

COE-C3005

Finite Element and Finite Difference Methods 2021

WEEK 16: INTRODUCTION

Fri 09:15-11:00 Calculation hours (JF & MÅ)

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- 2 BAR AND STRING MODELS**
- 3 FINITE DIFFERENCE METHOD (FDM)**
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- 6 MEMBRANE APPLICATION (FEM)**

LEARNING OUTCOMES

Introduction to approximate methods for *initial* and *boundary* value problems in solid mechanics. After the course, student understands the physical background of the bar and string model problems, knows the basic ideas of (1) particle surrogate, (2) finite difference, (3) finite element methods, is able to apply the methods to the model problems (1D), and knows the extensions to the thin slab and membrane models of solid mechanics (2D).

Prerequisites: Linear algebra, ordinary differential equations

1 INTRODUCTION

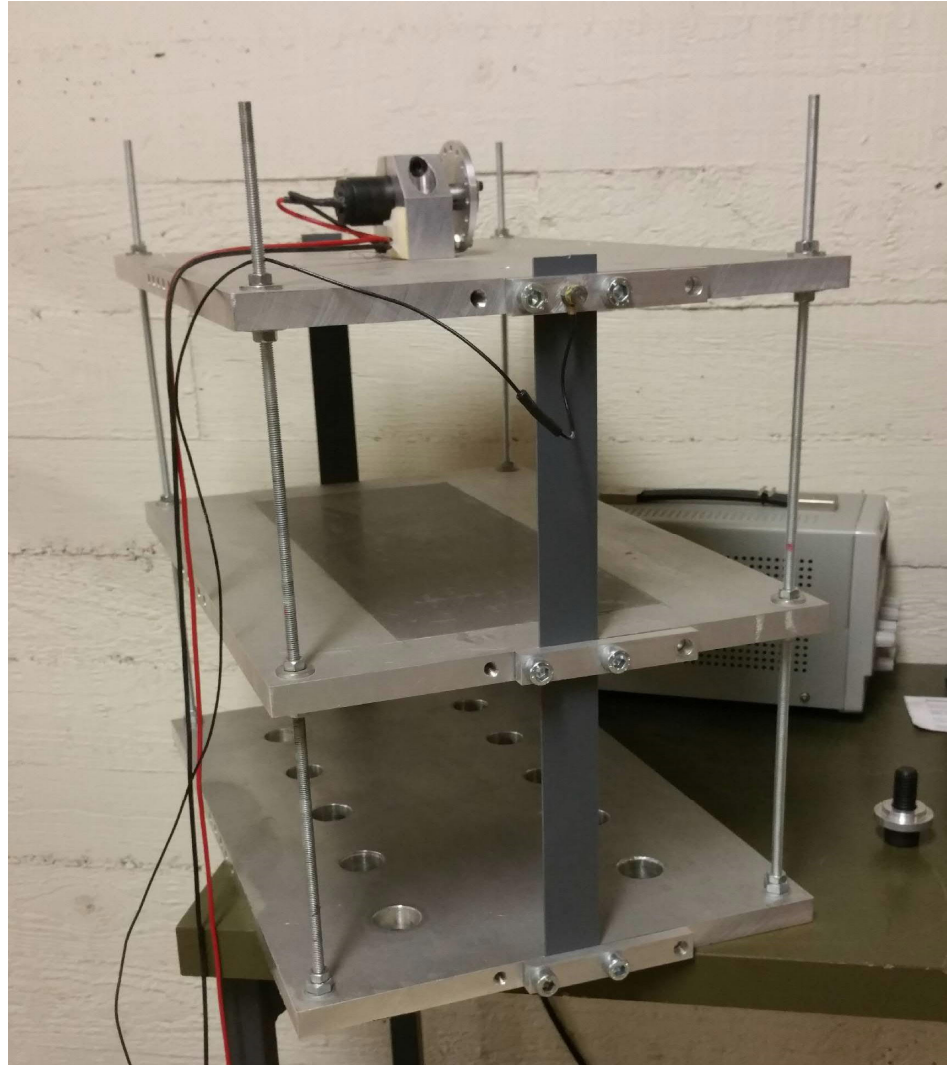
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VIBRATION OF 3-STORY BUILDING



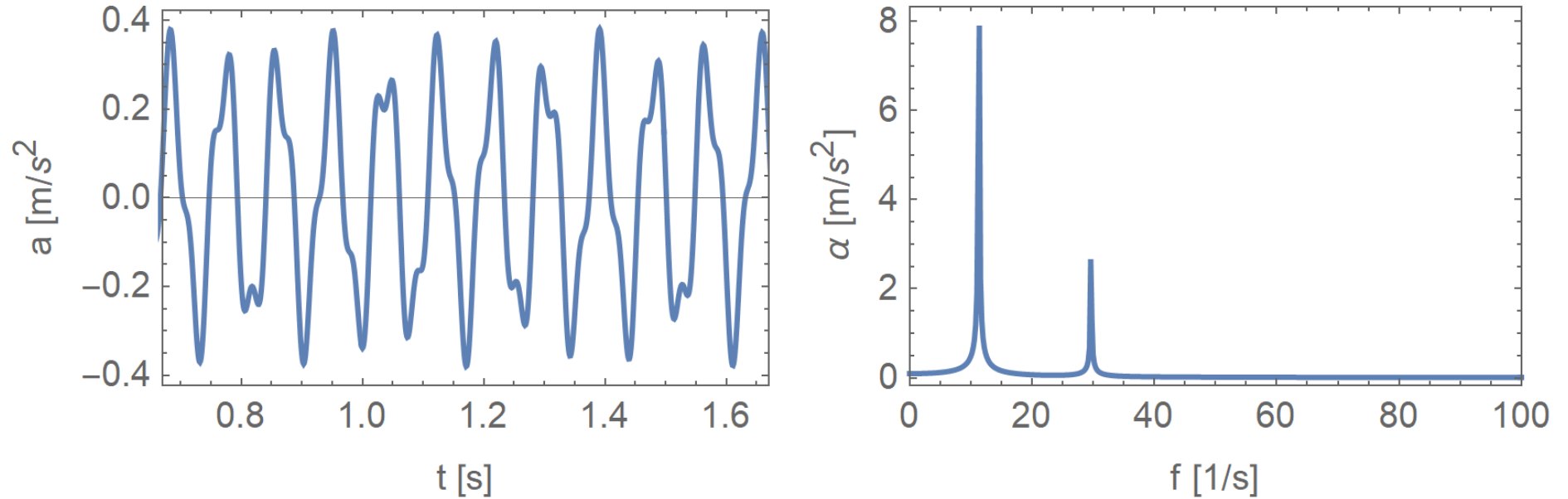
MODELLING ASSIGNMENT

In the modelling assignment, you will determine the two first frequencies of the free vibrations of the 3-story building using a model and

1. Particle Surrogate Method (PSM)
2. Finite Difference Method (FDM)
3. Finite Element Method (FEM)

To report the outcome, supplement the assignment paper with experimental results and the outcome of calculations (table for results in light blue shading). Return your report (in PDF) on Sun 25.04.2021 23:55 at the latest (MyCourses).

