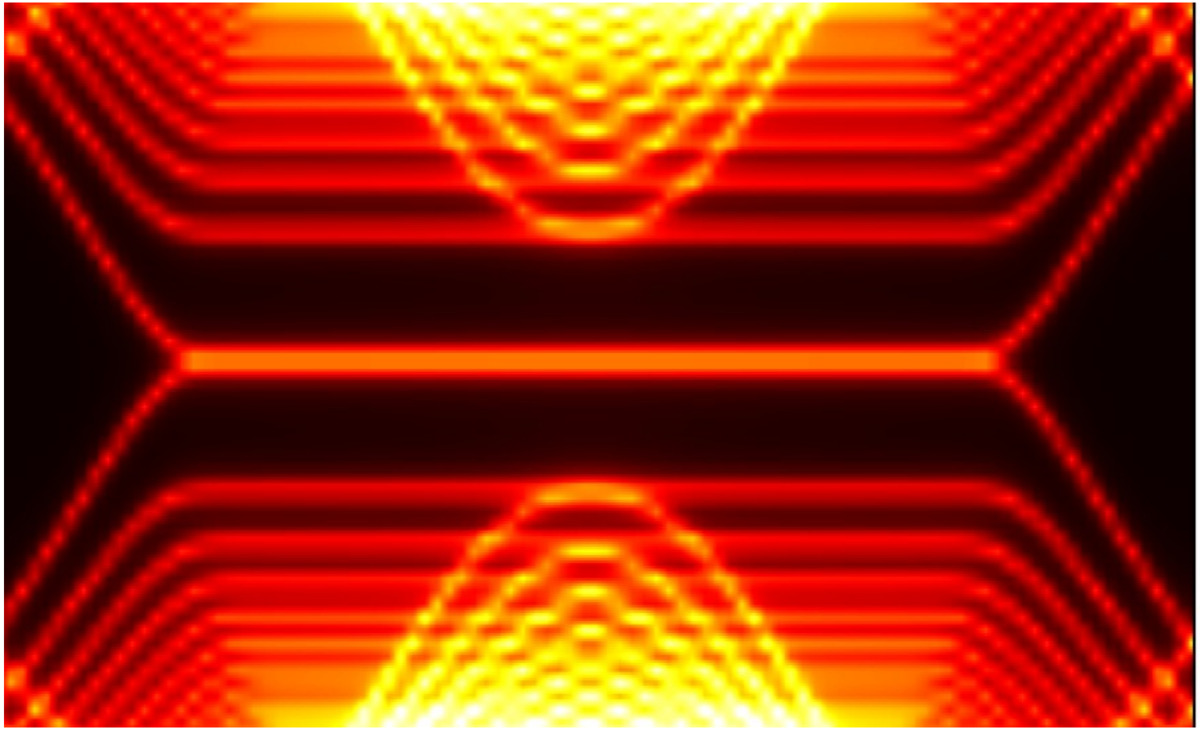
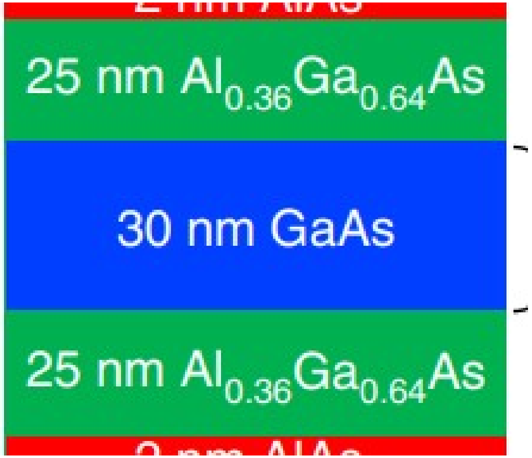
The quantum Hall effect



