COE-C3005 Finite Element and Finite difference methods

1. Determine the displacements w_i $i \in \{1,2,3\}$, if the vector of displacements $\mathbf{a}$, stiffness matrix $\mathbf{K}$, and the loading vector $\mathbf{F}$ of the equilibrium equations $-\mathbf{Ka}+\mathbf{F}=0$ are given by

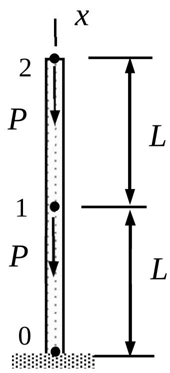
$$\mathbf{a}=\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}, \quad \mathbf{K}=k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \text{ and } \mathbf{F}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Answer $w_1=\dfrac{P}{k}$, $w_2=2\dfrac{P}{k}$, $w_3=3\dfrac{P}{k}$

2. Determine the angular velocities and modes $(\omega,\mathbf{A})$ of free vibrations, if the vector of unknowns $\mathbf{a}(t)$, stiffness matrix $\mathbf{K}$, and the mass matrix $\mathbf{M}$ of equations of motion $\mathbf{Ka}+\mathbf{M}\ddot{\mathbf{a}}=0$ are given by

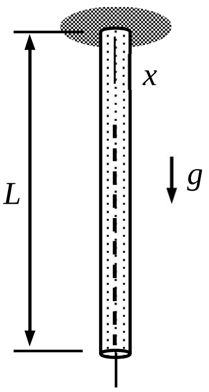
$$\mathbf{a}=\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad \mathbf{K}=k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \text{ and } \mathbf{M}=m\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Answer $(\omega,\mathbf{A})_1=(\sqrt{\dfrac{k}{m}},\begin{Bmatrix} 1 \\ -1 \end{Bmatrix})$, $(\omega,\mathbf{A})_2=(\sqrt{\dfrac{1}{5}\dfrac{k}{m}},\begin{Bmatrix} 1 \\ 1 \end{Bmatrix})$



3. The bar shown is loaded by point forces of equal magnitudes P but opposite directions acting on points 1 and 2. Use the Particle Surrogate Method on the regular grid shown to write the equilibrium equations of points 1 and 2. After that, solve the equations for the axial displacements u_1 and u_2. Cross-sectional area A and Young's modulus E of the material are constants.

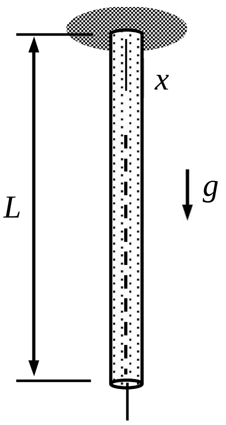
Answer $u_1=-2\dfrac{PL}{EA}$, $u_2=-3\dfrac{PL}{EA}$



4. Consider a bar of length L loaded by its own weight. Determine the displacements at the mid-point and at the free end using the continuum model. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.

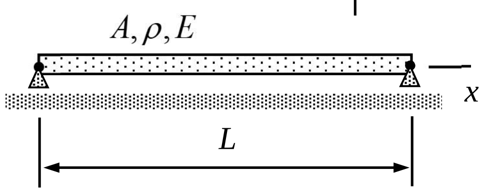
Answer $u(L/2)=\dfrac{3}{8}\dfrac{\rho g L^2}{E}$, $u(L)=\dfrac{1}{2}\dfrac{\rho g L^2}{E}$

5. Consider a bar of length L loaded by its own weight. Determine the displacements at the mid-point and at the free end using the Particle Surrogate Method of the bar with a regular grid $i \in \{0,1,2\}$. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.



Answer $u_1 = \dfrac{3}{8}\dfrac{\rho g L^2}{E}$, $u_2 = \dfrac{1}{2}\dfrac{\rho g L^2}{E}$

6. A bar is free to move in the horizontal direction as shown. Determine the angular velocities and modes $(\omega, \mathbf{A})$ of free vibrations using the Particle Surrogate Method and a regular grid $i \in \{0,1\}$. Cross-sectional are A, density ρ of the material, and Young's modulus E of the material are constants.