VIBRATION EXPERIMENT

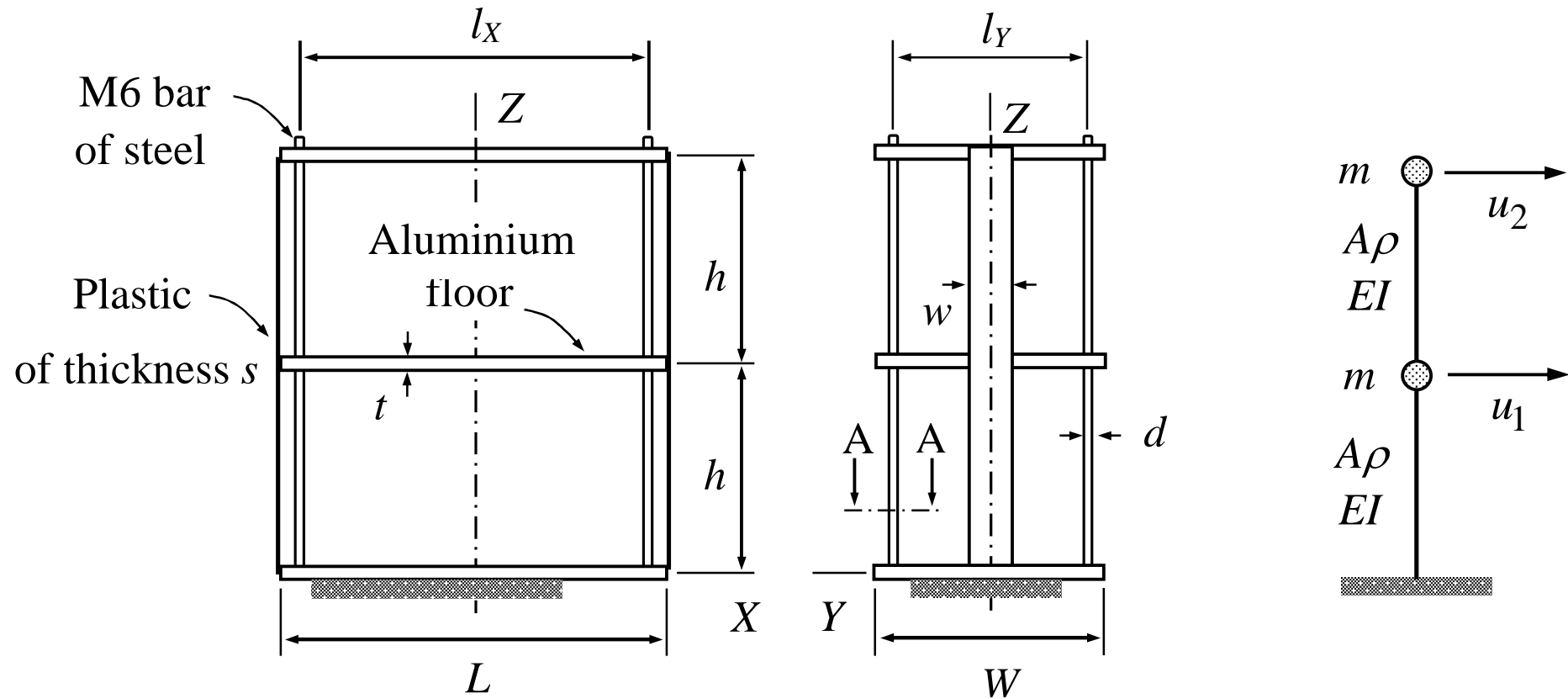


Experimental data consists of the acceleration time-series measured by the accelerometer at one point. In processing of data, the time-acceleration representation is transformed to frequency-mode magnitude form by Discrete Fourier Transform (DFT).

MODELLING STEPS

- **Crop:** Decide the boundary of a structure. Interaction with surroundings need to be described in terms of known forces, moments, displacements, and rotations. All uncertainties with this respect bring uncertainty to the model too.
- **Idealize and parameterize:** Simplify the geometry. Ignoring the details not likely to affect the outcome may simplify the analysis a lot. Assign symbols to geometric and material parameter of the idealized structure.
- **Model:** Write the equilibrium equations, constitutive equations, and boundary conditions of the structure.
- **Solve:** Use an analytical or approximate method and hand calculation or a code to find the solution.

STRUCTURE IDEALIZATION



The simplified model considers the columns as bending beams, floors as rigid bodies, omits the plastic strips, and assumes that the floors move horizontally in the XZ -plane. The horizontal displacements of the floors are denoted by $u_1(t)$ and $u_2(t)$.

APPROXIMATE METHODS

The simplest approximate equations of motion by Particle Surrogate Method, Finite Difference Method, and Finite Element Method, contain only the horizontal displacements of the first and second floors:

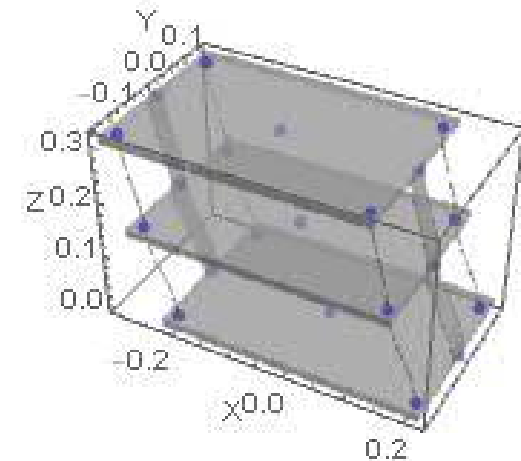
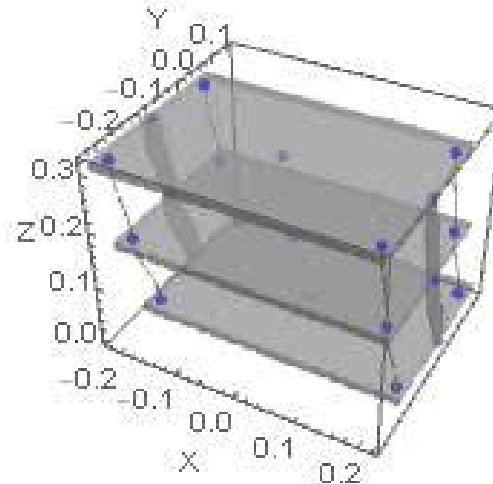
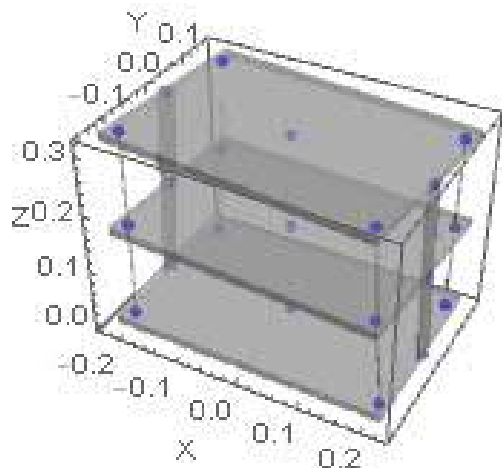
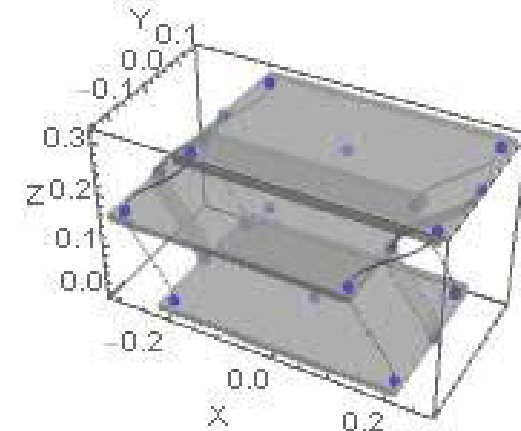
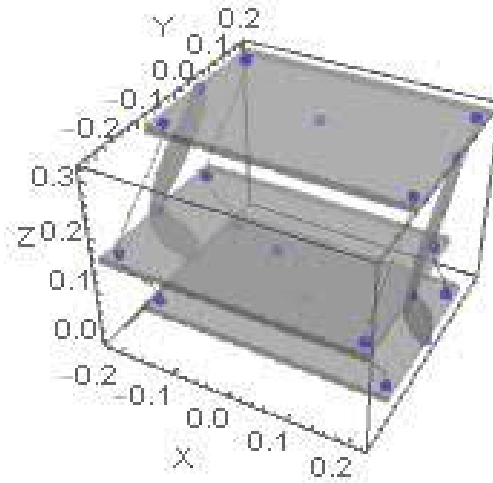
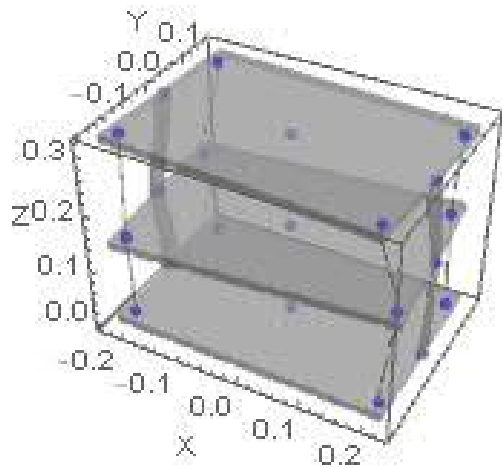
$$\text{PSM: } \left(m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4 \times \frac{1}{2} \rho A h \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) \frac{d^2}{dt^2} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + 4 \times 12 \frac{EI}{h^3} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

$$\text{FDM: } \left(\frac{m}{h} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4 \times \rho A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \frac{d^2}{dt^2} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + 4 \times 12 \frac{EI}{h^4} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

$$\text{FEM: } \left(m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4 \times \frac{\rho A h}{6} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \right) \frac{d^2}{dt^2} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + 4 \times 12 \frac{EI}{h^3} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

Mode analysis for the frequencies assumes solution of the form $\mathbf{a} = \mathbf{A} \exp(i\omega t)$ where $\omega = 2\pi f$, $i^2 = -1$, $\mathbf{a}(t) = \{u_1 \ u_2\}^T$, and $\mathbf{A} = \{A_1 \ A_2\}^T$ (some constants).