April 19th 2021

Materials showing quantum Hall effect

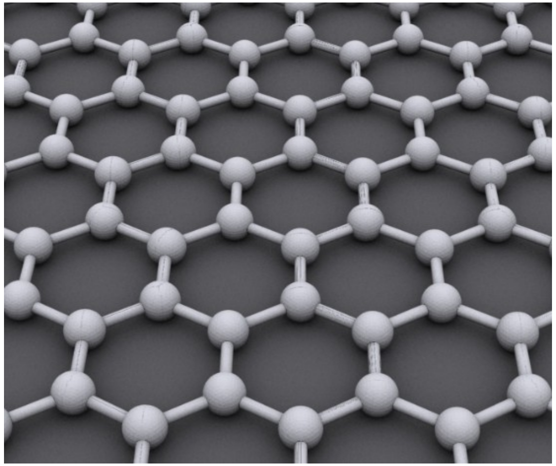
GaAs quantum wells



$$E \sim B$$

$$T \sim 1K$$

Graphene

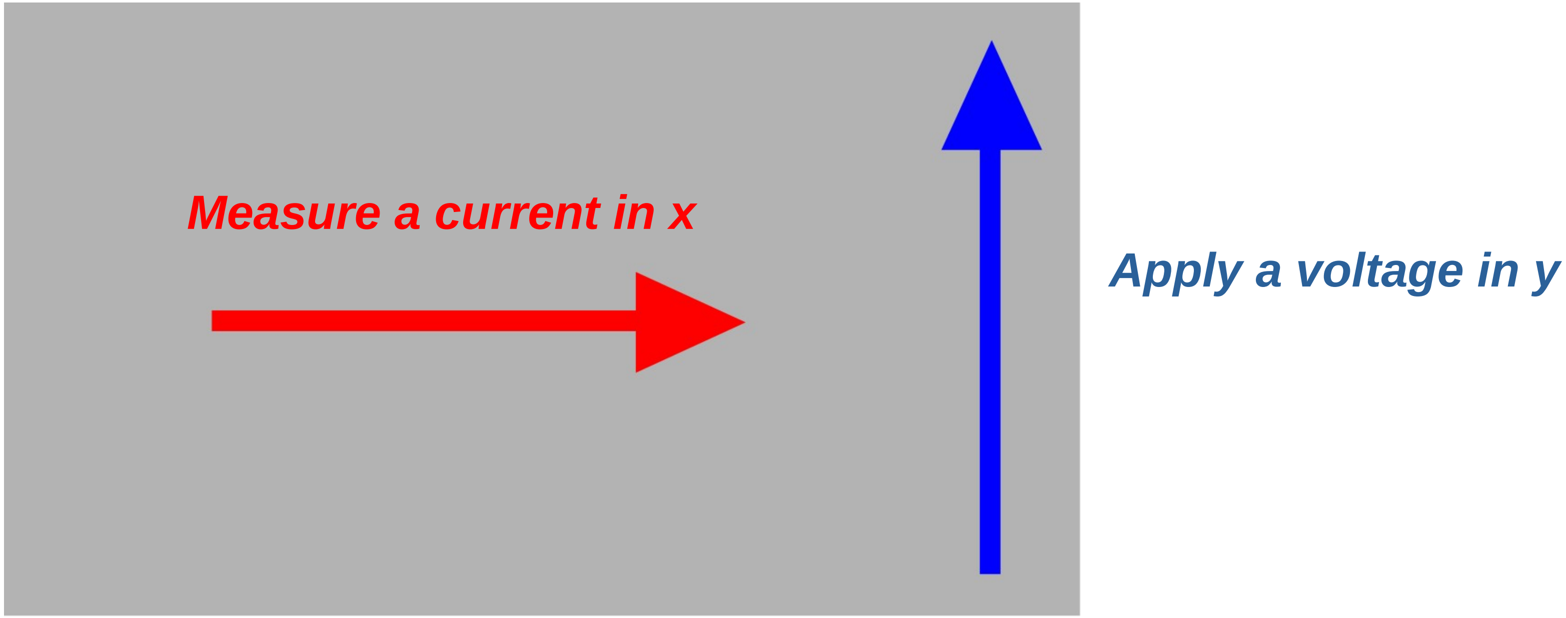


$$E \sim \sqrt{B}$$

$$T \sim 100K$$

The quantum Hall state

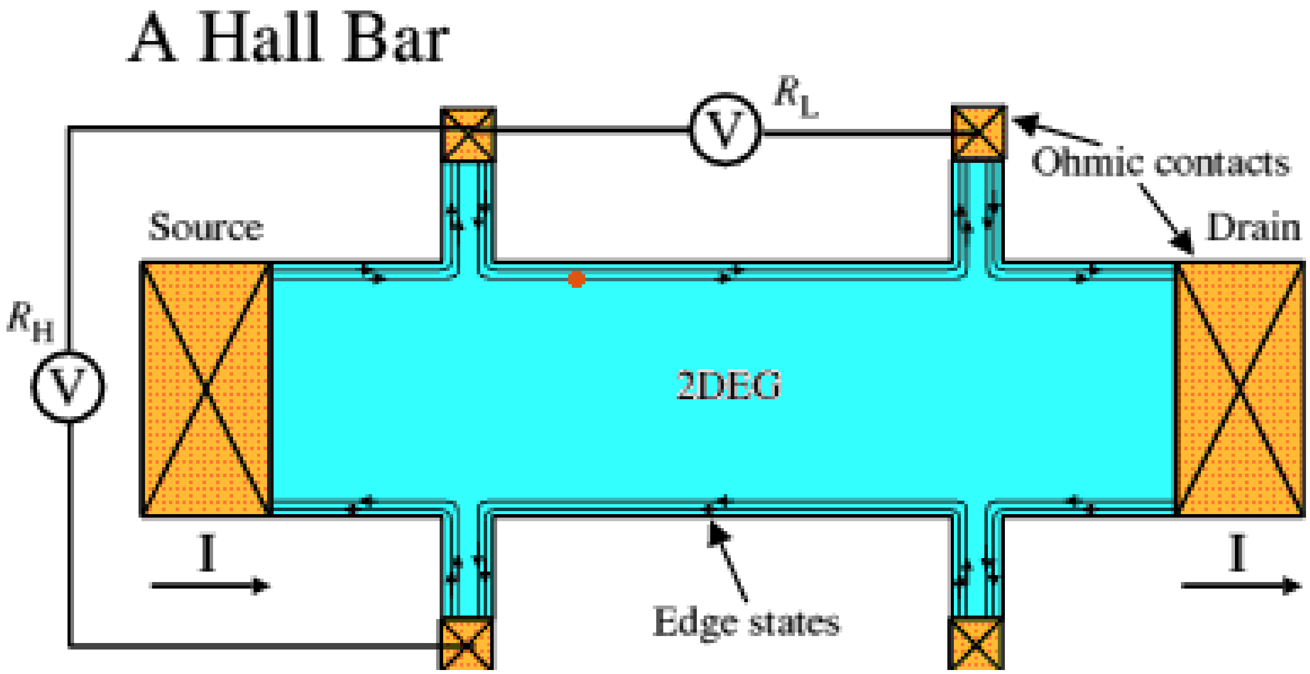
Take a two-dimensional material



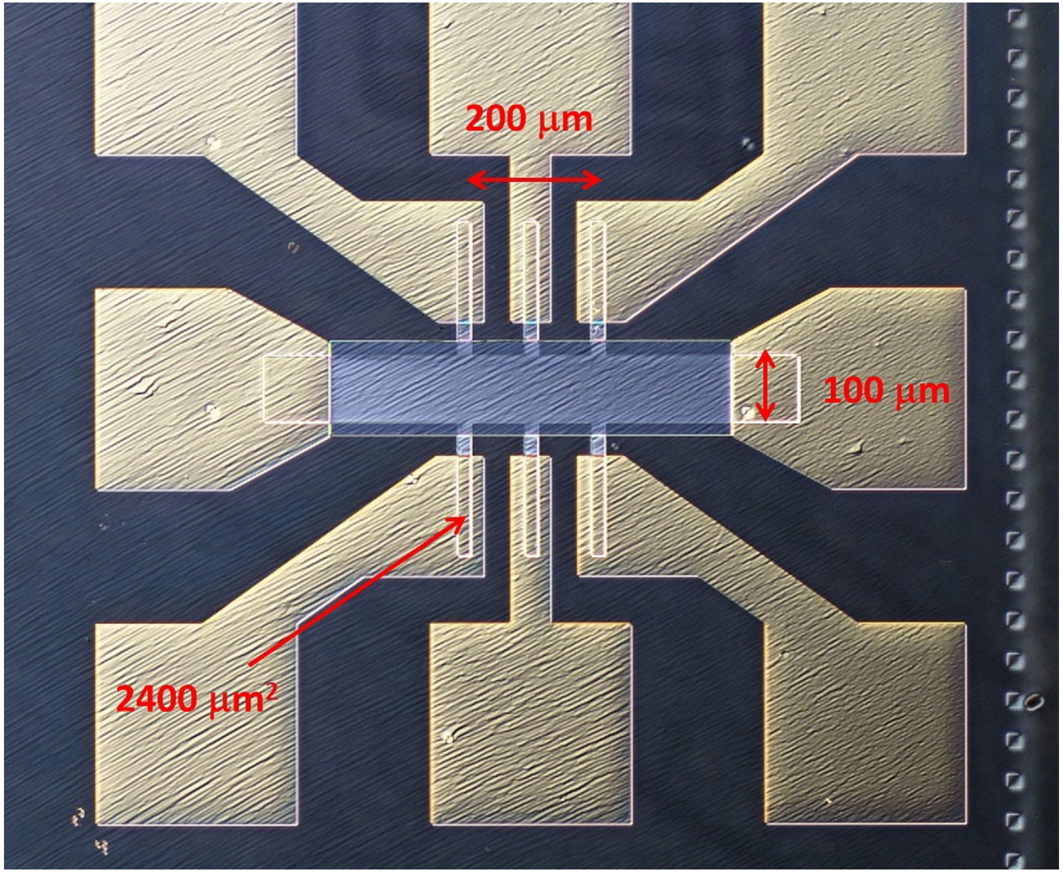
Hall conductance

$$J_x = \sigma_{xy} V_y$$

Quantum Hall devices



Quantum Hall devices



Reminder: Linear response for transverse current

$$J_x = \sigma_{xy} V_y$$

The Hall conductivity is obtained as $$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k}$$

with $$\Omega_\alpha(\mathbf{k}) = i \sum_{\beta \neq \alpha} \frac{\langle \Psi_\alpha | \partial H / \partial k_x | \Psi_\beta \rangle \langle \Psi_\beta | \partial H / \partial k_y | \Psi_\alpha \rangle}{(\epsilon_\alpha - \epsilon_\beta)^2} - \alpha \leftrightarrow \beta$$