Answer $(\omega, \mathbf{A})_1 = (0, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix})$ $(\omega, \mathbf{A})_2 = (\dfrac{2}{L}\sqrt{\dfrac{E}{\rho}}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix})$

Determine the displacements w_i $i \in \{1,2,3\}$, if the vector of displacements $\mathbf{a}$, stiffness matrix $\mathbf{K}$, and the loading vector $\mathbf{F}$ of the equilibrium equations $-\mathbf{Ka}+\mathbf{F}=0$ are given by

$$\mathbf{a}=\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}, \quad \mathbf{K}=k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \text{ and } \mathbf{F}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Solution

With linear equation systems $-\mathbf{Ka}+\mathbf{F}=0$ of more than two unknows, using the matrix inverse to get $\mathbf{a}=\mathbf{K}^{-1}\mathbf{F}$ is not practical in hand calculation. Gauss elimination is based on row operations aiming at an upper diagonal matrix. After that, solution for the unknowns is obtained step-by-step starting from the last equation. In standard form, the equation system is given by

$$k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Let us multiply the 2:nd equation by 2 and add to it equation 1 to get

$$k\begin{bmatrix} 2 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad \Rightarrow \quad k\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Let us multiply 3:rd equation by 3 and add to it the 2:nd equation to get the upper triangular matrix.

$$k\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix} \quad \Rightarrow \quad k\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix}.$$

Solution to the obtained equation system coincides with that of the original system as the equations are just linear combinations of the original one. After that, solution for the unknowns is obtained step-by-step starting from the last equation:

$$kw_3=3P \quad \Rightarrow \quad w_3=3\frac{P}{k}, \quad \leftarrow$$

$$k(3w_2-2w_3)=0 \quad \Rightarrow \quad w_2=\frac{2}{3}w_3=2\frac{P}{k}, \quad \leftarrow$$

$$k(2w_1-w_2)=0 \quad \Rightarrow \quad w_1=\frac{1}{2}w_2=\frac{P}{k}. \quad \leftarrow$$

Determine the angular velocities and modes $(\omega,\mathbf{A})$ of free vibrations, if the vector of unknowns $\mathbf{a}(t)$, stiffness matrix $\mathbf{K}$, and the mass matrix $\mathbf{M}$ of equations of motion $\mathbf{Ka}+\mathbf{M\ddot{a}}=0$ are given by

$$\mathbf{a}=\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},\quad \mathbf{K}=k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},\ \text{and}\quad \mathbf{M}=m\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Solution

The solution method for the second order ordinary differential equations is based on a trial solution on the form $\mathbf{a}=\mathbf{A}\exp(\mathrm{i}\omega t)$ in which $\mathbf{A}$ is the mode and ω the angular velocity associated with it. The goal of the modal analysis is to find all the possible pairs $(\omega,\mathbf{A})$ in the solution trial. Solution trial transforms the second order ordinary differential equation set into an algebraic one:

$$k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}+m\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}=0 \ \text{and}\ \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}=\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}e^{\mathrm{i}\omega t} \quad \Rightarrow$$

$$(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}-\lambda\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \quad \text{where}\quad \lambda=\frac{m\omega^2}{k}.$$

A homogeneous linear equation system can yield a non-zero solution only if the matrix is singular, i.e., its determinant vanishes. The condition can be used to find the possible values of λ

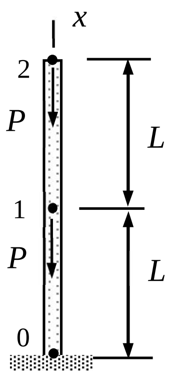
$$\det(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}-\lambda\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix})=\det(\begin{bmatrix} 2-4\lambda & -1-\lambda \\ -1-\lambda & 2-4\lambda \end{bmatrix})=(2-4\lambda)^2-(1+\lambda)^2=0 \quad \Rightarrow$$

$$(2-4\lambda)=(1+\lambda) \quad \text{or} \quad (2-4\lambda)=-(1+\lambda) \quad \Rightarrow \quad \lambda=1/5 \quad \text{or} \quad \lambda=1.$$