FEM SIMULATION WITH A DETAILED MODEL



1.1 PARTICLE SURROGATE MODELLING (PSM)

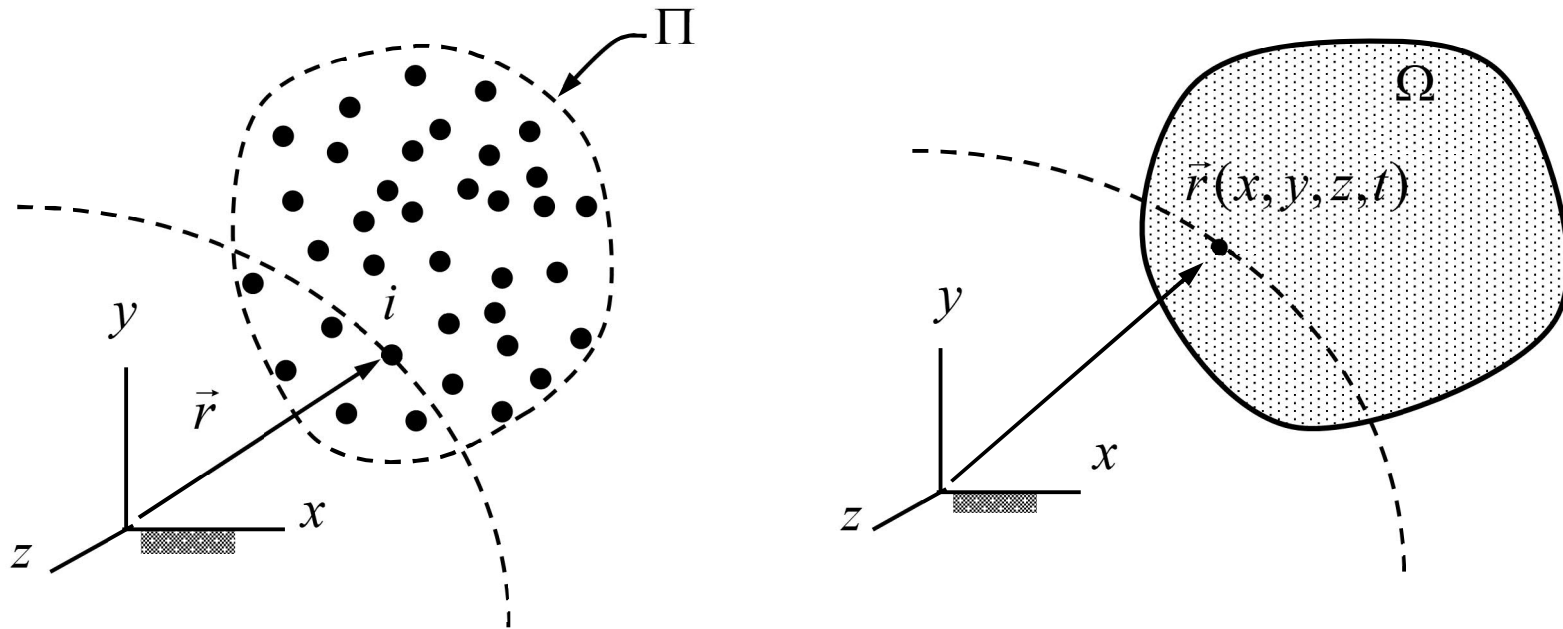
In a particle model surrogates, discretization replaces a continuum model by a particle model on a grid of spatial resolution h . There

- Inertia forces (actually mass) and external forces are lumped to the grid points. Elastic properties of the material are used to deduce an interaction model of particles at the grid points, i.e., internal forces of the particle system.
- The main unknowns are the displacement of the particles. The model consists of equations of motion of the particles and possible initial conditions (if known).

Discretization replaces the original problem with a computable problem whose complexity depends on resolution h . Particle discretization introduces modelling error compared with the continuous model which should reduce in h (by design of the method) and vanish in the limit $h \rightarrow 0$.

PARTICLE AND CONTINUUM MODELS

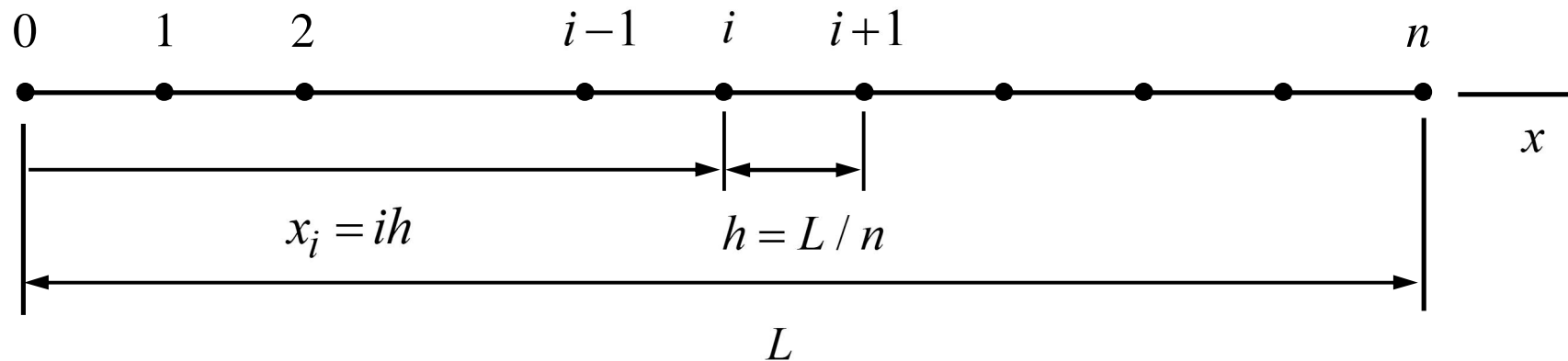
Particle surrogate model replaces the continuum model of solid mechanics by a particle model by reverting the reasoning used commonly in derivation of the continuum models.



In particle models, index $i \in \Pi \subset \mathbb{N}$ is used for labelling. In continuum models, material coordinates $(x, y, z) \in \Omega \subset \mathbb{R}^3$ are used for the purposes.

REGULAR GRID IN 1D

On a regular grid, the grid points are distributed evenly. Here, grid point numbering starts from 0 and increases without gaps in the direction of the x -axis the total number of grid points being $n+1$. The line segments are numbered from 1 in the same manner.



The numbering convention above fits well hand calculations in 1D case but it will be refined later for an unified geometrical description including, e.g., non-regular grids in several physical dimensions.

