2 BAR AND STRING MODELS

MATHEMATICAL PREREQUISITES

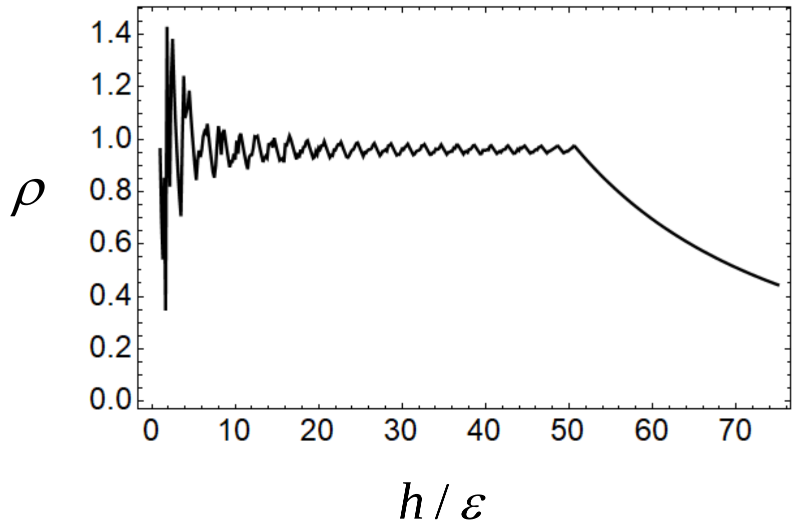
In analytical solution method, solution trial is used to transform a partial differential differentia equation into an ordinary differential equation, another solution trial is used to transform the ordinary differential equation into an algebraic equation etc.

Equation	Solution trial	Outcome
$k'\frac{\partial^2 a}{\partial x^2} - m'\frac{\partial^2 a}{\partial t^2} = 0$	$a(x,t) = A(x)e^{\mathrm{i}\omega t}$	$k'\frac{d^2 A}{dx^2} + m'\omega^2 A = 0$
$k'\frac{d^2 A}{dx^2} + m'\omega^2 A = 0$	$A(x) = ae^{\mathrm{i}\lambda x}$	$-\lambda^2 k' + m'\omega^2 = 0$

$$a(x,t) = \sum (\alpha \sin \omega_j t + \beta \cos \omega_j t)(\delta \sin \lambda_j x + \gamma \cos \lambda_j x) \quad \text{where} \quad \lambda_j = \omega_j \sqrt{\frac{m'}{k'}}$$

CONTINUUM MODEL

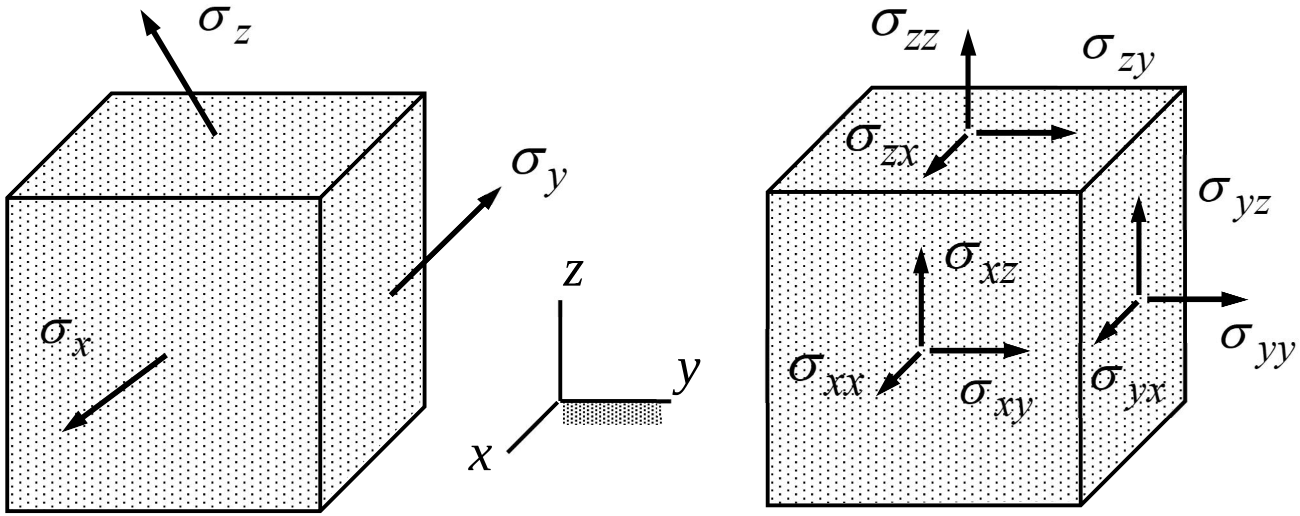
Continuum model is concerned with the average behavior of particles in a material element. The element is assumed to be small compared with the scale L of the body and large compared with the scale ε of the microstructure of the material.



Continuum model compromises modelling error with simplicity by assuming a scale $\varepsilon \ll h \ll L$ on which, e.g., material properties can be described by densities like mass per unit volume $\rho = \Delta m / \Delta V$ (mass density).

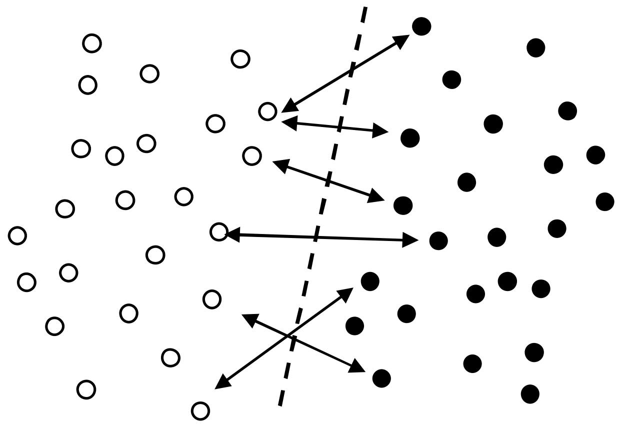
TRACTION AND STRESS

Traction $\vec{\sigma} = \Delta\vec{F} / \Delta A$ (a vector) describes the surface force between material elements of a body. Stress $\overleftrightarrow{\sigma}$ describes the surface forces acting on all edges of a material element. Traction and stress are related by $\vec{\sigma} = \vec{n} \cdot \overleftrightarrow{\sigma}$ in which $\vec{n}$ is the outward unit normal vector to face of the material element.



The first index of a stress component refers to the direction of the surface normal and the second that of the force component. On opposite edges components directions are opposite.

Traction and stress describe the internal forces. i.e., the interaction of the neighboring material elements. The white particles, belonging to certain material element, impose forces of resultant $\Delta\vec{F}$ to black particles of the neighboring element through area element ΔA .