Berry curvature of a band

Expression coming from perturbation theory

The Hall conductivity

The Hall conductivity is obtained as $$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k}$$

Using $$\langle \Psi_\alpha | \partial H / \partial k_\mu | \Psi_\beta \rangle = \langle \partial_{k_\mu} \Psi_\alpha | \Psi_\beta \rangle (\epsilon_\alpha - \epsilon_\beta)$$

the Hall conductivity can be expressed in terms of

Berry curvature

$$\Omega_\alpha = \partial_{k_x} A_y^\alpha - \partial_{k_y} A_x^\alpha$$

Berry connection

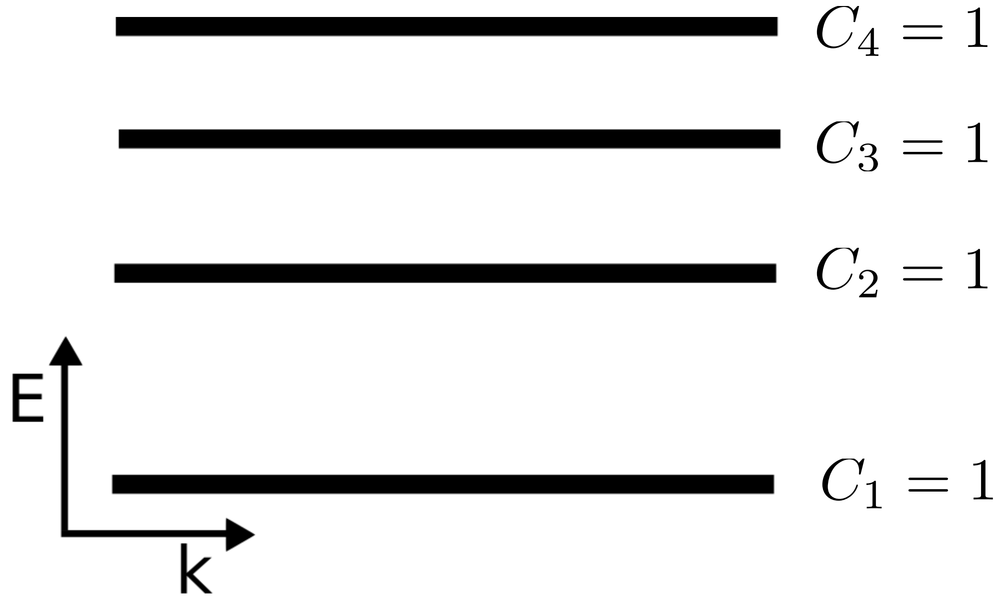
$$A_\mu^\alpha = i \langle \partial_{k_\mu} \Psi_\alpha | \Psi_\alpha \rangle$$

$$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k} = \sum_\alpha C_\alpha = C$$