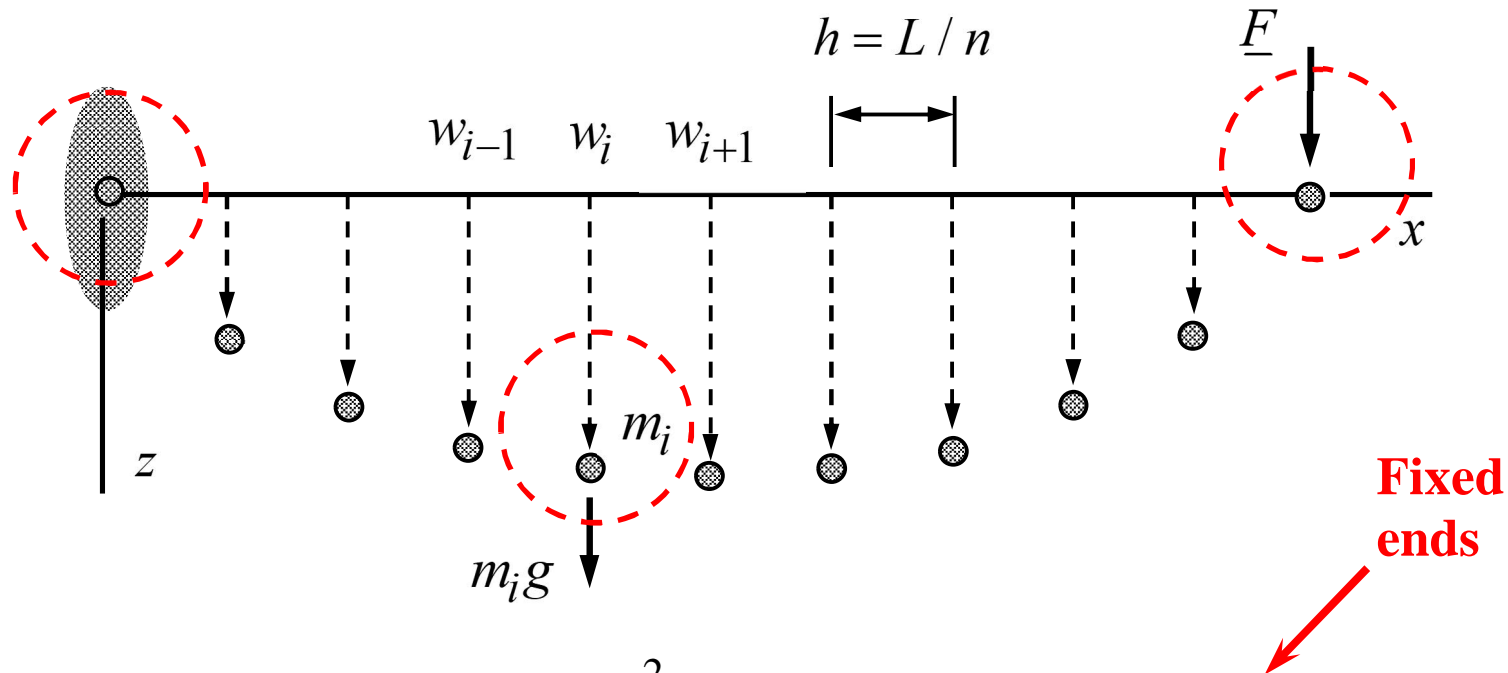
NEWTON'S LAWS OF MOTION

- I** In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.
- II** The vector sum of the forces on an object is equal to the mass of that object multiplied by the acceleration of the object (assuming that the mass is constant).
- III** When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

Newton's laws in their original forms above apply to each particle separately. The formulation for average behavior of particle systems, rigid bodies, deformable bodies, open system of particles etc. require slight modifications.

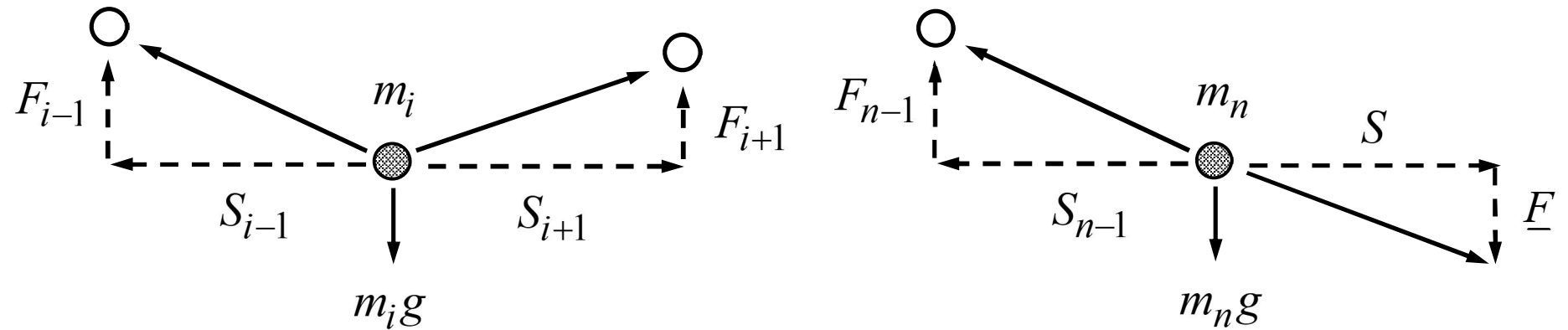
PARTICLE MODEL OF STRING

In the horizontal string model, particles move in vertical direction and forces between the particles are aligned with the string. In PSM, mass of the string and distributed transverse force are lumped as particles and point forces on a regular grid.



$$\frac{S}{h}(w_{i-1} - 2w_i + w_{i+1}) + m_i g = m_i \frac{d^2 w_i}{dt^2} \quad i \in \{1, 2, \dots, n-1\}, \quad w_i = 0 \quad i \in \{0, n\}$$

Free body diagrams for the typical particle and the particle at the end of the string



As particles move in the vertical direction, horizontal tightening must be constant S . Assuming also constant density of the material and constant cross-sectional area, external distributed force due to gravity $m_i g$ is the same for all particles. Using the free body diagrams and the geometry of the figures, where $m_i = \rho h A$, equations of motion in the transverse direction become

$$-F_{i-1} - F_{i+1} + m_i g = m_i \frac{d^2 w_i}{dt^2} \quad \text{where} \quad \frac{F_{i-1}}{S} = \frac{w_i - w_{i-1}}{h} \quad \text{and} \quad \frac{F_{i+1}}{S} = \frac{w_i - w_{i+1}}{h},$$

$$-F_{n-1} + \underline{F} + m_n g = m_n \frac{d^2 w_n}{dt^2} \quad \text{where} \quad \frac{F_{n-1}}{S} = \frac{w_n - w_{n-1}}{h}.$$

At the fixed boundary transverse displacement vanishes or coincides with that of the surroundings $w_0 = \underline{w}$ so the equations describing the displacement of the particles are

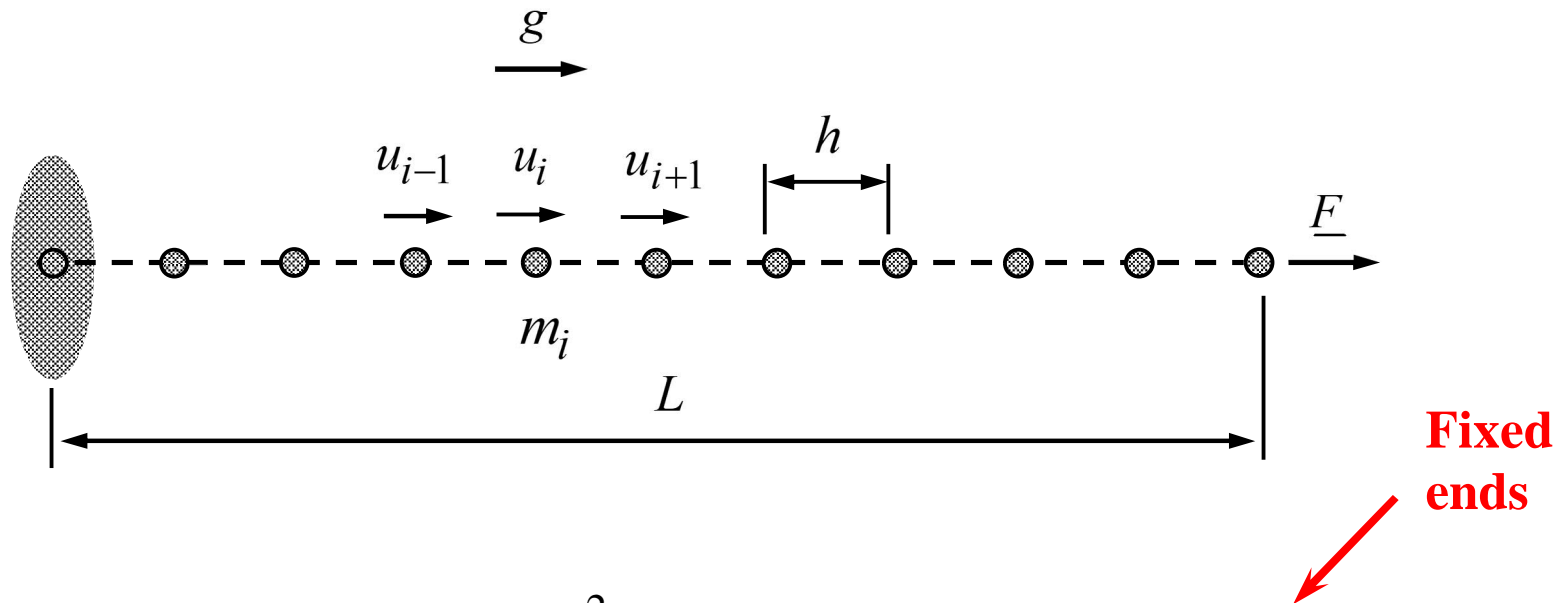
$$\frac{S}{h}(w_{i-1} - 2w_i + w_{i+1}) + m_i g = m_i \frac{d^2 w_i}{dt^2} \quad i \in \{1, 2, \dots, n-1\}$$

$$w_0 = \underline{w} \quad \text{and} \quad -S\left(\frac{w_n - w_{n-1}}{h}\right) + \underline{F} + m_n g = m_n \frac{d^2 w_n}{dt^2}.$$

For a unique solution, the second order ordinary differential equations in time require initial conditions specifying the positions and velocities of the particles at the initial time $t = 0$.