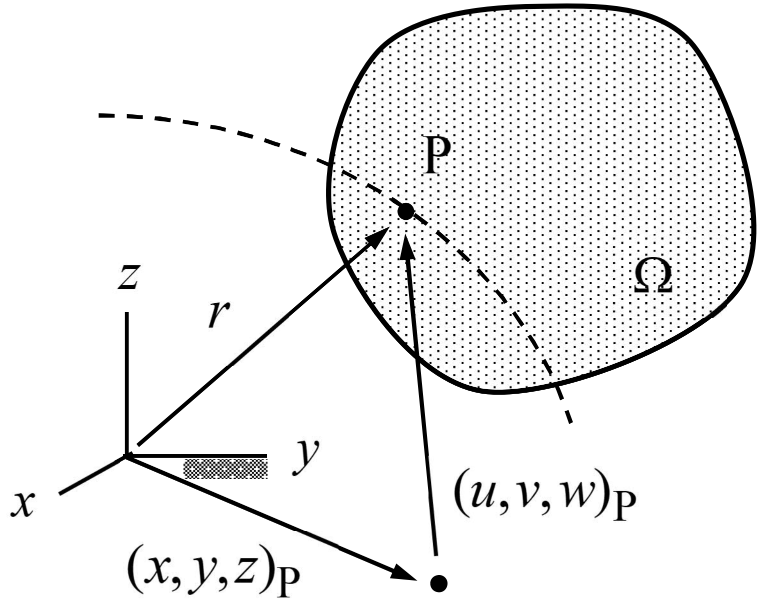
The ratio of $\Delta\vec{F}$ to area ΔA is assumed to be constant $\vec{\sigma}$, if the material element is small compared with the scale L of the body and large compared with the scale ε of the microstructure of the material so $\Delta\vec{F} = \vec{\sigma}\Delta A$. Theory assumes that the relationship holds also in form $d\vec{F} = \vec{\sigma} dA$ no matter the scale.

DISPLACEMENT

In continuum mechanics with solids, the motion of particle (x, y, z) is described by displacement components $u(x, y, z, t)$, $v(x, y, z, t)$, $w(x, y, z, t)$ in the directions of the coordinate axes relative to the initial position (x, y, z) at $t = 0$.

Non-stationary $\begin{Bmatrix} r_x \\ r_y \\ r_z \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} + \begin{Bmatrix} u(x, y, z, t) \\ v(x, y, z, t) \\ w(x, y, z, t) \end{Bmatrix}$

Stationary $\begin{Bmatrix} r_x \\ r_y \\ r_z \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} + \begin{Bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{Bmatrix}$



In stationary case, one considers only the initial and the final positions of the particles. In non-stationary case, the final position depends on parameter t (time).

FIRST PRINCIPLES

I Balance of mass Mass of a body is constant.

II Balance of linear momentum The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. ←

III Balance of angular momentum The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. ←

IV Balance of energy (Thermodynamics 1)

V Entropy growth (Thermodynamics 2)

The first principles apply in these simple forms to a closed set of particles, i.e., a set which consist of the same particles all the time.

2.1 ENGINEERINGS MODELS

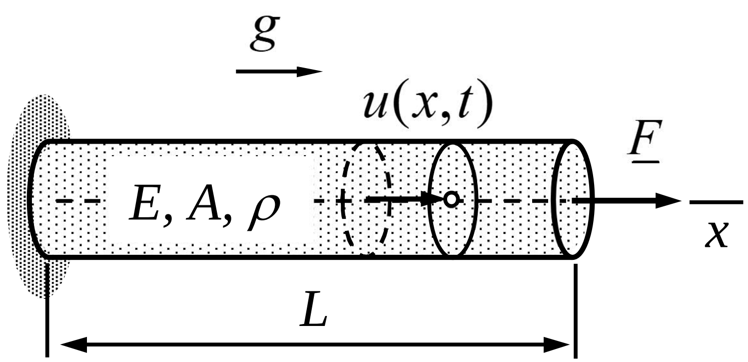
BAR is a body which is thin in two dimensions and has straight initial geometry. Displacement has only the axial component. Internal force is aligned with the axis.

STRING is a body which is thin in two dimensions and has straight initial geometry. Displacement has only the transverse component. Internal force is aligned with the axis (initial and deformed geometries).

THIN SLAB is a body which is thin in one dimension and has planar initial geometry. Displacement has only the mid-plane components. Internal force does not have transverse component.

MEMBRANE is a body which is very thin in one dimension and has planar initial geometry. Displacement has only the transverse component. Internal force does not have transverse component (initial and deformed geometries).

BAR MODEL

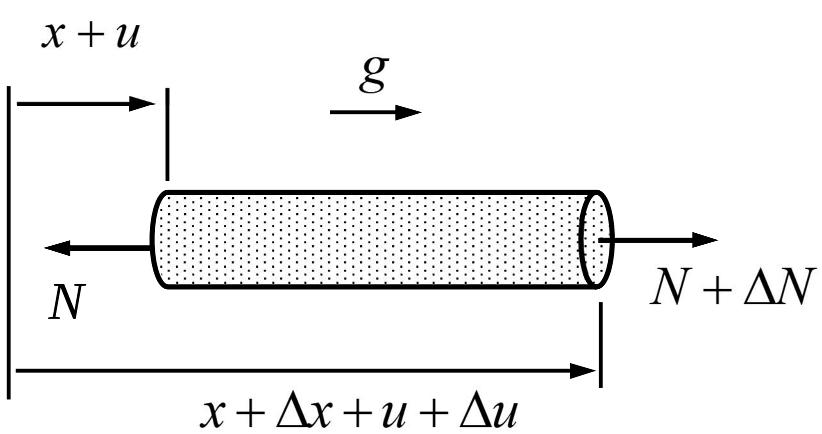


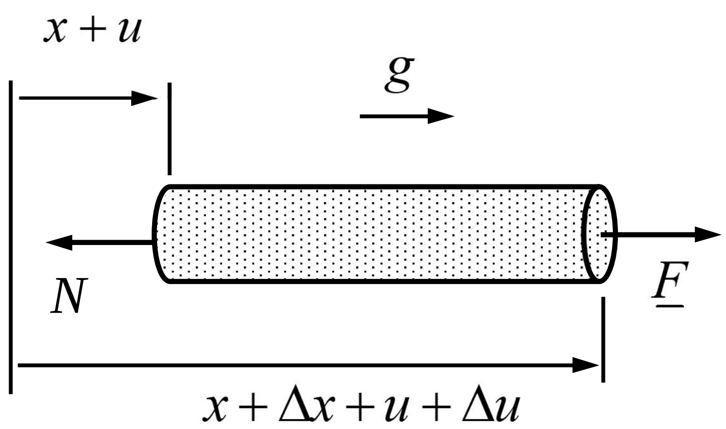
Equation of motion $$EA\frac{\partial^2 u}{\partial x^2} + \rho A g = \rho A \frac{\partial^2 u}{\partial t^2} \qquad x \in]0, L[\quad t > 0,$$

Boundary conditions $$u = \underline{u} \quad \text{or} \quad EA\frac{\partial u}{\partial x} = \underline{F} \qquad x \in \{0, L\} \quad t > 0,$$

Initial conditions $$u = g(x) \quad \text{and} \quad \frac{\partial u}{\partial t} = h(x) \qquad x \in [0, L] \quad t = 0.$$

Let us apply the first principles to a material element of initial length Δx at the initial and final geometries.





The cases where the material element is inside the bar and located at the free boundary differ.