Chern number

Chern numbers in the quantum Hall state

Band-structure in the quantum Hall state



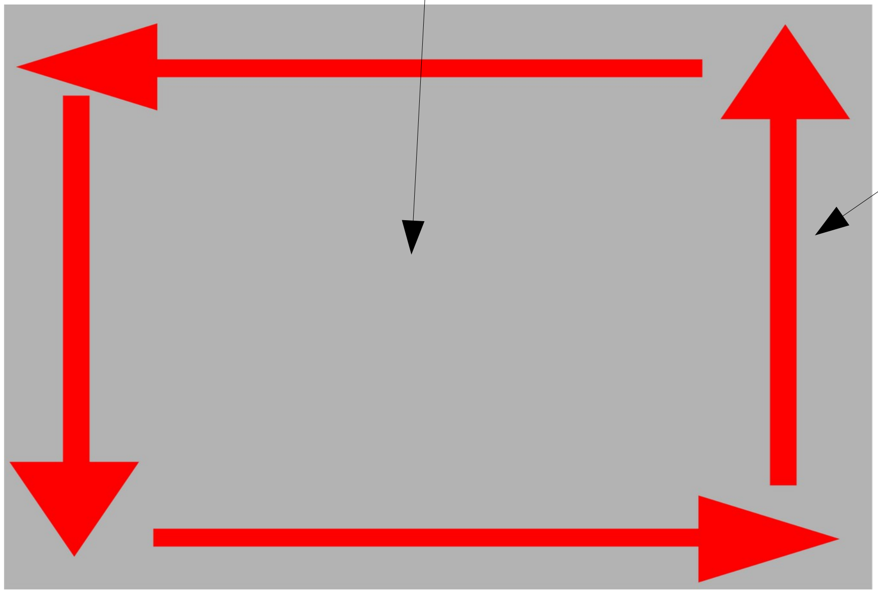
Hall conductivity

$$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k} = \sum_\alpha C_\alpha = C$$

Each band (a.k.a Landau level), contributes with Chern number +1

The puzzling quantum Hall effect

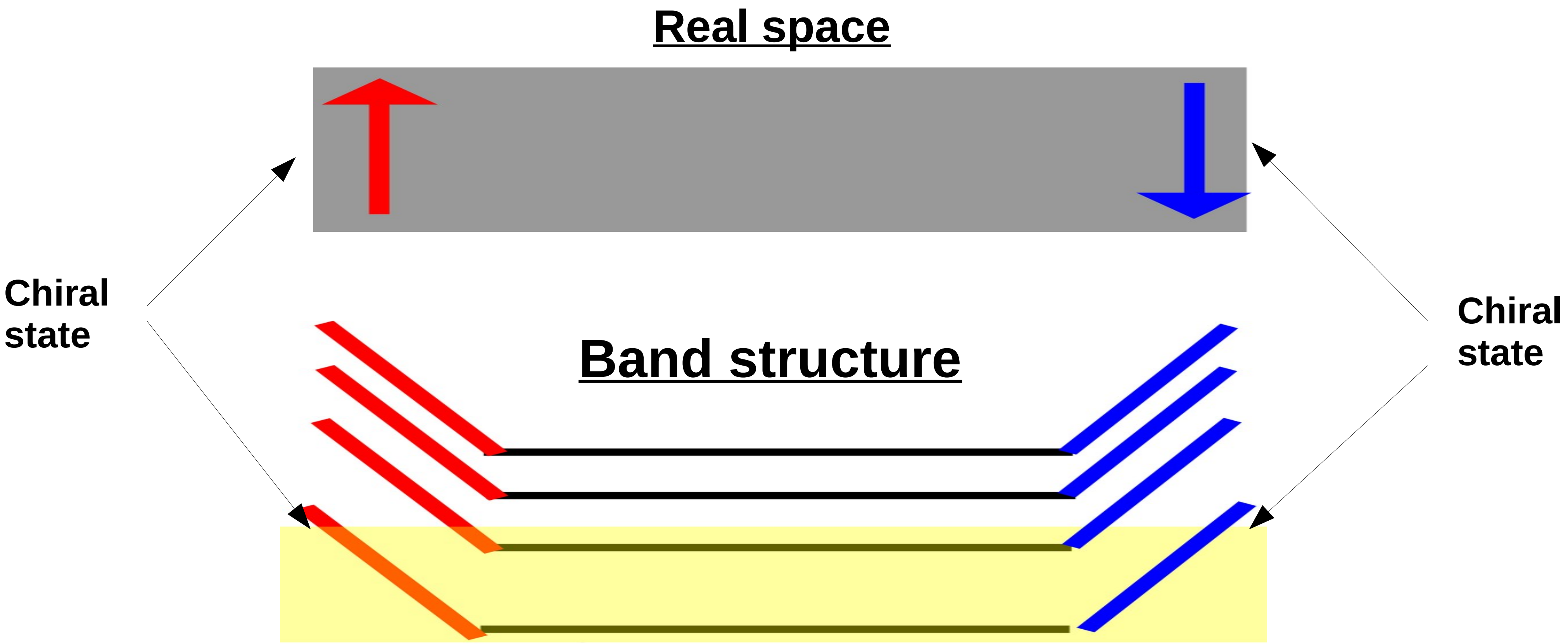
The bulk of a quantum Hall state is insulating