PARTICLE MODEL OF BAR

In the horizontal bar model, particles move in the horizontal direction and forces between the particles are aligned with the bar. In PSM, mass of the bar and the distributed horizontal force are lumped as particles and point forces on a regular grid.



$$\frac{EA}{h}(u_{i-1} - 2u_i + u_{i+1}) + m_i g = m_i \frac{d^2 u_i}{dt^2} \quad i \in \{1, 2, \dots, n-1\} \quad \text{and} \quad u_i = 0 \quad i \in \{0, n\}$$

In the particle surrogate model of an elastic bar, interaction of the particles are modelled by using elastic springs of spring constant $k = EA/h$, where A is the cross-sectional area and E the Young's modulus of the material. The external forces acting on particles are due to gravity.



Using the free body diagrams for the typical particle and the particle at the free end of the string

$$-F_{i-1} - F_{i+1} + m_i g = m_i \frac{d^2 u_i}{dt^2} \quad \text{where} \quad F_{i-1} = \frac{EA}{h}(u_i - u_{i-1}) \quad \text{and} \quad F_{i+1} = \frac{EA}{h}(u_i - u_{i+1}),$$

$$-F_{n-1} + \underline{F} + m_n g = m_n \frac{d^2 u_n}{dt^2} \quad \text{where} \quad F_{n-1} = \frac{EA}{h} (u_n - u_{n-1}).$$

At the fixed boundary transverse displacement vanishes $u_0 = 0$ or coincides with that of the surroundings $u_0 = \underline{u}$ so the equations describing the displacement of the particles are

$$\frac{EA}{h} (u_{i-1} - 2u_i + u_{i+1}) + m_i g = m_i \frac{d^2 u_i}{dt^2} \quad i \in \{1, 2, \dots, n-1\},$$

$$u_0 = \underline{u} \quad \text{and} \quad -\frac{EA}{h} (u_n - u_{n-1}) + \underline{F} + m_n g = m_n \frac{d^2 u_n}{dt^2}.$$

For a unique solution, the second order ordinary differential equations in time require initial conditions specifying the positions and velocities of the particles at the initial time $t = 0$.

DIFFERENCE AND MATRIX REPRESENTATIONS

In their mathematical forms, the particle models for string and bar coincide. Assuming fixed ends, both can be considered as particular cases of a bit more generic set of ordinary second order difference-differential equations (notation $\dot{a} \equiv da / dt$, $\ddot{a} \equiv d^2a / dt^2$)

$$k_i(a_{i-1} - 2a_i + a_{i+1}) + F_i = m_i\ddot{a}_i \quad i \in \{1, 2, \dots, n-1\} \quad \text{and} \quad a_i = 0 \quad i \in \{0, n\} \quad t > 0$$

$$a_i = g_i \quad \text{and} \quad \dot{a}_i = h_i \quad i \in \{1, 2, \dots, n-1\} \quad t = 0$$

or, using the more concise matrix representation, as a set of ordinary second order differential equations

$$-\mathbf{Ka} + \mathbf{F} = \mathbf{M}\ddot{\mathbf{a}} \quad t > 0, \quad \mathbf{a} = \mathbf{g} \quad \text{and} \quad \dot{\mathbf{a}} = \mathbf{h} \quad t = 0.$$

The two (mathematically equivalent) representations are the starting points for difference equation and matrix based solution methods for displacement and vibration analyses.

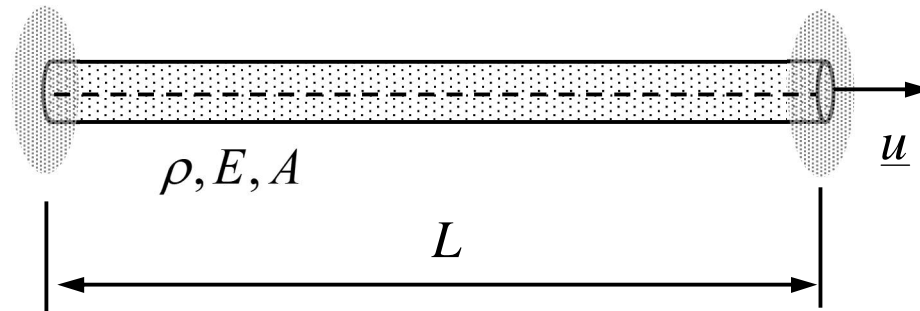
The $(n-1) \times (n-1)$ stiffness and mass matrices of the matrix representation are given by

$$\mathbf{K} = \begin{bmatrix} 2k_1 & -k_1 & & & \\ -k_2 & 2k_2 & -k_2 & & \\ & & \ddots & & \\ & & & -k_{n-1} & 2k_{n-1} \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & \ddots & & \\ & & & & m_{n-1} \end{bmatrix}$$

and the $(n-1)$ column matrices for the displacement of free particles, external forces, initial displacements and velocities

$$\mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{Bmatrix}, \quad \mathbf{F} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{n-1} \end{Bmatrix}, \quad \mathbf{g} = \begin{Bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n-1} \end{Bmatrix}, \quad \mathbf{h} = \begin{Bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-1} \end{Bmatrix}.$$

EXAMPLE A connector bar is welded at its ends to rigid walls. If the right end wall displacement is \underline{u} , write the matrix representation for the stationary particle surrogate model. Cross sectional area A and Young's modulus of the material E are constants Use a regular grid of points $i \in \{0,1,2,3\}$. Also, find the solution to the axial displacements.



Answer $u_0 = 0$, $u_1 = \frac{1}{3}\underline{u}$, $u_2 = \frac{2}{3}\underline{u}$, and $u_3 = \underline{u}$

In time independent problem without external distributed forces, the two difference equations for the free interior particles and the conditions for the boundary particles simplify to

$$u_0 = 0, \quad \frac{EA}{h}(u_0 - 2u_1 + u_2) = 0, \quad \frac{EA}{h}(u_1 - 2u_2 + u_3) = 0, \quad \text{and} \quad u_3 = \underline{u}.$$

In matrix representation, one considers the equations for the free particles and uses the known displacements of the fixed particles in their expressions to get

$$-\frac{EA}{h} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \frac{EA}{h} \begin{Bmatrix} 0 \\ \underline{u} \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ \underline{u} \end{Bmatrix} = \underline{u} \begin{Bmatrix} 1/3 \\ 2/3 \end{Bmatrix}. \quad \leftarrow$$

1.2 DISPLACEMENT ANALYSIS

A stationary surrogate particle model is composed of equilibrium equations for the free particles and equations defining the displacement of fixed particles (displacement boundary conditions). The equilibrium equations for the free interior end boundary particles differ (see the derivation)

Free interior $k(a_{i-1} - 2a_i + a_{i+1}) + F_i = 0 \quad i \in \{1, 2, \dots, n-1\}$

Free boundary $k(a_0 - a_1) = \underline{F}_0$ or $k(a_n - a_{n-1}) = \underline{F}_n$

Fixed $a_i = \underline{a}_i \quad i \in \{0, n\}$

For a boundary particle, one may give the force acting on a particle or displacement of the particle but not both. Also, displacement condition should be specified for one particle to make the solution unique (otherwise rigid body motion is not constrained).

MATRIX REPRESENTATION

Representing the displacement of the free particles by column matrix \mathbf{a} , the coefficient of internal force, and the external force terms by square stiffness matrix \mathbf{K} and column matrix \mathbf{F} , respectively, gives the set of algebraic equations in their “standard” forms

$$-\mathbf{K}\mathbf{a} + \mathbf{F} = \mathbf{0} \quad \text{where} \quad \mathbf{K} = k \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}, \quad \mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{Bmatrix}, \quad \text{and} \quad \mathbf{F} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{n-1} \end{Bmatrix}.$$

Matrix representation allows difference equations of various forms in the same model. Solution to the displacement follows with the standard methods for linear equations system. Therefore, the computational work in calculations is some power (1...3) of n and $n \sim 1/h$ (depends on the method used).

DIFFERENCE EQUATION REPRESENTATION

Difference equation is a mathematical equality involving the differences between successive values of a function of a discrete variable, typically values of a function at discrete spatial or temporal domain. Assuming constant coefficients α, β, γ and a polynomial δ_i

Difference equation $\alpha a_{i-1} + \beta a_i + \gamma a_{i+1} + \delta_i = 0 \Rightarrow$

Generic solution $a_i = Ar_1^i + Br_2^i$ or $a_i = r_1^i (A + Bi)$ where $\alpha + \beta r + \gamma r^2 = 0$

Particular solution $a_i = C + Di + Ei^2 + \dots$

Difference equations of discrete variables correspond to differential equations of continuous variables: Solution is composed of the generic solution to homogeneous equations and a particular solution. Uniqueness of the solution require additional (boundary) conditions of number indicated by the order.