Mass balance: $\Delta m = (\rho A)^{\circ} \Delta x = (\rho A)(\Delta x + \Delta u)$

Momentum balance $\rightarrow$: $N + \Delta N - N + g\Delta m = \Delta m \dfrac{\partial^2 u}{\partial t^2}$

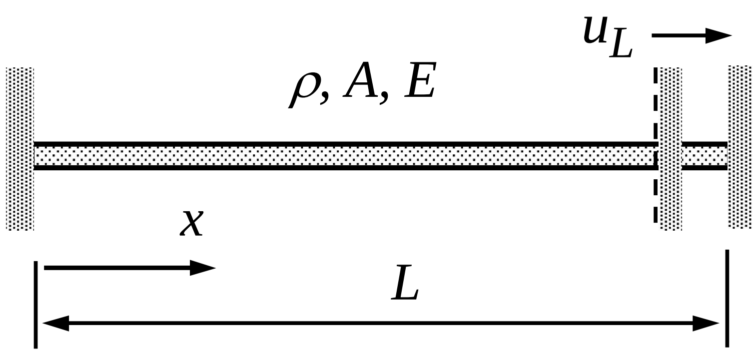
Momentum balance $\rightarrow$: $F - N + g\Delta m = \Delta m \dfrac{\partial^2 u}{\partial t^2}$

The limit model assumes that $\Delta N / \Delta x$ exists also when $\Delta x \rightarrow 0$. In case of a discontinuity, like a point force P at x_0, one obtains the "jump" condition $[\![N]\!] + P = 0$ where $[\![a]\!] = \lim_{\varepsilon \rightarrow 0}[a(x_0 + \varepsilon) - a(x_0 - \varepsilon)]$.

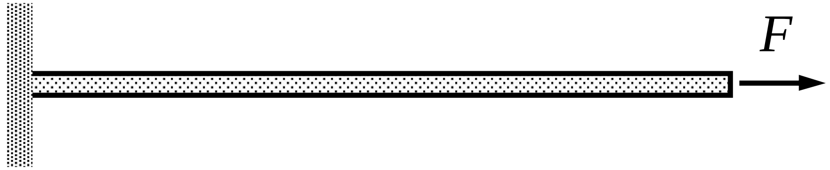
Hooke's law: $\sigma = E \dfrac{\Delta u}{\Delta x} \;\Rightarrow\; N = EA \dfrac{\Delta u}{\Delta x}$

relates the length change and force acting on a material element. Material model is the weakest element of the model as it lacks the generality of the first principles and brings the major portion of the modelling error!

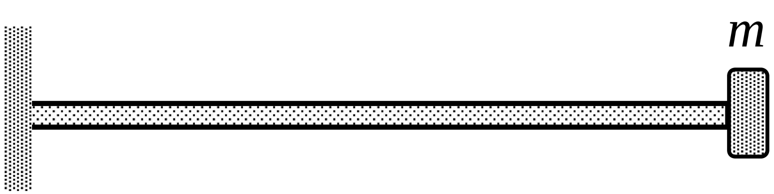
BOUNDARY CONDITIONS



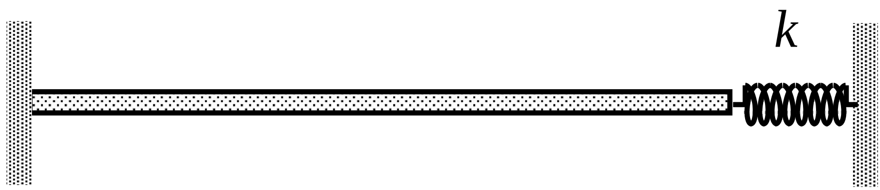
$$u(0,t) = u(L,t) - u_L(t) = 0$$



$$u(0,t) = EA\frac{\partial}{\partial x}u(L,t) - F(t) = 0$$

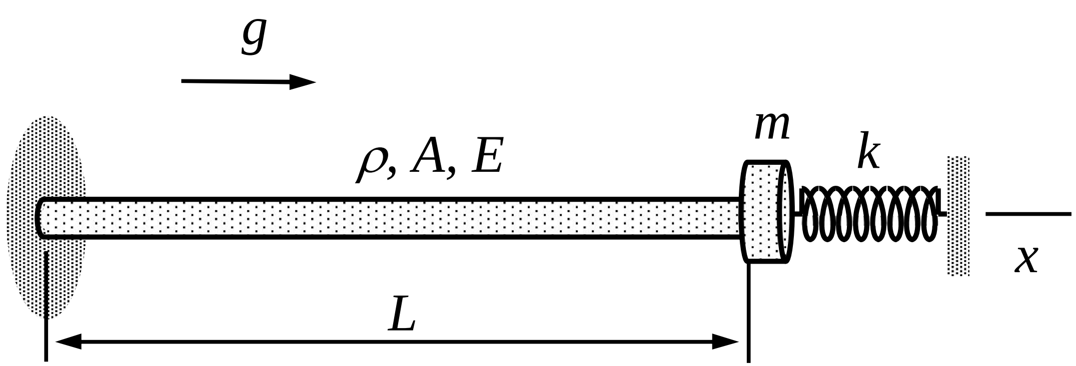


$$u(0,t) = EA\frac{\partial}{\partial x}u(L,t) + m\frac{\partial^2}{\partial t^2}u(L,t) = 0$$



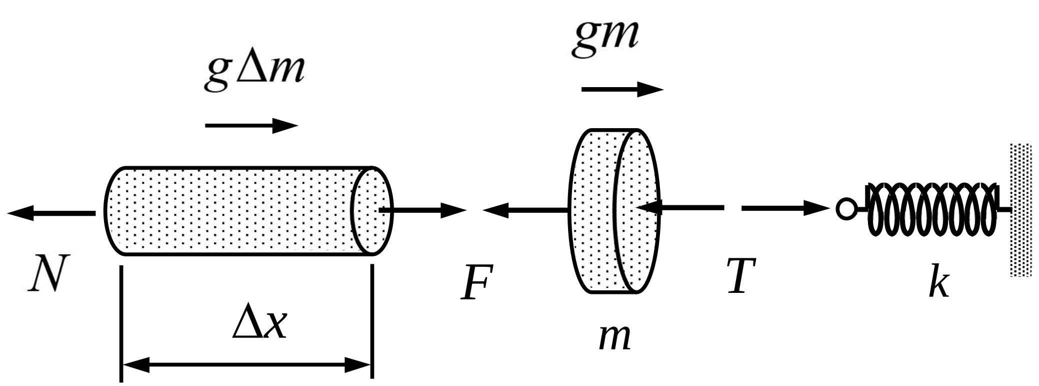
$$u(0,t) = EA\frac{\partial}{\partial x}u(L,t) + ku(L,t) = 0$$

EXAMPLE Derive the boundary condition for the case shown starting from the first principles. The particle at the right end is connected to rigid wall with a spring. Spring force vanishes when displacement at the end $u(L,t)=0$.