Knowing the possible values for a non-zero solution, the modes follow from the linear equation system when the values of λ:s are substituted there (one at the time):

$$\lambda=1:\ \omega=\sqrt{\frac{k}{m}} \quad \text{and} \quad \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \quad \text{so} \quad (\omega,\mathbf{A})_1=(\sqrt{\frac{k}{m}},\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}),\ \leftarrow$$

$$\lambda=\frac{1}{5}:\ \omega=\sqrt{\frac{1}{5}\frac{k}{m}} \quad \text{and} \quad \frac{1}{5}\begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \quad \text{so} \quad (\omega,\mathbf{A})_2=(\sqrt{\frac{1}{5}\frac{k}{m}},\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}).\ \leftarrow$$



The bar shown is loaded by point forces of equal magnitudes P but opposite directions acting on points 1 and 2. Use the Particle Surrogate Method on the regular grid shown to write the equilibrium equations of points 1 and 2. After that, solve the equations for the axial displacements u_1 and u_2. Cross-sectional area A and Young's modulus E of the material are constants.

Solution

The equilibrium equations of the two free particles and one fixed for the bar model according to the particle surrogate method are given by (formulae collection)

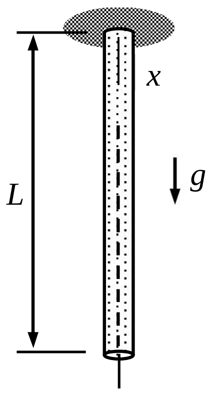
$$u_0 = 0, \quad \frac{EA}{h}(u_0 - 2u_1 + u_2) - P = 0, \quad \frac{EA}{h}(u_1 - u_2) - P = 0$$

where $h = L$. Notice that the given point force needs to be taken into account in the sum of forces on the left-hand side of equation of motion for particle 1 (in the formulae collection, only the effect of gravity is considered). The matrix notation uses only the equations of the free particles and the boundary condition given by particle 0 to eliminate u_0 from the equilibrium equations $-\mathbf{Ka} + \mathbf{F} = \mathbf{0}$

$$-\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + P\begin{Bmatrix} -1 \\ -1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{PL}{EA}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{Bmatrix} -1 \\ -1 \end{Bmatrix} = \frac{PL}{EA}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} -1 \\ -1 \end{Bmatrix} = -\frac{PL}{EA}\begin{Bmatrix} 2 \\ 3 \end{Bmatrix}. \quad \leftarrow$$

Consider a bar of length L loaded by its own weight. Determine the displacements at the mid-point and at the free end using the continuum model. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.



Solution

In stationary case, the continuum model for the problem is given by equations

$$EA\frac{d^2u}{dx^2}+\rho Ag=0 \quad x\in]0,L[\ , \ \ u=0 \quad x=0\text{, and } \ EA\frac{du}{dx}=0 \quad x=L\,.$$

Repetitive integrations in the differential equation give the generic solution containing two integration constants:

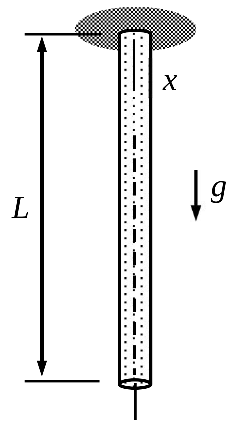
$$\frac{d^2u}{dx^2}=-\frac{\rho g}{E} \ \Rightarrow \ \frac{du}{dx}=-\frac{\rho g}{E}x+a \ \Rightarrow \ u=-\frac{\rho g}{E}\frac{1}{2}x^2+ax+b\,.$$

Then, substituting the generic solution into the boundary conditions

$$u(0)=b=0 \text{ and } \frac{du}{dx}(L)=-\frac{\rho g}{E}L+a=0 \quad \Leftrightarrow \quad a=\frac{\rho g}{E}L \text{ and } b=0\,.$$

Solution to the problem for all points give also the values at the center and end points

$$u(x)=\frac{\rho g}{E}x(L-\frac{1}{2}x) \quad \Rightarrow \quad u(\frac{L}{2})=\frac{3}{8}\frac{\rho gL^2}{E} \text{ and } u(L)=\frac{\rho gL^2}{2E}\,.$$ ←



Consider a bar of length L loaded by its own weight. Determine the displacements at the mid-point and at the free end using the Particle Surrogate Method of the bar with a regular grid $i \in \{0,1,2\}$. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.