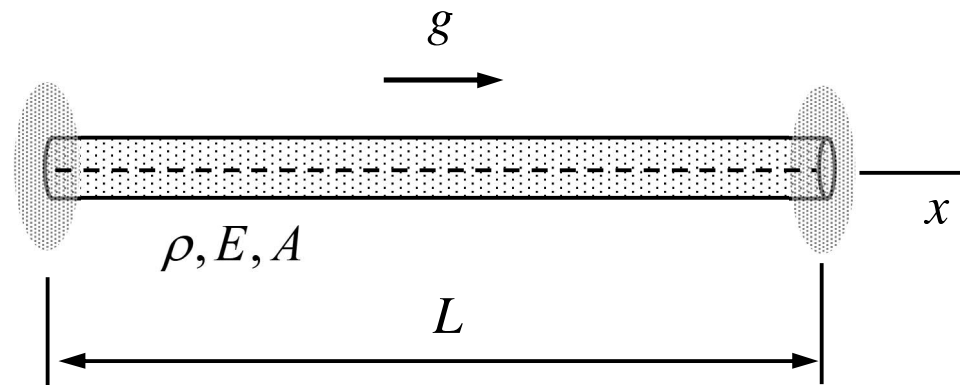
The well-known methods for 2nd order ordinary differential equations work with slight modifications. Solution to the homogeneous equation $\delta_i = 0$ is obtained with $a_i = Ar^i$:

$$\alpha Ar^{i-1} + \beta Ar^i + \gamma Ar^{i+1} = Ar^{i-1}(\alpha + \beta r + \gamma r^2) = 0 \Rightarrow \alpha + \beta r + \gamma r^2 = 0.$$

Separate roots imply the generic solution of the form $a_i = Ar_1^i + Br_2^i$. A double root implies the generic solution of the form $a_i = (A + Bi)r_1^i$. Assuming a polynomial δ_i , particular solution follows with a higher order polynomial trial etc. Finally, the two parameters A and B of the solution follow from the equations for the boundary particles $i \in \{0, n\}$.

Representation by difference equation is particularly useful on regular grids and a simple δ_i as the difference equation can be solved analytically in the same manner as the underlying ordinary differential equation. Therefore, effort for solving the problem does not depend on n at all.

EXAMPLE A connector bar, which is loaded by its own weight, is welded at its ends to rigid walls. Use a particle surrogate model on a regular grid of points $i \in \{0,1,2,3,4\}$ to find the displacements at the grid points. Use first the difference equation method to determine displacements on the generic grid and apply that to get the solution for the case $n = 4$. Cross sectional area A , density of the material ρ , and Young's modulus of the material E are constants.



Answer $u_i = \frac{\rho L^2 g}{E} \frac{i(n-i)}{2n^2}$ and $u_i = \frac{\rho L^2 g}{E} \frac{i(4-i)}{32}$ when $n = 4$

Let us use the difference equation method for generic n

$$k(u_{i-1} - 2u_i + u_{i+1}) + F = 0 \quad i \in \{1, 2, \dots, n-1\} \quad \text{and} \quad u_i = 0 \quad i \in \{0, n\},$$

where $k = EA/h$, $m = \rho Ah$, $F = mg$, and $h = L/n$. Solution to the homogeneous equation follows with $a_i = Ar^i$ giving, when substituted to the difference equation,

$$r^2 - 2r + 1 = (r-1)^2 = 0.$$

The double root $r = 1$ implies the generic solution $u_i = A + Bi$. As loading F is constant, i.e., a zero-order polynomial in i , a second order polynomial $u_i = Ci^2$ might work as the particular solution. Substitution into the difference equation gives

$$k[C(i-1)^2 - 2Ci^2 + C(i+1)^2] + F = 0 \Rightarrow C = -\frac{F}{2k}.$$

When the solution is substituted there, the two displacement conditions $u_0 = u_n = 0$ give equations for the two parameters of the generic solution part

$$u_0 = A = 0 \quad \text{and} \quad u_n = A + Bn - \frac{F}{2k}n^2 = 0 \quad \Leftrightarrow \quad A = 0 \quad \text{and} \quad B = \frac{F}{2k}n.$$

Therefore, the analytic solution to the bar problem by PSM becomes

$$u_i = \frac{F}{2k}i(n-i) \quad \text{where} \quad \frac{F}{k} = \frac{\rho L^2 g}{En^2}.$$

NOTICE: In terms of the coordinates of the grid points $x_i = hi$ and $L = hn$

$$u_i = \frac{\rho L^2 g}{E} \frac{i(n-i)}{2n^2} = \frac{\rho g}{E} \frac{x_i(L-x_i)}{2} \quad (\text{continuous model } u(x) = \frac{\rho g}{E} \frac{x(L-x)}{2}).$$

1.3 VIBRATION ANALYSIS

In time dependent case, the model is composed of equations for the interior and boundary particles, and two initial conditions for the free particles. Considering displacement component $a(t)$ on a regular grid

$$\mathbf{Particles } i \in \{1, 2, \dots, n-1\}: \quad k(a_{i-1} - 2a_i + a_{i+1}) + F_i = m_i \ddot{a}_i \quad i \in \{1, 2, \dots, n-1\} \quad t > 0$$

$$\mathbf{Particle } 0: \quad a_0 = \underline{a}_0 \quad \text{or} \quad -k(a_0 - a_1) + \underline{F}_0 + m_0 g = m_0 \ddot{w}_0 \quad t > 0$$

$$\mathbf{Particle } n: \quad a_n = \underline{a}_n \quad \text{or} \quad -k(a_n - a_{n-1}) + \underline{F}_n + m_n g = m_n \ddot{w}_n \quad t > 0$$

$$\mathbf{Initial conditions: } a_i = g_i \quad \text{and} \quad \dot{a}_i = h_i \quad i \in \{1, 2, \dots, n-1\} \quad t = 0$$

In solid mechanics, one may give the force acting on a particle or displacement of the particle as the boundary condition but not both.

MATRIX REPRESENTATION

Representing the displacement of the free particles by column matrix $\mathbf{a}(t)$, coefficients of $\ddot{\mathbf{a}}(t)$ by square mass matrix \mathbf{M} , coefficients of $\mathbf{a}(t)$ by square stiffness matrix \mathbf{K} , and the external force terms by column matrix \mathbf{F} in the difference equations of the free particles gives the second order initial value problem of ordinary differential equations

$$-\mathbf{K}\mathbf{a} + \mathbf{F} = \mathbf{M}\ddot{\mathbf{a}} \quad t > 0 \quad \mathbf{a} = \mathbf{g} \quad \text{and} \quad \dot{\mathbf{a}} = \mathbf{h} \quad t = 0.$$

The column matrices \mathbf{g} and \mathbf{h} represent the initial positions and velocities of the free particles. Matrix representation is the concise starting point for

- (1) mode analysis for frequencies and modes of free vibrations
- (2) displacement solutions based on the frequencies and modes
- (3) step-by-step time integration methods on temporal grid of time instants

The $n-1$ by $n-1$ matrices and the $n-1$ column matrix corresponding to a problem of free particles $i \in \{1, 2, \dots, n-1\}$, conditions $a_0 = a_n = 0$, and $k_i = k$, $m_i = m$, $F_i = F$

$$\mathbf{K} = k \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{M} = m \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and } \mathbf{F} = F \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{Bmatrix}.$$

The coefficients for the bar of length $L = nh$ are $k = EA/h$, $m = \rho Ah$, and $F = g\rho Ah$ (gravity in the direction of the axis) and the coefficients for the string of length $L = nh$ are $k = S/h$, $m = \rho Ah$, and $F = g\rho Ah$ (gravity in the transverse direction).

MODAL ANALYSIS

For constant \mathbf{M} , \mathbf{K} and $\mathbf{F} = \mathbf{0}$, displacement can be considered as the sum of harmonic components. In mode analysis, a harmonic trial solution is used to transform the ordinary differential equations into algebraic one for the angular velocity and mode pairs (ω_j, \mathbf{A}_j) :

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{0} \quad \text{and} \quad \mathbf{a}(t) = \mathbf{A}e^{i\omega t} \quad \Rightarrow \quad (-\mathbf{M}\omega^2 + \mathbf{K})\mathbf{A} = \mathbf{0}.$$

The necessary condition for a non-zero solution to \mathbf{A} is $\det(-\mathbf{M}\omega^2 + \mathbf{K}) = 0$. The algebraic polynomial equation gives ω_j $j \in \{1, 2, \dots\}$ of number of the free particles as its solutions (positive square roots of ω^2). After that, the modes follow from

$$(-\mathbf{M}\omega_j^2 + \mathbf{K})\mathbf{A}_j = \mathbf{0}$$

up to an arbitrary multiplier. The angular velocity ω and frequency f are related by $\omega = 2\pi f$.