Answer $EA\frac{\partial u}{\partial x}+ku-mg+m\frac{\partial^2 u}{\partial t^2}=0 \quad x=L \quad t>0$

Let us apply the balance of momentum to a material element using the free body diagrams of the bar element and particle

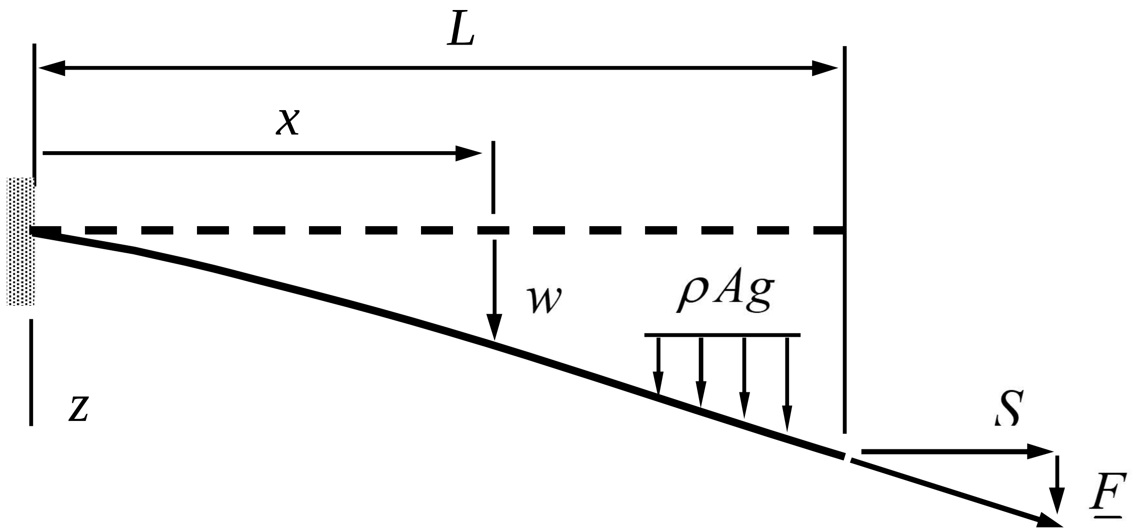


$$-N + \rho A g \Delta x + F = \rho A \Delta x \frac{\partial^2 u}{\partial t^2}\,, \qquad -F - T + mg = m\frac{\partial^2 u}{\partial t^2}$$

Eliminating the internal force F, using $N = EA\partial u / \partial x$ and $T = ku$, and considering ja the limit $\Delta x \to 0$

$$-EA\frac{\partial u}{\partial x} - ku + mg - m\frac{\partial^2 u}{\partial t^2} = 0 \quad x = L \quad t > 0.$$ ➔

STRING MODEL

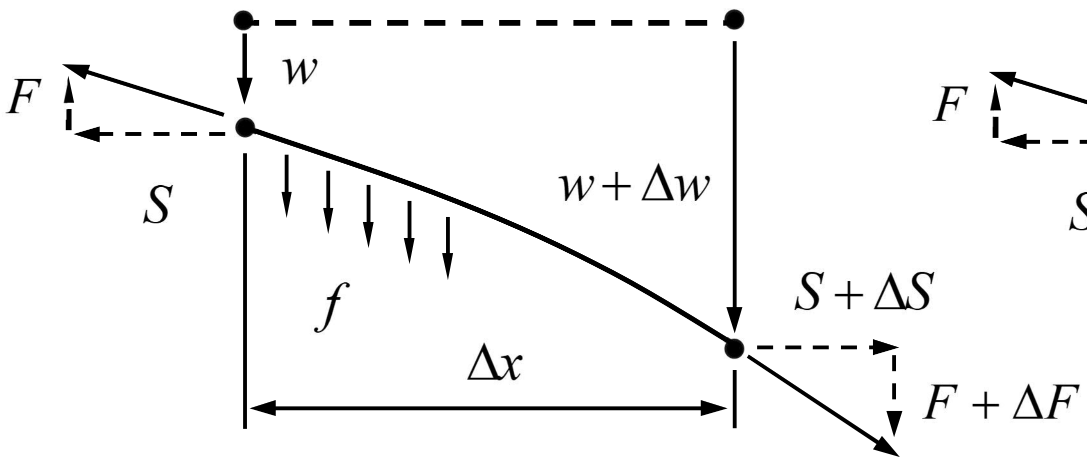


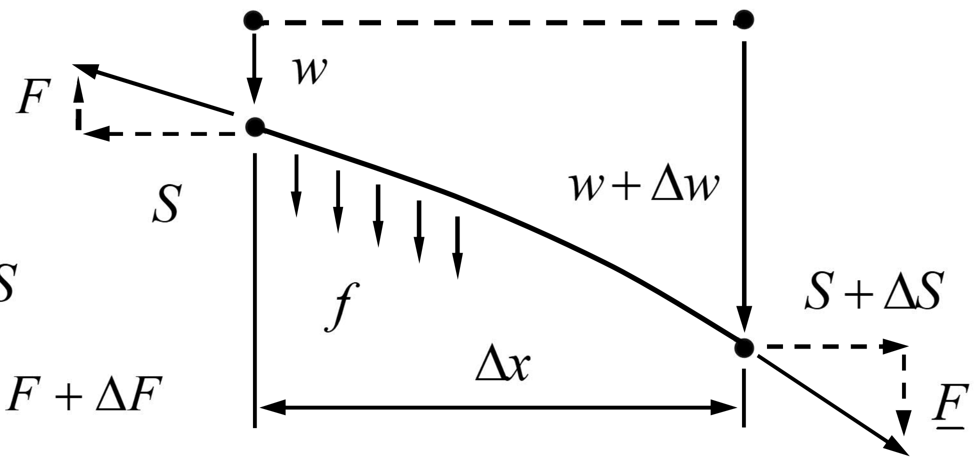
Equation of motion $S\dfrac{\partial^2 w}{\partial x^2} + \rho A g = \rho A \dfrac{\partial^2 w}{\partial t^2} \quad x \in]0, L[\quad t > 0$

Boundary conditions $w = \underline{w}$ or $S\dfrac{\partial w}{\partial x} - \underline{F} = 0 \quad x \in \{0, L\} \quad t > 0$

Initial conditions $w = g(x)$ and $\dfrac{\partial w}{\partial t} = h(x) \quad x \in]0, L[\quad t = 0$

Let us apply principles **I** and **II** to an originally straight material element $[x, x+\Delta x]$ of initial length Δx by assuming a planar problem and only the transverse displacement $w(x,t)$.





Mass balance: $\Delta m = \Delta x(\rho A)^{\circ} = (\rho A)\Delta x\sqrt{1+(\Delta w/\Delta x)^2}$

Momentum balance $\rightarrow$: $S + \Delta S - S = 0$

Momentum balance $\downarrow$: $F+\Delta F-F+f^{\circ}\Delta x=(\rho A)^{\circ}\Delta x\dfrac{\partial^2 w}{\partial t^2}$

Momentum balance $\downarrow$: $-F-\underline{F}+f^{\circ}\Delta x=(\rho A)^{\circ}\Delta x\dfrac{\partial^2 w}{\partial t^2}$

The constitutive equations is also required. The momentum balance in the horizontal direction implies that tightening S of string is constant. Therefore, according to the figure

$$\frac{F}{S}=\frac{\partial w}{\partial x} \Leftrightarrow F=S\frac{\partial w}{\partial x} \quad \text{and} \quad \frac{F+\Delta F}{S}=\frac{\partial w}{\partial x}+\frac{\partial^2 w}{\partial x^2}\Delta x \Leftrightarrow F+\Delta F=S(\frac{\partial w}{\partial x}+\frac{\partial^2 w}{\partial x^2}\Delta x)$$

The second order limit problem $\Delta x \to 0$ assumes continuous derivatives at all the points inside the domain. In case of a vertical point force P acting on the string, the limit equation for the point of action becomes $[\![F]\!]+P=0$.

BOUNDARY-INITIAL VALUE PROBLEM

In their mathematical forms, the continuum models for string and bar coincide. Denoting the displacement by $a(x,t)$, the boundary-initial value problems are given by equations

Differential equation: $k'\dfrac{\partial^2 a}{\partial x^2}+f'=m'\dfrac{\partial^2 a}{\partial t^2} \quad x\in\Omega \quad t>0$

Boundary conditions: $a=\underline{a}$ or $n(k'\dfrac{\partial a}{\partial x})-\underline{F}=0 \quad x\in\partial\Omega \quad t>0$

Initial conditions: $a=g(x)$ and $\dfrac{\partial a}{\partial t}=h(x) \quad x\in\Omega \quad t=0$

In a boundary condition, one may specify displacement or force but not both simultaneously. For a unique solution in stationary case, the displacement needs to be specified at least one point. In the force boundary condition, $n=\pm 1$ depending on the boundary point.

