad (b)\, V(x) = \gamma x^4$.