MODE SUPERPOSITION

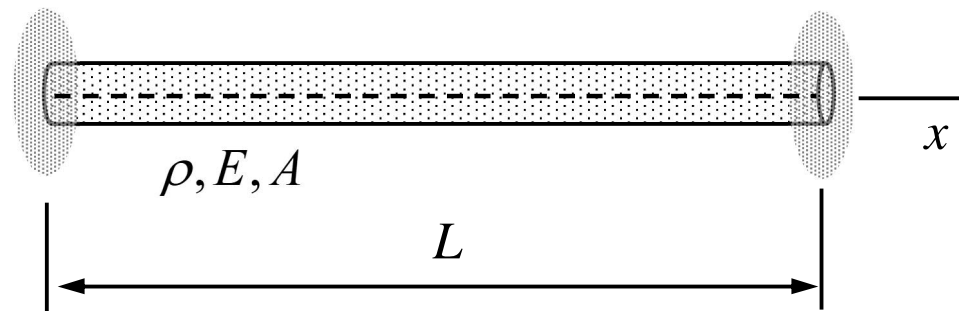
If the initial conditions concerning position and displacement of the particles are known (quite exceptional case), the outcome of the modal analysis (ω_j, \mathbf{A}_j) $j \in \{1, 2, \dots\}$ can be used to construct a displacement solution for the given initial data. The combination of the modes for $\mathbf{a} = \mathbf{g}$ and $\dot{\mathbf{a}} = \mathbf{h}$ is given by

$$\mathbf{a}(t) = \sum_{j \in \{1, 2, \dots\}} \mathbf{A}_j \left[\frac{1}{\omega_j} \alpha_j \sin(\omega_j t) + \beta_j \cos(\omega_j t) \right] \quad \text{where}$$

$$\alpha_j = \frac{\mathbf{A}_j^T \mathbf{h}}{\mathbf{A}_j^T \mathbf{A}_j} \quad \text{and} \quad \beta_j = \frac{\mathbf{A}_j^T \mathbf{g}}{\mathbf{A}_j^T \mathbf{A}_j}.$$

As the first term contains division by ω_k , one should use $\lim_{\omega \rightarrow 0} \sin(\omega t) / \omega = t$ if $\omega_k = 0$. The simple formula relies on orthogonality of the modes $\mathbf{A}_j^T \mathbf{A}_l = 0$ whenever $j \neq l$. One may think that the coefficients α_j and β_j are given by discrete Fourier series.

EXAMPLE A connector bar is welded at its ends to rigid walls. Use a particle surrogate model on a regular grid of points $i \in \{0, 1, \dots, n\}$ to find displacements at the grid points as functions of time for the initial data $g_i = U$ and $h_i = 0$, respectively. Cross sectional area A , density of the material ρ , and Young's modulus of the material E are constants. Use the matrix method and consider the case $n = 3$



Answer $\mathbf{a}(t) = U \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos\left(\sqrt{\frac{k}{m}}t\right) = U \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos\left(\frac{3}{L} \sqrt{\frac{E}{\rho}}t\right)$

Let us start with modal analysis with the stiffness and mass matrix of the example problem where $k = EA/h$, $m = \rho Ah$, and $h = L/3$. The number of free particles is 2 so

$$\left(m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \omega^2 + k \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\right) \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = k \begin{bmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad (\text{denote } \lambda = \frac{m\omega^2}{k})$$

The homogeneous linear equation system can yield a non-zero solution to the mode only if the matrix in parenthesis is singular, i.e., its determinant vanishes

$$\det \begin{bmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 2 \end{bmatrix} = (\lambda - 2)^2 - 1 = 0 \quad \text{so } \lambda_1 = 1 \text{ or } \lambda_2 = 3.$$

Knowing the possible angular velocities, solution to the modes are given by the linear equation systems:

$$\lambda_1 = 1: \quad \omega_1 = \sqrt{\lambda \frac{k}{m}} \quad \text{and} \quad \begin{bmatrix} 1-2 & 1 \\ 1 & 1-2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{so} \quad (\omega_1, \mathbf{A}_1) = \left(\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right).$$

$$\lambda_2 = 3: \quad \omega_2 = \sqrt{3 \frac{k}{m}} \quad \text{and} \quad \begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{so} \quad (\omega_2, \mathbf{A}_2) = \left(\sqrt{3 \frac{k}{m}}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \right).$$

The coefficients of the series solution are $\alpha_1 = \alpha_2 = 0$ and

$$\beta_1 = \frac{\mathbf{A}_1^T \mathbf{g}}{\mathbf{A}_1^T \mathbf{A}_1} = \frac{\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}^T \begin{Bmatrix} U \\ U \end{Bmatrix}}{\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}^T \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}} = U, \quad \beta_2 = \frac{\mathbf{A}_2^T \mathbf{g}}{\mathbf{A}_2^T \mathbf{A}_2} = \frac{\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}^T \begin{Bmatrix} U \\ U \end{Bmatrix}}{\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}^T \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}} = 0 \quad \Rightarrow$$

$$\mathbf{a}(t) = \sum_{k \in \{1,2\}} \mathbf{A}_j \left[\frac{1}{\omega_j} \alpha_j \sin(\omega_j t) + \beta_j \cos(\omega_j t) \right] = U \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos\left(\sqrt{\frac{k}{m}} t\right).$$

DIFFERENTIAL-DIFFERENCE EQUATION METHOD

The analytical solution method for differential-difference equations on a regular grid uses a trial solution mode which gives an algebraic equation for the corresponding angular velocity. Then the outcome of modal analysis takes the form

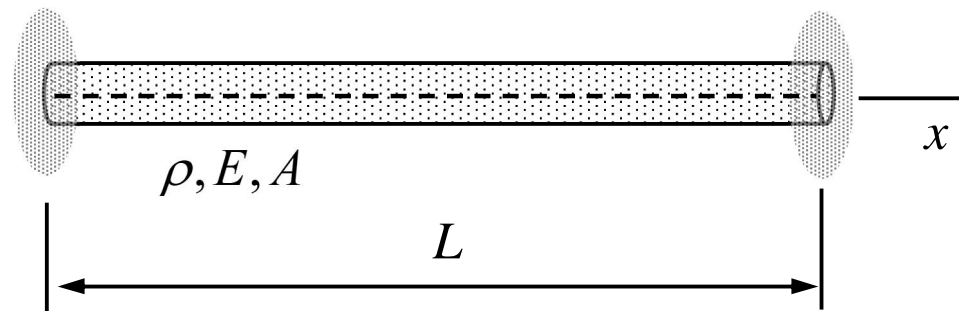
$$(\omega_j, \mathbf{A}_j) \text{ where } \omega_j = \sqrt{2 \frac{k}{m} [1 - \cos(\frac{j\pi}{n})]} \text{ and } (\mathbf{A}_j)_i = \alpha \cos(\pi j \frac{i}{n}) + \beta \sin(\pi j \frac{i}{n}),$$

where α and β are determined by the (homogeneous) boundary conditions. The combination of the modes for $\mathbf{a} = \mathbf{g}$ and $\dot{\mathbf{a}} = \mathbf{h}$ at $t = 0$

$$\mathbf{a}(t) = \sum_{j \in \{1, 2, \dots\}} \mathbf{A}_j \left[\frac{1}{\omega_j} \alpha_j \sin(\omega_j t) + \beta_j \cos(\omega_j t) \right], \quad \alpha_j = \frac{\mathbf{A}_j^T \mathbf{h}}{\mathbf{A}_j^T \mathbf{A}_j} \text{ and } \beta_k = \frac{\mathbf{A}_j^T \mathbf{g}}{\mathbf{A}_j^T \mathbf{A}_j}$$

is the same as with the matrix formulation.

EXAMPLE A connector bar is welded at its ends to rigid walls. Use a particle surrogate model on a regular grid of points with $n = 3$ to find displacements at the grid points as functions of time for the initial data $g_i = U$ and $h_i = 0$, respectively. Cross sectional area A , density of the material ρ , and Young's modulus of the material E are constants. Use the differential-difference equation method.



Answer $\mathbf{a}(t) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos\left(\frac{3}{L} \sqrt{\frac{E}{\rho}} t\right)$

The method is based on the closed form solution to the modal analysis

$$(\omega_j, \mathbf{A}_j) \text{ where } \omega_j = \sqrt{2 \frac{k}{m} [1 - \cos(\frac{j\pi}{n})]} \text{ and } (\mathbf{A}_j)_i = \gamma \cos(\pi j \frac{i}{n}) + \delta \sin(\pi j \frac{i}{n})$$

As both ends are fixed, the parameters of the modes are chosen to be $\gamma = 0$ and $\delta = 1$ (say). As $n = 3$ $j \in \{1, 2\}$ so

$$\omega_1 = \sqrt{2 \frac{k}{m} [1 - \cos(\pi \frac{1}{3})]} = \sqrt{\frac{k}{m}} \text{ and } \mathbf{A}_1 = \begin{Bmatrix} \sin(\pi / 3) \\ \sin(2\pi / 3) \end{Bmatrix} = \frac{\sqrt{3}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\omega_2 = \sqrt{2 \frac{k}{m} [1 - \cos(2\pi \frac{1}{3})]} = \sqrt{3 \frac{k}{m}} \text{ and } \mathbf{A}_2 = \begin{Bmatrix} \sin(2\pi / 3) \\ \sin(4\pi / 3) \end{Bmatrix} = \frac{\sqrt{3}}{2} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Notice that the modes differ only in scaling from the ones of the matrix method. Therefore, the solution by the two methods coincide as should be the case.