The edge has chiral states

Chiral: propagating
only in one direction

The quantum Hall effect

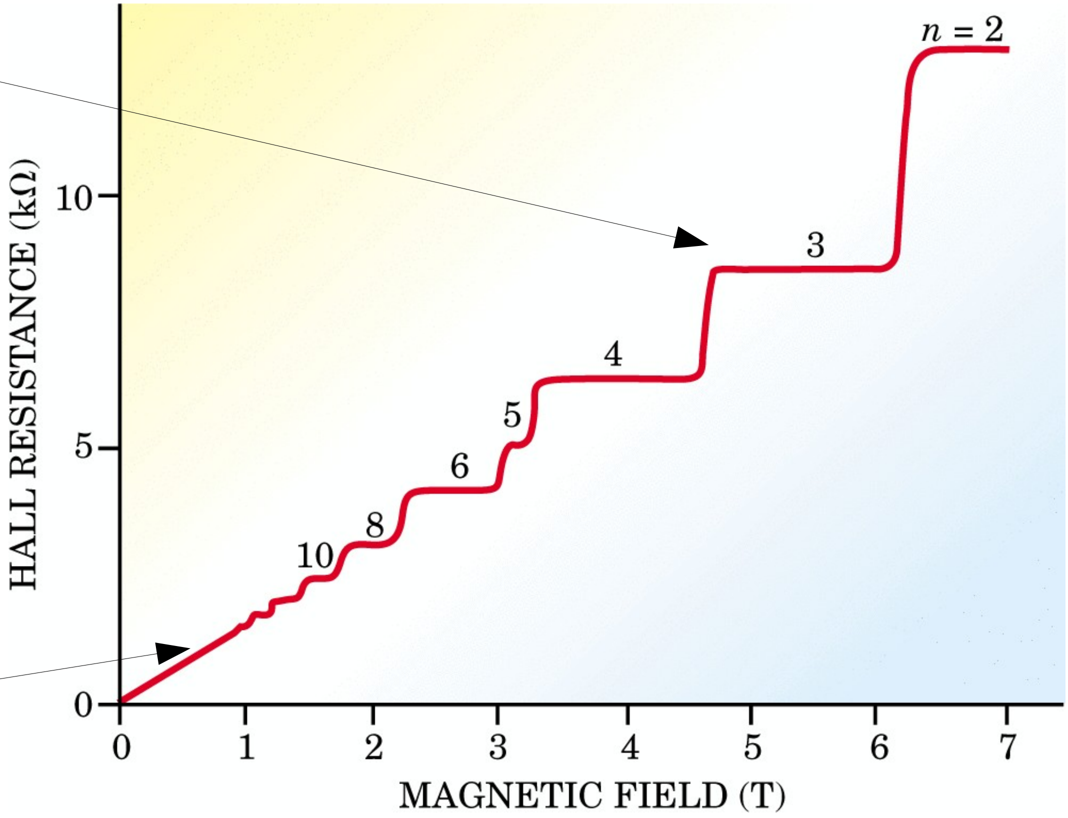


The quantum Hall state

Quantum regime

In the quantum Hall regime

$$\rho_{xy} \sim \sigma_{xy}^{-1}$$

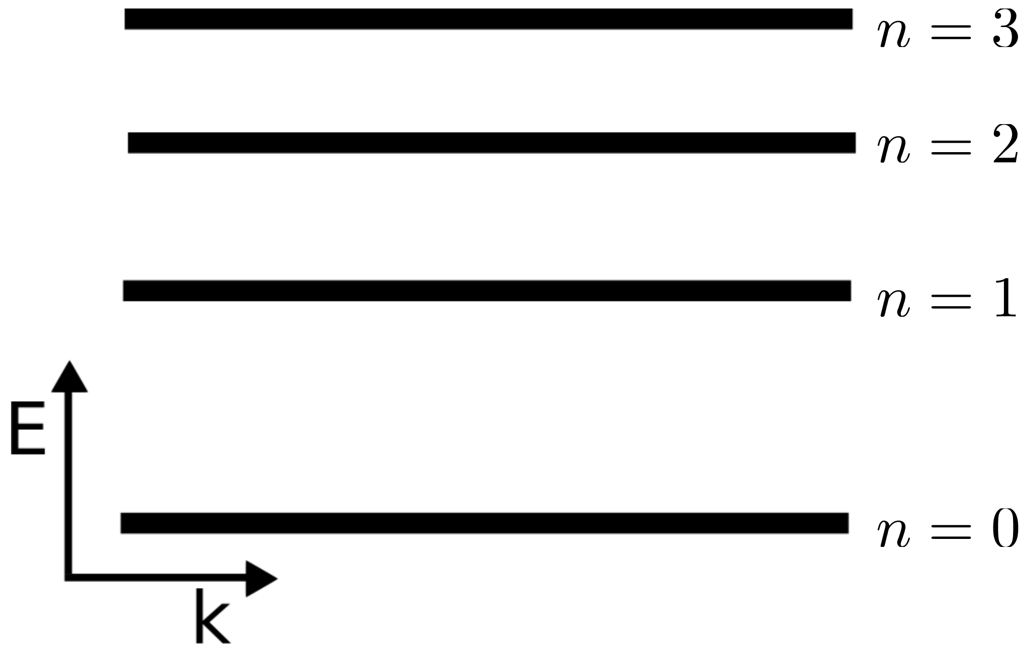


Classical regime

Landau levels

Landau levels in a nutshell

Band-structure in the quantum Hall state



The energy levels are

$$E \sim \left(n + \frac{1}{2}\right) B$$

For a Dirac equation they would be

$$E \sim \sqrt{nB}$$

Electrons coupled to a magnetic field

Let us take a conventional electron gas coupled to a gauge field

$$\widehat{H} = \frac{1}{2m}\left[(\widehat{p}_x - eA_x(\widehat{\boldsymbol{r}}))^2 + (\widehat{p}_y - eA_y(\widehat{\boldsymbol{r}}))^2\right]$$

