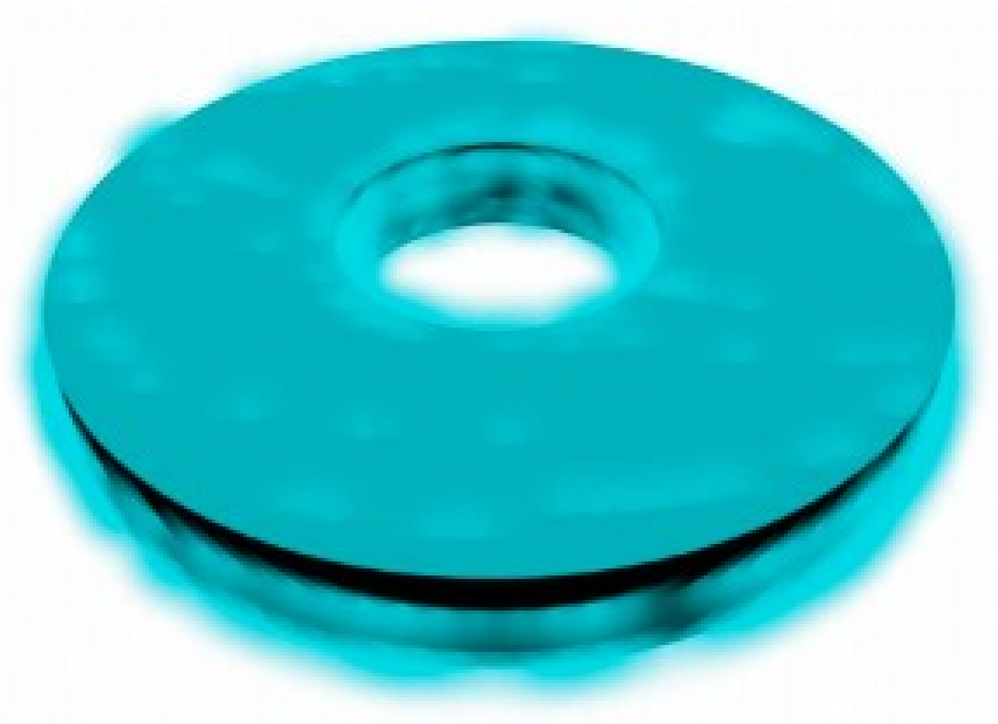
Superconductivity, Nambu representation and Majorana physics



May 3rd 2021

Today’s learning outcomes

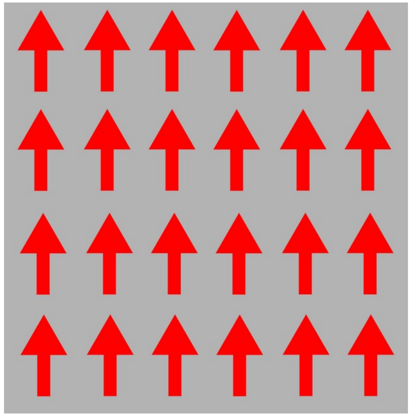
- The physical nature of superconductivity
- The Nambu representation of superconducting excitations
- The emergence of Majorana excitations in topological superconductors

A reminder from previous sessions

Electronic interactions are responsible for symmetry breaking

Broken time-reversal symmetry

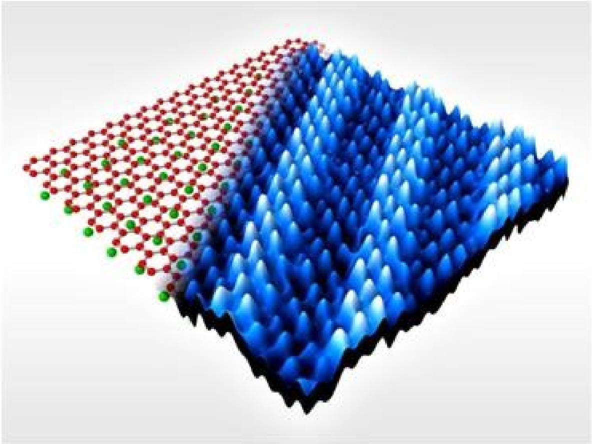
Classical magnets



$$\mathbf{M} \rightarrow -\mathbf{M}$$

Broken crystal symmetry

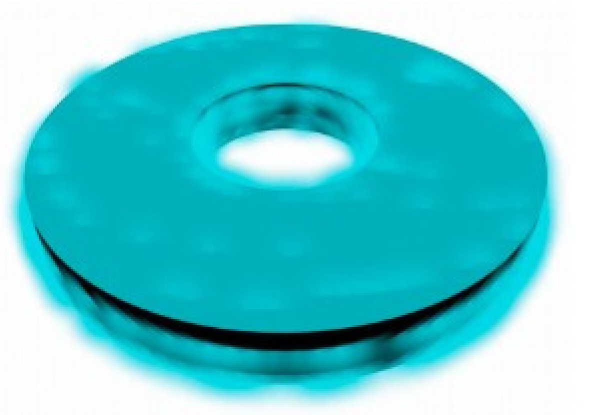
Charge density wave



$$\mathbf{r} \rightarrow \mathbf{r} + \mathbf{R}$$

Broken gauge symmetry

Superconductors



$$\langle c_\uparrow c_\downarrow \rangle \rightarrow e^{i\phi} \langle c_\uparrow c_\downarrow \rangle$$

Quantum matter with interactions

We can consider two broad groups of interacting quantum matter

$$H = \sum_{ij} t_{ij} c_i^\dagger c_j + \sum_{ijkl} V_{ijkl} c_i^\dagger c_j c_k^\dagger c_l$$

With a mean field description

$$H \approx \sum_{ij} \bar{t}_{ij} c_i^\dagger c_j + \sum_{ij} \Delta_{ij} c_i c_j$$

Approximate quadratic Hamiltonian
Effective single particle description

Weakly correlated matter

Without a mean field description

????