2.2 DISPLACEMENT ANALYSIS

In the displacement problem, one assumed that all quantities are independent in time so derivatives with respect to time vanish and partial derivatives with respct to the spatiol coordinate become ordinary one. Assuming that point forces, if any, act on I

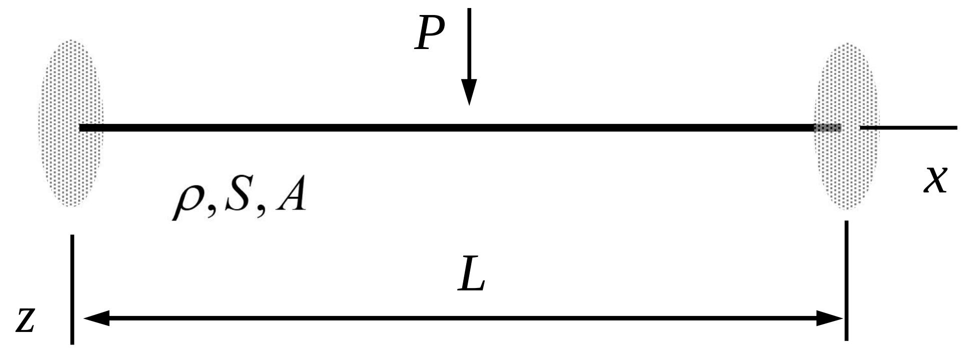
Differential equation $k'\frac{d^2a}{dx^2} + f' = 0 \quad x \in \Omega \setminus I$

Continuity conditions $k'\left[\!\left[\frac{da}{dx}\right]\!\right] + P = 0$ and $[\![a]\!] = 0 \quad x \in I$

Boundary conditions $a = \underline{a}$ or $n(k'\frac{da}{dx}) - \underline{F} = 0 \quad x \in \partial\Omega$

At the boundary, one may specify the force or displacement but not both. Also, displacement should be specified at one point for an unique solution.

EXAMPLE A string of length L, tightening S, cross-sectional area A, and density ρ, is loaded by a point force P at its center point. If the ands are fixed and the initial geometry without loading is straight, find the solution to the transverse displacement as function of x using the continuum model.



Answer $$w(x) = \begin{cases} \dfrac{P}{2S}x & 0 \le x < \dfrac{L}{2} \\ \dfrac{P}{2S}(L-x) & \dfrac{L}{2} < x \le L \end{cases}$$

The boundary value problem is given by equilibrium equations for the interior points, continuity conditions at the center point, and boundary conditions for the end points

$$S\frac{d^2w}{dx^2}=0 \quad x\in]0,\frac{L}{2}[\quad \text{or} \quad x\in]\frac{L}{2},L[,$$

$$S\left[\!\left[\frac{dw}{dx}\right]\!\right]+P=0\ ,\ [\![w]\!]=0 \quad x=\frac{L}{2}, \qquad \text{and} \qquad w(x)=0 \quad x\in\{0,L\}.$$

The generic solution to the differential equation is the same form $w=a+bx$ in both parts of the domain with different integration constants. Using the remaining conditions

$$w(x)=\begin{cases} a_1+b_1x & 0<x<\frac{L}{2} \\ a_2+b_2x & \frac{L}{2}<x<L \end{cases} \quad \Rightarrow \quad w(x)=\begin{cases} \frac{P}{2S}x & 0\le x<\frac{L}{2} \\ \frac{P}{2S}(L-x) & \frac{L}{2}<x\le L \end{cases}.$$

DISPLACEMENT SOLUTION

Displacement calculation with the bar or string model means solving a boundary value problem of ordinary linear second order differential equation. In the present context, coefficient are constants so finding the displacement is straightforward.

(a) Find the generic solution to the differential equation by repetitive integrations on both sides. Each integration creates a new integration constant. In case of point forces etc. apply the integration separately in subdomains.

(b) Use the boundary conditions conerning given displacement and forces on the boundaries to find the values of two integration constants. In case of point forces etc. glue the solutions to the subdomains together with continuity of displacement and the condition implied by momentum equilibrium written for, e.g., the point of action of a point force.

2.3 VIBRATION ANALYSIS

The equations describing vibration consist of equation of motion, boundary conditions for displacement or force, and initial conditions for the displacement and velocity

Equation of motion $k'\frac{\partial^2 a}{\partial x^2} + f' = m'\frac{\partial^2 a}{\partial t^2} \quad x \in \Omega \quad t > 0$

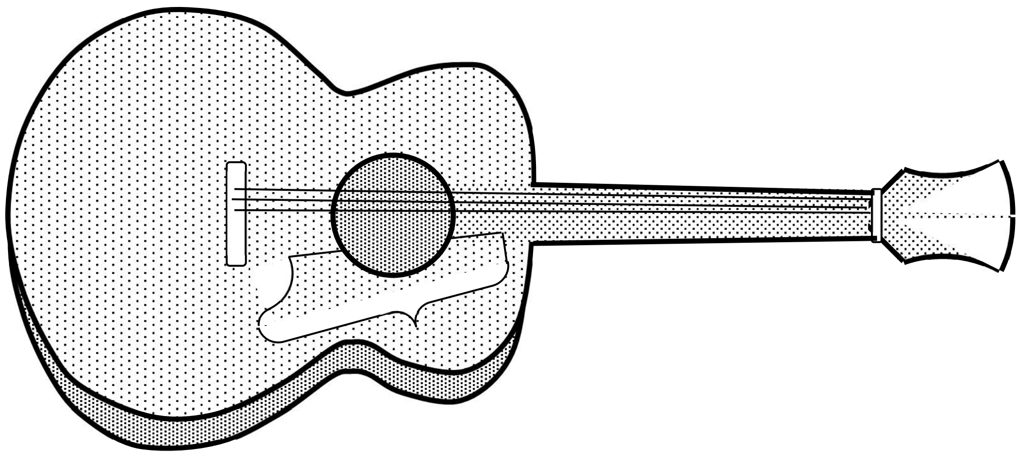
Boundary conditions $a = \underline{a}$ or $n(k'\frac{\partial a}{\partial x}) - \underline{F} = 0 \quad x \in \partial\Omega \quad t > 0$

Initial conditions $a = g(x)$ and $\frac{\partial a}{\partial t} = h(x) \quad x \in \Omega \quad t = 0$

Initial and boundary conditions for displacement should match for a regular solution of a vibration problem (but not always, like when a bar moving with a constant velocity hits a rigid wall). In vibration problems, the boundary conditions are homogeneous so $\underline{a} = \underline{F} = 0$.