Gauge potential

Momentum

Minimal gauge coupling

Landau levels in a nutshell

Lets take a quadratic Hamiltonian $$H \sim p_x^2 + p_y^2$$

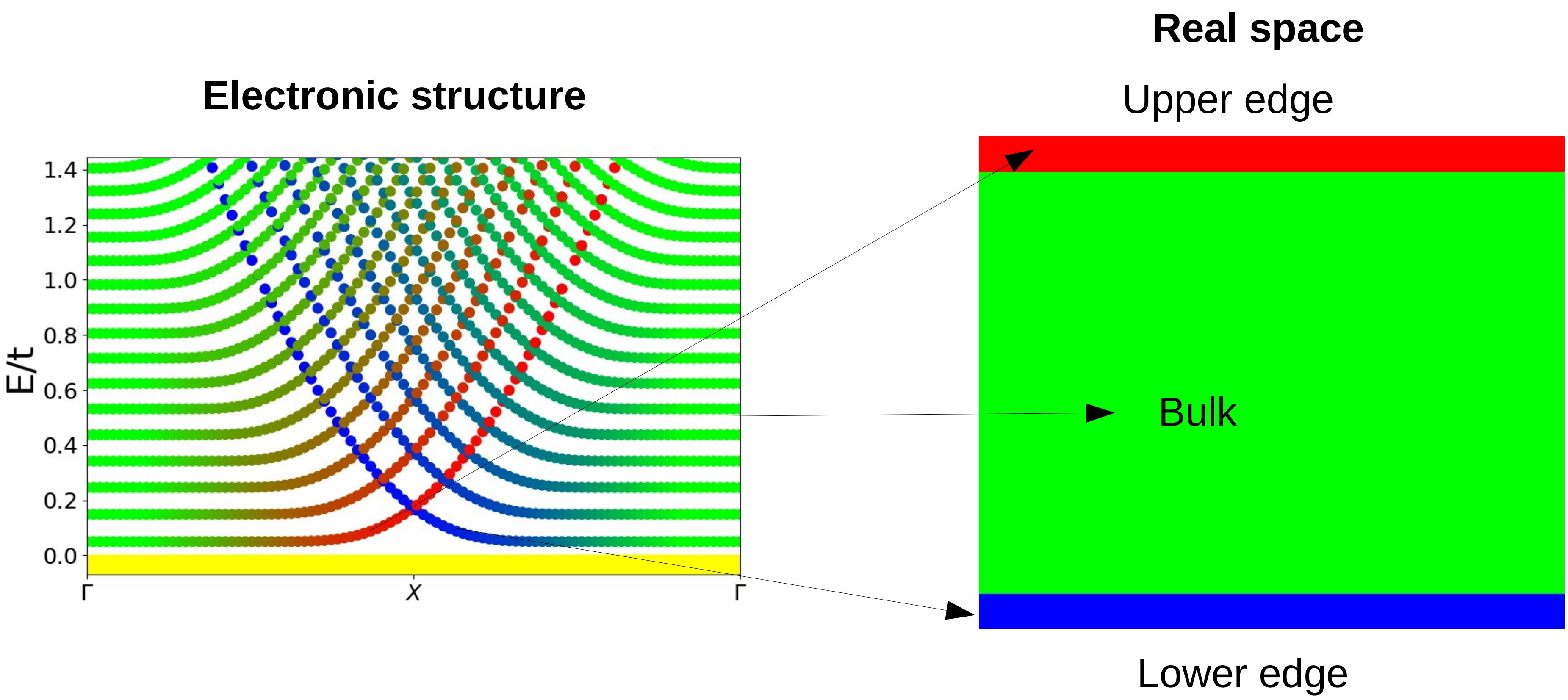
And add a magnetic field (minimal coupling) $$\mathbf{p} \to \mathbf{p} + \mathbf{A}$$

Take the Landau gauge
$$\mathbf{A} = (0, Bx, 0)$$
$$\nabla \times \mathbf{A} = (0, 0, B)$$

Plugging the magnetic potential in we get $$H \sim p_x^2 + B^2x^2$$

This looks like an
harmonic oscillator

Landau levels in a nutshell



Ladder operators for Landau levels

Let us define new operators for the full Hamiltonian

$$\widehat{a}^{\dagger} = \frac{\ell}{\sqrt{2}}\left[(\widehat{p}_x - eA_x(\widehat{\boldsymbol{r}})) + \mathrm{i}(\widehat{p}_y - eA_y(\widehat{\boldsymbol{r}}))\right]$$

"creation"

$$\widehat{a} = \frac{\ell}{\sqrt{2}}\left[(\widehat{p}_x - eA_x(\widehat{\boldsymbol{r}})) - \mathrm{i}(\widehat{p}_y - eA_y(\widehat{\boldsymbol{r}}))\right]$$

"annihilation"

$$\ell^2 = (eB)^{-1}$$

Magnetic length

How do these operators help us solve the Hamiltonian?

Ladder operators for Landau levels

Let us define new operators for the full Hamiltonian

$$\widehat{a}^{\,\dagger} = \frac{\ell}{\sqrt{2}}\left[(\widehat{p}_x - eA_x(\widehat{\boldsymbol{r}})) + \mathrm{i}(\widehat{p}_y - eA_y(\widehat{\boldsymbol{r}}))\right]$$

"creation"

$$\widehat{a} = \frac{\ell}{\sqrt{2}}\left[(\widehat{p}_x - eA_x(\widehat{\boldsymbol{r}})) - \mathrm{i}(\widehat{p}_y - eA_y(\widehat{\boldsymbol{r}}))\right]$$

"annihilation"

Canonical commutation relation

$$\widehat{a}\,\widehat{a}^{\,\dagger} - \widehat{a}^{\,\dagger}\widehat{a} = 1$$

Ladder operators for Landau levels

The Hamiltonian for the quantum Hall state

$$\widehat{H} = \omega_{\mathrm{c}} \left(\widehat{a}^{\dagger} \widehat{a} + \frac{1}{2} \right)$$

With eigenenergies

$$E_n = \omega_{\mathrm{c}}(n + 1/2),\ n = 0, 1, 2, \cdots .$$

Cyclotron frequency $\omega_{\mathrm{c}} = eB/m$

The Landau levels in the symmetric gauge

Take the symmetric gauge

$$\mathbf{A} = \frac{B}{2}(-y, x, 0) \qquad \nabla \times \mathbf{A} = (0, 0, B)$$