Solution

In Particle Surrogate Method, mass is lumped at the grid points and external forces act on the particles. Interaction of the particles corresponds to a spring of coefficient $k = EA/h$ where $h = L/2$ with the present grid. Equations for particles $i \in \{0,1,2\}$ are

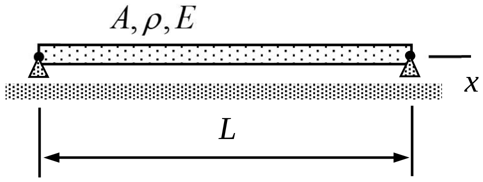
$$u_0 = 0, \quad 2\frac{EA}{L}(u_0 - 2u_1 + u_2) + \frac{L}{2}A\rho g = 0, \quad \text{and} \quad 2\frac{EA}{L}(u_1 - u_2) + \frac{L}{4}A\rho g = 0.$$

In matrix form for the last two equations, in which the first equations is used to eliminate u_0,

$$-2\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \frac{L}{4}A\rho g\begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad 2\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{L}{4}A\rho g\begin{Bmatrix} 2 \\ 1 \end{Bmatrix} \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{L^2}{8}\frac{\rho g}{E}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = \frac{L^2}{8}\frac{\rho g}{E}\frac{1}{1}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = \frac{L^2}{8}\frac{\rho g}{E}\begin{Bmatrix} 3 \\ 4 \end{Bmatrix}. \quad \leftarrow$$

A bar is free to move in the horizontal direction as shown. Determine the angular velocities and modes $(\omega,\mathbf{A})$ of free vibrations using the Particle Surrogate Method and a regular grid $i\in\{0,1\}$. Cross-sectional are A, density ρ of the material, and Young's modulus E of the material are constants.



Solution

In Particle Surrogate Method, mass is lumped at the grid points and external and internal forces act on the particles. Interaction of the particles corresponds to a spring of coefficient $k = EA/h$ where the spacing $h = L$ in the present case. The equations for particles $i\in\{0,1\}$ take the forms

$$\frac{EA}{h}(u_1 - u_0) = \frac{h}{2}\rho Ah\ddot{u}_0 \text{ and } \frac{EA}{h}(u_0 - u_1) = \frac{h}{2}\rho Ah\ddot{u}_1$$

or using the matrix notation

$$\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_0 \\ u_1 \end{Bmatrix} + \frac{h}{2}\rho A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_0 \\ \ddot{u}_1 \end{Bmatrix} = 0 .$$

The solution method for the second order ordinary differential equations is based on a trial solution of the form $\mathbf{a} = \mathbf{A}\exp(\mathrm{i}\omega t)$ in which $\mathbf{A}$ is the mode and ω the angular velocity associated with it. The goal of the modal analysis is to find all the possible pairs $(\omega,\mathbf{A})$. When substituted there, solution trial transforms the second order ordinary differential equation set into an algebraic one:

$$(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \text{ where } \lambda = \frac{1}{2}\omega^2\frac{h^2\rho}{E} = \frac{1}{2}\omega^2\frac{L^2\rho}{E} .$$

A homogeneous linear equation system can give a non-zero solution only if the matrix is singular, i.e., its determinant vanishes. The condition can be used to find the possible values of λ

$$\det(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = \det(\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}) = (1-\lambda)^2 - 1 = 0 \quad \Rightarrow \quad \lambda = 0 \text{ or } \lambda = 2 .$$

Knowing the possible values of λ for a non-zero solution, the modes follow from the linear equation system when the values of λ:s are substituted there (one at the time):

$$\lambda = 0 : \ \omega = 0 \text{ and } \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{so} \quad (\omega,\mathbf{A})_1 = (0,\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}) , \leftarrow$$

$$\lambda = 2 : \ \omega = \frac{2}{L}\sqrt{\frac{E}{\rho}} \text{ and } \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{so} \quad (\omega,\mathbf{A})_2 = (\frac{2}{L}\sqrt{\frac{E}{\rho}},\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}) . \leftarrow$$