PYTHAGORAS' DISCOVERY

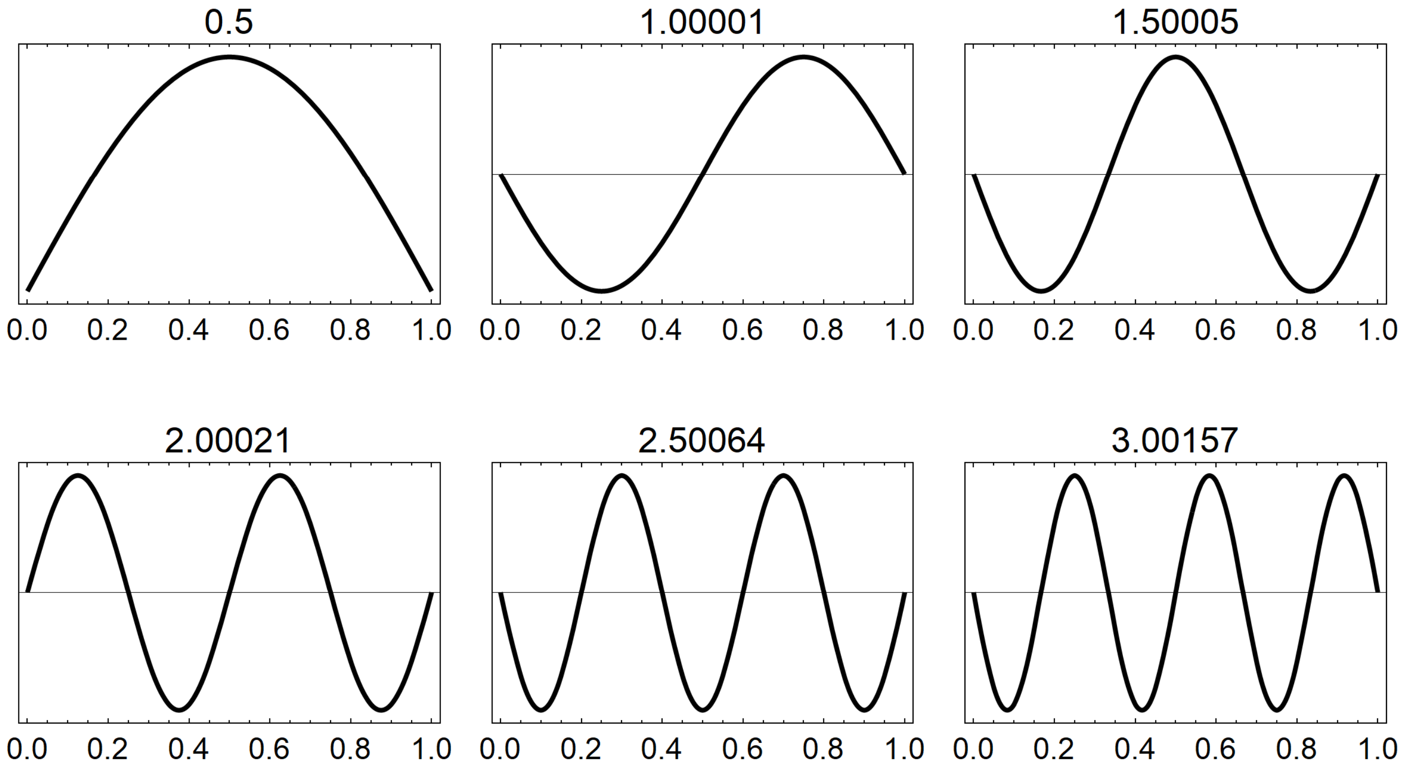
According to the experiments by Pythagoras, pitch of a note f depends on the length of string L, stress σ and density ρ of material!



String model $f_{\min} = \frac{1}{2L}\sqrt{\frac{S}{\rho A}} = \frac{1}{2L}\sqrt{\frac{\sigma}{\rho}}$

VIBRATION MODES OF STRING

Displacement of string in vibration can be represented as sum of harmonic modes each vibrating in its own frequency f [1/s]. For a string of fixed ends, the frequency-mode pairs are (dimensionless value $Lf / \sqrt{S / \rho A}$)



MODAL ANALYSIS

In vibration analysis of structures, one is usually interested in the frequencies f ($\omega = 2\pi f$) or angular velocity-mode pairs (ω, A) for the free vibrations. Using the generic form valid for the bar and string models, the modes and angular velocities are related by

(a) $A(x) = \delta \sin(\lambda x) + \gamma \cos(\lambda x)$ and $\lambda = \omega \sqrt{\dfrac{m'}{k'}}$

The (assumedly homogeneous) boundary conditions at the ends

(b) $A(x) = 0$ or $n(k' \dfrac{dA}{dx}) = 0 \quad x \in \partial\Omega$

give 2 conditions for the 3 parameters δ, γ, λ and, thereby, the possible pairs $(\omega, A)_j$. Notice that the modes are determined up to an arbitrary scaling only (3 parameters and 2 equations).

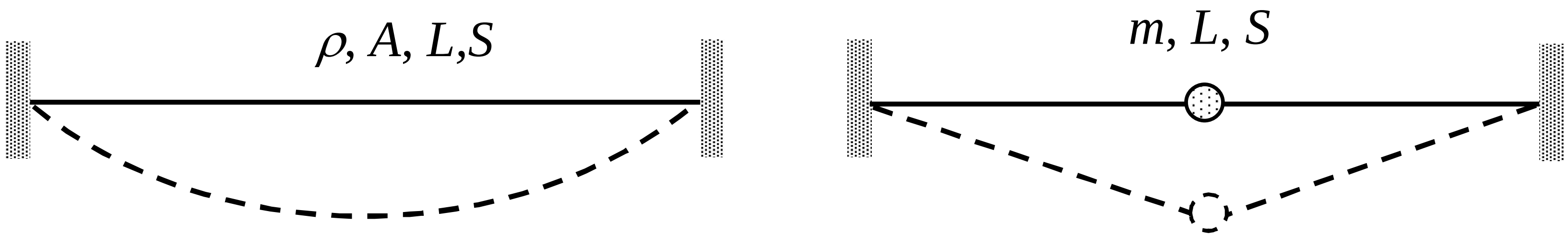
In mode detail, the possible angular velocity-mode pairs (ω, A) follow from displacement assumption $a(x,t) = A(x)e^{\mathrm{i}\omega t}$ which transforms the partial differential equation into an ordinary one. Using the generic form for both model problems, the steps to the generic form of the modes

$$k' \frac{\partial^2 a}{\partial x^2} = m' \frac{\partial^2 a}{\partial t^2} \quad \Rightarrow \quad \frac{d^2 A}{dx^2} + \omega^2 \frac{k'}{m'} A = 0.$$

The generic solution to the angular velocity-mode pairs (ω, A) is therefore given by

$$A(x) = \delta \sin(\lambda x) + \gamma \cos(\lambda x) \text{ where } \lambda = \omega \sqrt{\frac{k'}{m'}}. \quad \leftarrow$$

EXAMPLE What is relative error if the smallest frequency of the free vibrations f_{min} is calculated with a particle surrogate model of the figure ($\bar{f}_{\text{min}}$), where the mass of the particle is half of that of the string?