Landau level wavefunctions $$\Psi_m(z) \sim z^m e^{-|z|^2/(4\ell^2)}$$

$$z = x + iy \qquad \ell \sim 1/\sqrt{B}$$

This wavefunction will be our starting point for the fractional quantum Hall state

The quantum Hall effect without Landau levels

Quantum Hall effect without Landau levels

- Net magnetic field is not necessary to have QH physics
- To have a non-zero Chern number, we just need to break time-reversal symmetry

Berry curvature

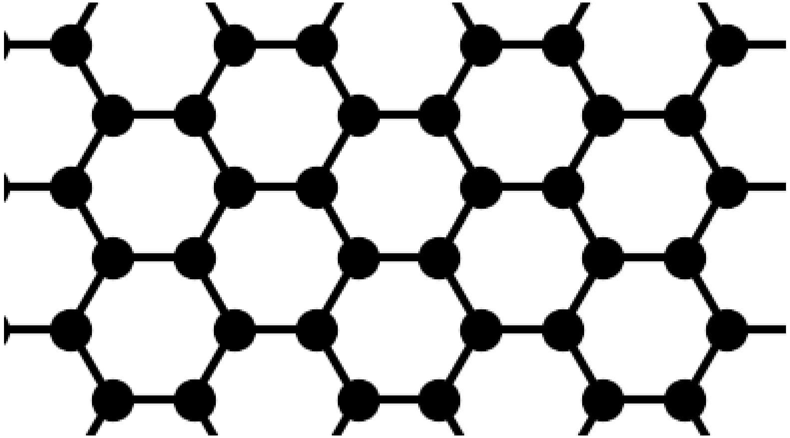
$$\Omega_\alpha = \partial_{k_x} A_y^\alpha - \partial_{k_y} A_x^\alpha$$

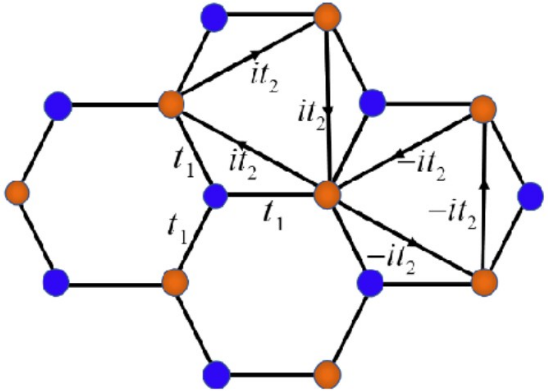
Chern number

$$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k} = \sum_\alpha C_\alpha = C$$

The Haldane model

Take a model in the honeycomb lattice and include second-neighbor hopping breaking time-reversal symmetry

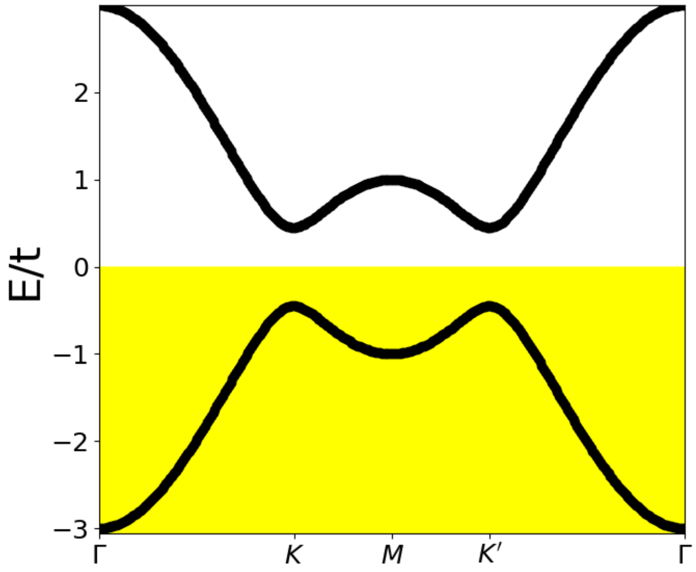




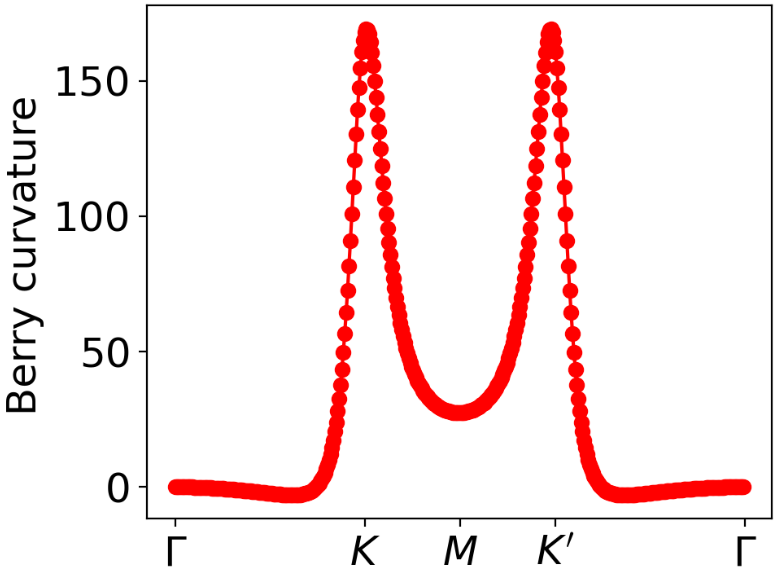
Equivalent to having a spatially dependent field that averages to zero in space