DISCRETE SINE SERIES

The discrete Fourier series (various forms exist) can be used to represent a list as the sum of lists of harmonic terms. For example, the sine-transformation pair for a list a_i $i \in \{1, 2, \dots, n-1\}$ is given by

$$\alpha_j = \frac{2}{n} \sum_{i \in \{1, 2, \dots, n-1\}} \sin(j\pi \frac{i}{n}) a_i \quad j \in \{1, 2, \dots, n-1\}$$

$$a_i = \sum_{j \in \{1, 2, \dots, n-1\}} \alpha_j \sin(j\pi \frac{i}{n}) \quad i \in \{1, 2, \dots, n-1\}$$

The transformation pair is based on the orthogonality of the modes (Cronecker delta $\delta_{jl} = 1$ if $j = l$ and $\delta_{jl} = 0$ if $j \neq l$)

$$\sum_{j \in \{1, 2, \dots, n-1\}} \sin(j\pi \frac{i}{n}) \sin(l\pi \frac{i}{n}) = \delta_{jl} \frac{n}{2}.$$

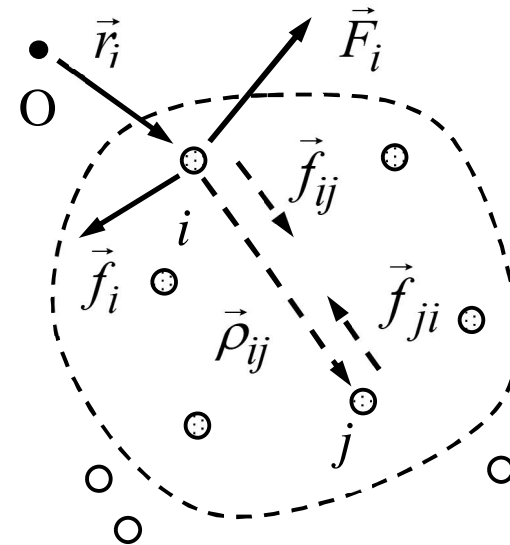
1.4 PRINCIPLE OF VIRTUAL WORK

Principle of virtual work is one of the equivalent forms of equilibrium equations and equations of motion (an important one). According to the principle, work of forces acting on the particles vanishes in all virtual displacements of the particles. In short $\delta W = \delta W^{\text{ext}} + \delta W^{\text{int}} + \delta W^{\text{ine}} = 0 \quad \forall \delta \vec{r}_i$.

External $\delta W^{\text{ext}} = \sum \vec{F}_i \cdot \delta \vec{r}_i$

Internal $\delta W^{\text{int}} = \sum \vec{f}_i \cdot \delta \vec{r}_i = -\sum f_p \delta \rho_p$

Inertia $\delta W^{\text{ine}} = -\sum m_i \vec{a}_i \cdot \delta \vec{r}_i$



Principle of virtual work has a physical interpretation which is useful in connection with the variety of particle and continuum models and numerical methods in solid mechanics.

Let us consider the set of particles $i \in I$ and the set of interacting pairs $p \in P \subset I \times I$. The two equivalent representations for the equations of motion are

$$\vec{F}_i + \vec{f}_i = m_i \vec{a}_i \quad \forall i \in I \quad \Leftrightarrow \quad \sum_{i \in I} \delta \vec{r}_i \cdot (\vec{F}_i + \vec{f}_i - m_i \vec{a}_i) = 0 \quad \forall \delta \vec{r}_i,$$

where $\delta \vec{r}_i$ is the virtual displacement (a virtual offset) of particle i . The two-ways implication follows from the fundamental lemma of variation calculus. Works of the forces in the virtual displacement (forces are not affected by the virtual offset)

$$\delta W^{\text{ext}} = \sum_{i \in I} \delta \vec{r}_i \cdot \vec{F}_i, \quad \delta W^{\text{int}} = \sum_{i \in I} \delta \vec{r}_i \cdot \vec{f}_i, \quad \text{and} \quad \delta W^{\text{ine}} = -\sum_{i \in I} \delta \vec{r}_i \cdot m_i \vec{a}_i.$$

Above, \vec{f}_i denotes the sum of the internal forces acting on particle i . Let us consider a typical pair $p = (i, j)$ of particles and interaction $\vec{f}_{ij} = -\vec{f}_{ji}$. The contribution to δW^{int} can be written in a more concise form

$$\delta W_p^{\text{int}} = \vec{f}_{ij} \cdot \delta \vec{r}_i + \vec{f}_{ji} \cdot \delta \vec{r}_j = \vec{f}_{ij} \cdot \delta(\vec{r}_i - \vec{r}_j) = -\vec{f}_{ij} \cdot \delta \vec{\rho}_{ij} = -f_p \delta \rho_p,$$

where $\vec{r}_j = \vec{r}_i + \vec{\rho}_{ij}$ (assuming a simple force interaction). The overall work of the internal forces is obtained as the sum over all the interacting pairs, i.e.,

$$\delta W^{\text{int}} = \sum_{p \in P} \delta W_p^{\text{int}} = -\sum_{p \in P} f_p \delta \rho_p.$$

VIRTUAL WORK EXPRESSIONS FOR STRING

Let us consider particles $i \in I = \{0, 1, \dots, n\}$, interacting particle pairs $p \in P \subset I \times I$, choose $\delta w_i = 0$ whenever $w_i = \underline{w}_i$ (known), and denote $\Delta w_p = w_i - w_j$ when $p = (i, j)$

Internal forces:
$$\delta W^{\text{int}} = - \sum_{p \in P} \delta \Delta w_p \frac{S}{h} \Delta w_p$$

External forces:
$$\delta W^{\text{ext}} = \sum_{i \in I} \delta w_i F_i$$

Inertia forces:
$$\delta W^{\text{ine}} = - \sum_{i \in I} \delta w_i \rho A h \ddot{w}_i$$

Principle of virtual work and the virtual work expressions give a concise representation of the string and bar equations of PSM. Various different boundary conditions can be included by modification of the expression using the physical work interpretation. The representation is almost indispensable with membrane and thin slab models of the course and irregular grids on generic solution domains.

The external and inertia parts are obvious. In the string model, particle i interacts with the neighbors $i-1$ and $i+1$ only. Therefore, virtual work of the internal forces (all particles accounted for)

$$-\delta W^{\text{int}} = \delta w_0 F_1 + \delta w_1 (F_0 + F_3) + \delta w_2 (F_1 + F_3) + \dots + \delta w_n F_{n-1}.$$

Substituting expressions $F_{i-1} = S(w_i - w_{i-1})/h$ and $F_{i+1} = S(w_i - w_{i+1})/h$ for the left and right neighbour interactions and rearranging

$$-\delta W^{\text{int}} \frac{h}{S} = (\delta w_1 - \delta w_0)(w_1 - w_0) + (\delta w_2 - \delta w_1)(w_2 - w_1) + \dots + (\delta w_n - \delta w_{n-1})(w_n - w_{n-1})$$

and, finally, using the concise sum notation

$$\delta W^{\text{int}} = -\sum_{i \in \{1, 2, \dots, n\}} (\delta w_i - \delta w_{i-1}) \frac{S}{h} (w_i - w_{i-1}) = -\sum_{p \in P} \delta \Delta w_p \frac{S}{h} \Delta w_p. \quad \leftarrow$$

VIRTUAL WORK EXPRESSIONS

Principle of virtual work is just a concise representation of the equations-of-motion (or equilibrium equations) and boundary conditions of a particle surrogate model. Virtual work expression depends on the problem, but the principle does not.

Virtual work	Bar	String
δW^{int}	$-\sum_{p \in P} \delta \Delta u_p \frac{EA}{h} \Delta u_p$	$-\sum_{p \in P} \delta \Delta w_p \frac{S}{h} \Delta w_p$
δW^{ext}	$\sum_{i \in I} \delta u_i F_i$	$\sum_{i \in I} \delta w_i F_i$
δW^{ine}	$-\sum_{i \in I} \delta u_i \rho A h \ddot{u}_i$	$-\sum_{i \in I} \delta w_i \rho A h \ddot{w}_i$