Answer $e = \dfrac{\bar{f}_{\text{min}} - f_{\text{min}}}{f_{\text{min}}} \times 100\% = (\dfrac{2\sqrt{2}}{\pi} - 1) \times 100\% \approx -10\%$

Let us start with the continuum model considered as the precise model. In string equations, $k' = S$, $m' = \rho A$, and $a = w(x,t)$. External force and displacement at the boundaries vanish and $\Omega =]0, L[$. The possible angular velocity-mode pairs follow from the generic solution

$$A(x) = \delta \sin(\lambda x) + \gamma \cos(\lambda x) \text{ and } \lambda = \omega \sqrt{\rho A / S}.$$

Boundary conditions $A(0) = A(L) = 0$ give 2 conditions for the 3 parameters γ, δ, λ

$$A(0) = \gamma = 0 \text{ and } A(L) = \delta \sin(\lambda L).$$

A non-zero solution is possible only if $\sin(\lambda L) = 0$ so $\lambda L = \pi j \quad j \in \{1, 2, \ldots\}$. The smallest angular velocity, and therefore the frequency, follows with selection $j = 1$ so

$$f_j = \frac{\omega_j}{2\pi} = \frac{1}{L}\sqrt{\frac{S}{\rho A}}\frac{1}{2} j \quad \Rightarrow \quad f_{\min} = \frac{1}{L}\sqrt{\frac{S}{\rho A}}\frac{1}{2}.$$ ←

Particle surrogate model for string with $i = \{0,1,2\}$ and fixed particles at the end points gives the equation of motion:

$$2\frac{S}{h}w_1 + \rho A h \frac{d^2 w_1}{dt^2} = 0 \iff \frac{d^2 w_1}{dt^2} + \omega_e^2 w_1 = 0 \quad \text{where} \quad \omega = 2\pi f = \sqrt{2\frac{S}{\rho A h^2}}.$$

As $h = L/2$, the prediction by PSM $\bar{f}_{\min} = \frac{\sqrt{2}}{\pi L}\sqrt{\frac{S}{\rho A}}$. ←

MODE SUPERPOSITION

If the initial conditions concerning position and displacement of the particles are known (quite exceptional case in practice), the outcome of the modal analysis $(\omega, A)_j$ can be used to construct a displacement solution for the given initial data starting with the series

(a) $$a(x,t) = \sum A_j(x)[\alpha_j \frac{1}{\omega_j} \sin(\omega_j t) + \beta_j \cos(\omega_j t)].$$

The combination of the modes giving $a = g(x)$ and $\partial a / \partial t = h(x)$ at $t = 0$ follow with the coefficient expression

(b) $$\alpha_j = \frac{1}{\mathrm{A}_j^2} \int_\Omega A_j(x) h dx \text{ and } \beta_j = \frac{1}{\mathrm{A}_j^2} \int_\Omega A_j(x) g dx \text{ where } \mathrm{A}_j^2 = \int_\Omega A_j(x) A_j(x) dx.$$

The coefficients correspond to the spatial Fourier series of the initial data obtained with the orthogonal harmonic modes from the modal analysis.

The series representation follows directly from the displacement assumption written with the polar representation $e^{\mathrm{i}\alpha} = \cos\alpha + \mathrm{i}\cos\alpha$ and real valued coefficients

$$a(x,t) = \sum A_j(x)e^{\mathrm{i}\omega_j t} = \sum A_j(x)[\alpha_j \frac{1}{\omega_j}\sin(\omega_j t) + \beta_j \cos(\omega_j t)].$$

Initial conditions imply that $\sum A_j(x)\beta_j = g(x)$ and $\sum A_j(x)\alpha_j = g(x)$. Multiplying both sides with $A_l(x)$, integrating over the domain, and using the orthogonality gives

$$\alpha_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x)h(x)dx, \quad \beta_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x)g(x)dx, \quad \mathrm{A}_j^2 = \int_\Omega A_j(x)A_j(x)dx. \text{ ⇦}$$

FOURIER SERIES

The Fourier series (various forms exist) can be used to represent a function as the sum of harmonic terms. For example, the sine-transformation pair for a function $a(x)$ $x \in [0, L]$ with vanishing values at the end points is given by

$$\alpha_j = \frac{2}{L}\int_0^L \sin(j\pi\frac{x}{L})a(x)dx \quad j \in \{1, 2, \ldots\} \quad \Leftrightarrow \quad a(x) = \sum_{j\in\{1,2,\ldots\}} \alpha_j \sin(j\pi\frac{x}{L}).$$

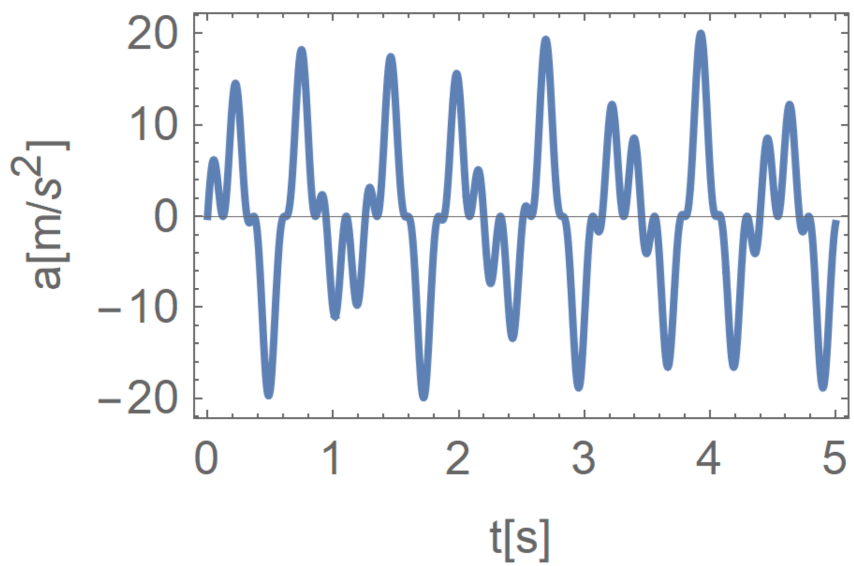
The transformation pair is based on the orthogonality of the modes

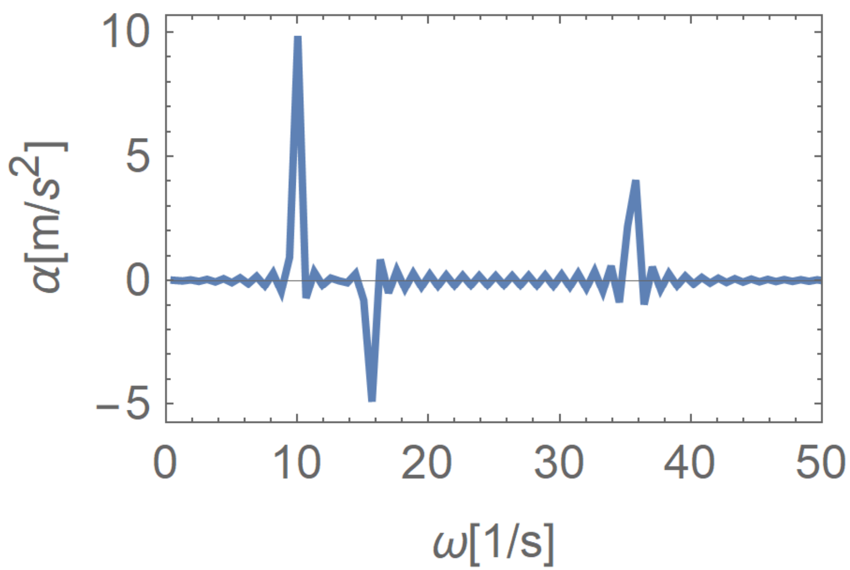
$$\int_0^L \sin(j\pi\frac{x}{L})\sin(l\pi\frac{x}{L})dx = \frac{L}{2}\delta_{jl} \text{ (Kronecker delta).}$$

The transformation (with respect to time) can be used to analyze frequency contents of data, filtering, to find the combination of the terms of the generic series solution for bar and string models satisfying the initial conditions, etc.