The Haldane model: bulk electronic structure

Electronic structure

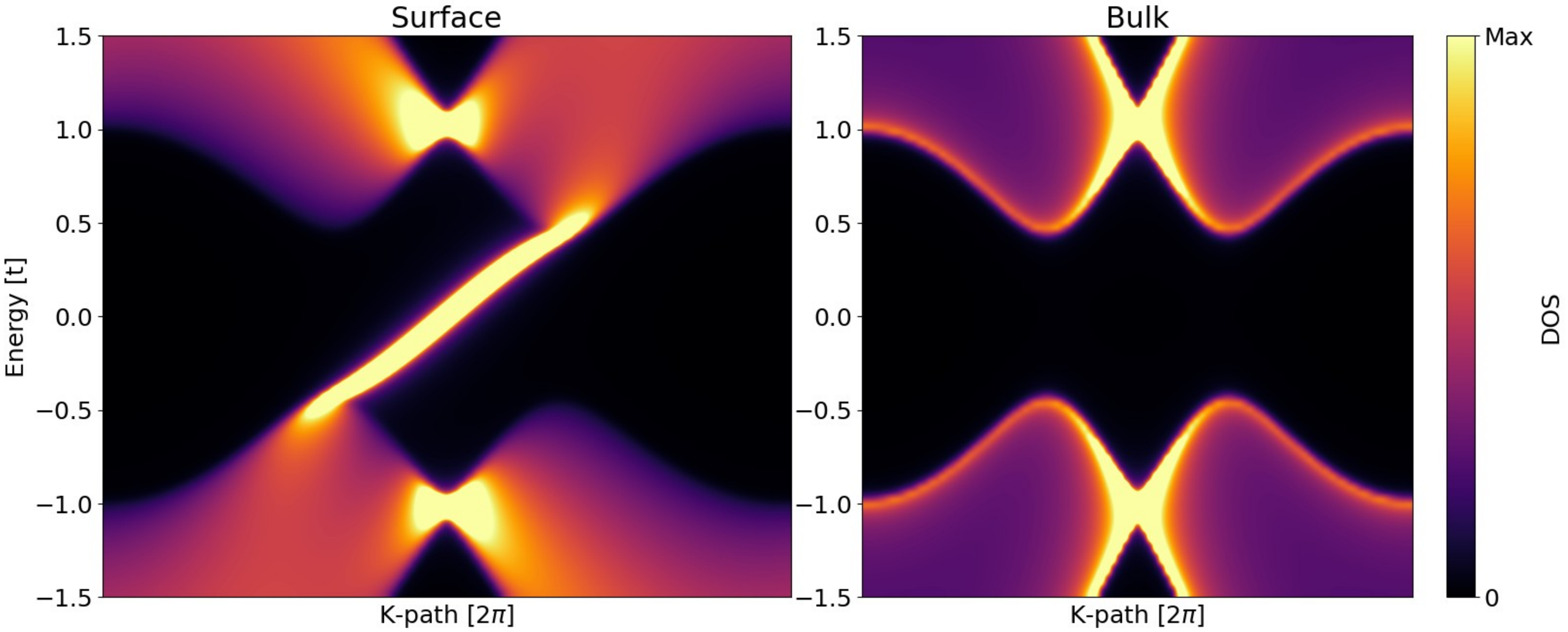


Berry curvature



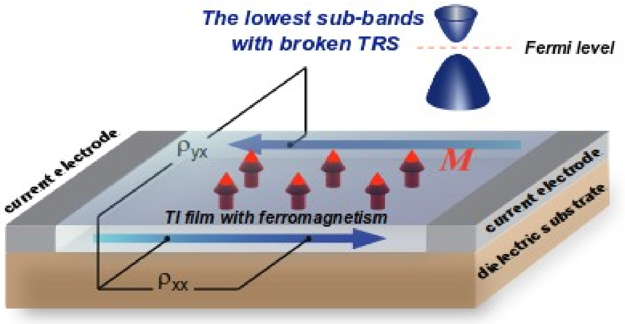
The Haldane model: edge electronic structure

Surface and bulk spectral functions



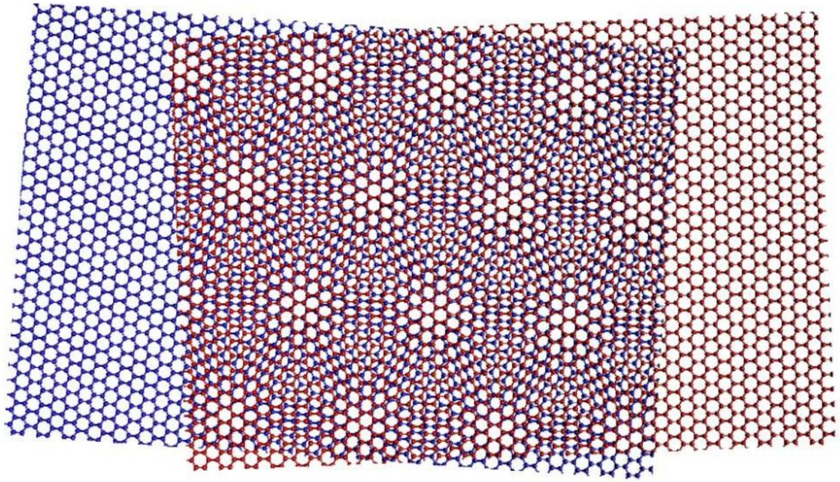
Real Chern insulating materials

Magnetically doped topological insulators



Cr-doped
$(Bi,Sb)_2Te_3$

Twisted graphene bilayers



(aligned with BN substrate)

Take home

- Quantum Hall effect can exist with and without external magnetic field
- Reading material
  - Titus Neupert notes: pages 51-56
  - Bernevig & Hughes: pages 104-107

In the next session: the fractional quantum Hall effect

<u>Lets put the Fermi energy in the 0 Landau level</u>

and assume that we have a certain filling?

Single particle wavefunction for the lowest Landau level

$$\Psi(z) \sim p(z) e^{-|z|^2/(4\ell^2)}$$

Polynomial

What is the wavefunction when we have repulsive interactions?

$$\Psi(z_1, z_2, \ldots, z_n)$$