FREQUENCY CONTENTS OF DATA

Fourier transform (with respect to time) can be used, e.g., to analyze frequency contents of data, filtering of data etc. As an example, transform (right) of the measured acceleration (left), imply $a(t) = 10\sin(10t) - 5\sin(15.6t) + 5\sin(35.6t)$ (in appropriate units)

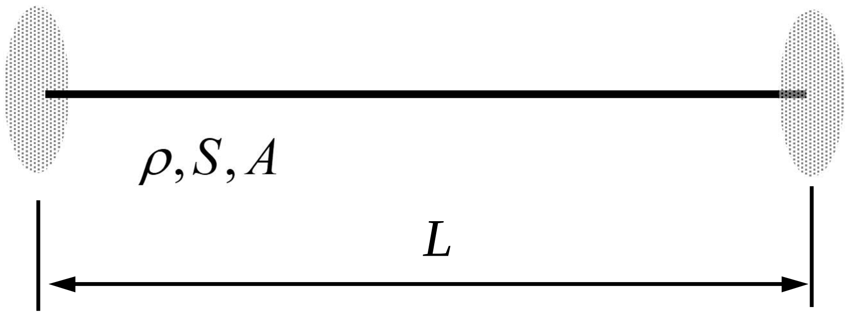




In filtering, one may just omit, e.g., components having frequencies over some value or maybe components of amplitudes of small values depending on the application.

EXAMPLE Find the transverse displacement of the string shown as function of time. The initial displacement is sinusoidal $g = W\sin(\pi x / L)$ (W is constant) and the initial speed vanishes, i.e., $h = 0$. Assume that the string is tightened with S in the horizontal direction and mass per unit length ρA is constant. Use the outcome of the modal analysis for a string fixed at both ends $A_j(x) = \sin(\lambda_j x)$ and $\omega_j = \lambda_j \sqrt{S/(\rho A)}$, where $\lambda_j = \pi j / L \quad j \in \{1, 2, \ldots\}$

.



Answer $w(x,t) = W\cos(\frac{\pi}{L}\sqrt{\frac{S}{\rho A}}t)\sin(\frac{\pi}{L}x)$

After modal analysis, one may superpose the modes to find a particular combination satisfying the initial conditions. According to the recipe

$$w(x,t) = A_j(x)[\alpha_j \frac{1}{\omega_j}\sin(\omega_j t) + \beta_j \cos(\omega_j t)]$$

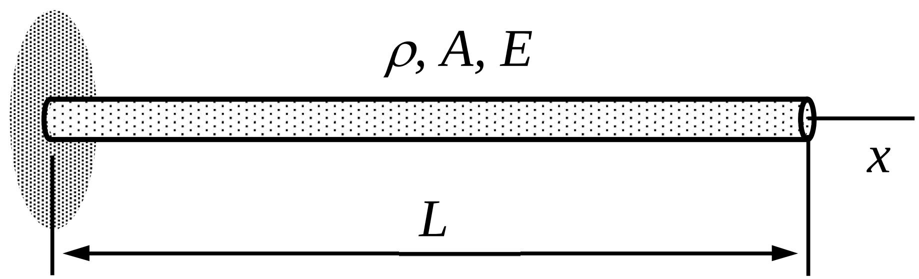
where

$$\alpha_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x) h dx \quad \text{and} \quad \beta_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x) g dx \text{ where } \mathrm{A}_j^2 = \int_\Omega A_j(x) A_j(x) dx$$

The initial data $g(x) = W\sin(\pi x / L)$, $h(x) = 0$ and modes $A_j(x) = \sin(j\pi x / L)$ give $\mathrm{A}_j^2 = L/2$, $\alpha_j = 0$, and $\beta_j = W\delta_{1j}$ $j \in \{1, 2, \ldots\}$ and, thereby, the solution

$$w(x,t) = W\sin(\pi \frac{x}{L})\cos(\frac{\pi}{L}\sqrt{\frac{S}{\rho A}}t).$$ ←

EXAMPLE Find the axial displacement of the bar shown as the functions of time t and position x, if the initial displacement and velocity at $t = 0$ are given by $g(x) = U\sin(\pi x / 2L)$ and $h(x) = 0$, respectively.



Answer $u(x,t) = U\cos(\sqrt{\frac{E}{\rho}}\frac{\pi}{2L}t)\sin\frac{\pi x}{2L}$

Let us start with the modal analysis. According to the recipe, the modes are given by

$$A(x) = \delta \sin(\lambda x) + \gamma \cos(\lambda x),$$

where the proper combination of parameters δ, γ, λ follow from the boundary conditions. In the present case

$$A(0) = \gamma = 0 \quad \text{and} \quad (\frac{dA}{dx})_{x=L} = \delta\lambda \cos(\lambda L) - \gamma\lambda \sin(\lambda L) = 0.$$

The angular velocity-mode pairs (bar model $m' = \rho A$ and $k' = EA$) implied by the boundary conditions ($\cos(\lambda L) = 0$) are

$$A_j(x) = \sin(\lambda_j x) \quad \text{and} \quad \omega_j = \lambda_j \sqrt{\frac{E}{\rho}} \quad \text{where} \quad \lambda_j L = \frac{\pi}{2}(1 + 2j) \quad j \in \{0,1,2,\ldots\}.$$

Knowing the angular velocity-mode pairs, the particular combination satisfying the initial conditions follows from

$$u(x,t) = \sum_{j\in\{0,1,2,\ldots\}} A_j(x)[\alpha_j \frac{1}{\omega_j}\sin(\omega_j t) + \beta_j \cos(\omega_j t)]$$

where

$$\alpha_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x) h dx, \ \ \beta_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x) g dx \ \text{ and } \ \mathrm{A}_j^2 = \int_\Omega A_j(x) A_j(x) dx.$$

The initial data $g(x) = U\sin(\pi x/2L)$, $h(x) = 0$ give the coefficients $\alpha_j = 0$ and $\beta_j = W\delta_{0j} \quad j \in \{0,1,2,\ldots\}$ and, thereby, the solution

$$u(x,t) = U\cos(\sqrt{\frac{E}{\rho}}\frac{\pi}{2L}t)\sin\frac{\pi x}{2L}.$$ ←

2.4 PRINCIPLE OF VIRTUAL WORK

Principle of virtual work for particle and continuum models are just concise representations of equilibrium equations or equations-of-motion and boundary conditions of the particle and continuum models.

Virtual work	**Particle**	**Continuum**
δW^{int}	$-\sum_{e\in P}\left(\frac{\Delta\delta u_e}{\Delta x}EA\frac{\Delta u_e}{\Delta x}\right)h$	$-\int_\Omega\left(\frac{\partial\delta u}{\partial x}EA\frac{\partial u}{\partial x}\right)dx$
δW^{ext}	$\sum_{i\in I}(\delta u_i f)h$	$\int_\Omega(\delta u f)dx$
δW^{ine}	$-\sum_{i\in I}\left(\delta u_i\rho A\frac{\partial^2 u_i}{\partial t^2}\right)h$	$-\int_\Omega\left(\delta u\rho A\frac{\partial^2 u}{\partial t^2}\right)dx$

The expressions for the continuum can be considered as the limit cases of the particle model ones when $n \to \infty$ and $h = L / n$. The limit expression for the bar, string, thin slab membrane etc. models of solid mechanics play an important role in the Finite Element Method to be discussed later.