No good quadratic approximation
Requires exact solutions or numerical

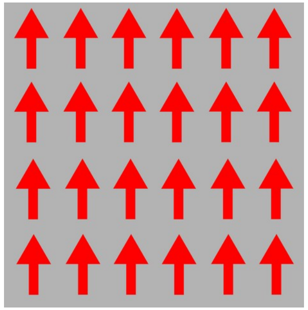
Strongly correlated matter

Correlations and mean field

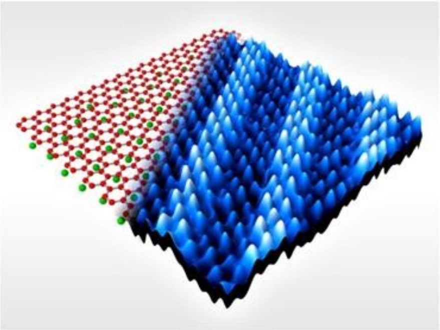
Many quantum states can be approximately described by mean field theories

$$H \approx \sum_{ij} \bar{t}_{ij} c_i^\dagger c_j + \sum_{ij} \Delta_{ij} c_i c_j + h.c.$$

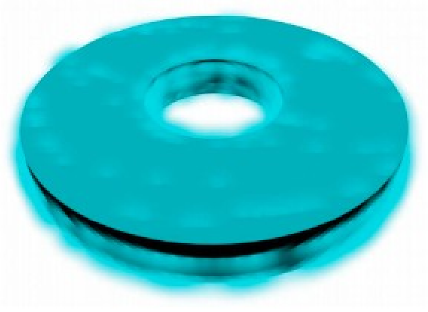
Magnets



Charge density waves



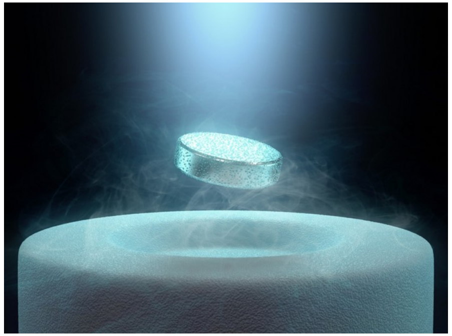
Superconductors



Forms of orrelated matter

Superconductors

with a mean field description

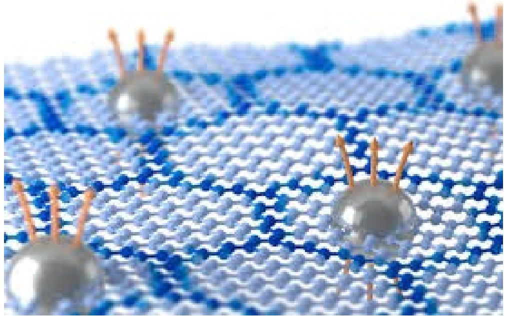


Attractive interactions
Destroyed by magnetic fields

Majorana excitations

Fractional quantum Hall states

without a mean field description

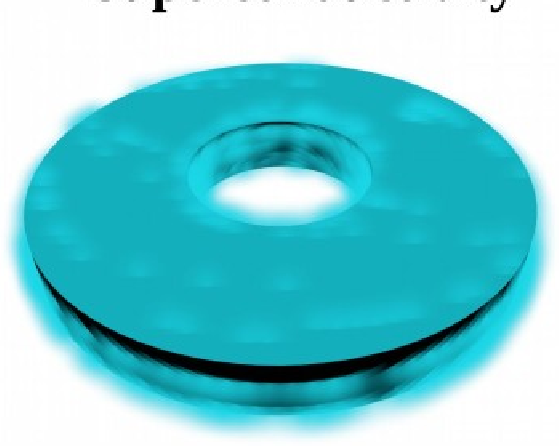


Repulsive interactions
Require strong magnetic fields

Fractional charged excitations

Macroscopic quantum phenomena

Superconductivity

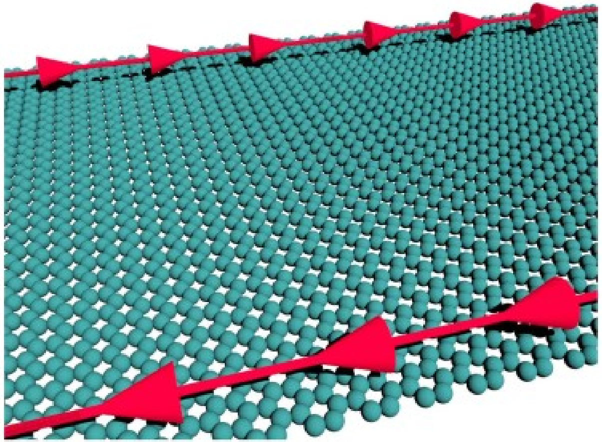


$$\Phi = \frac{h}{2e}$$

Many-body state

Quantization of flux

Quantum Hall effect



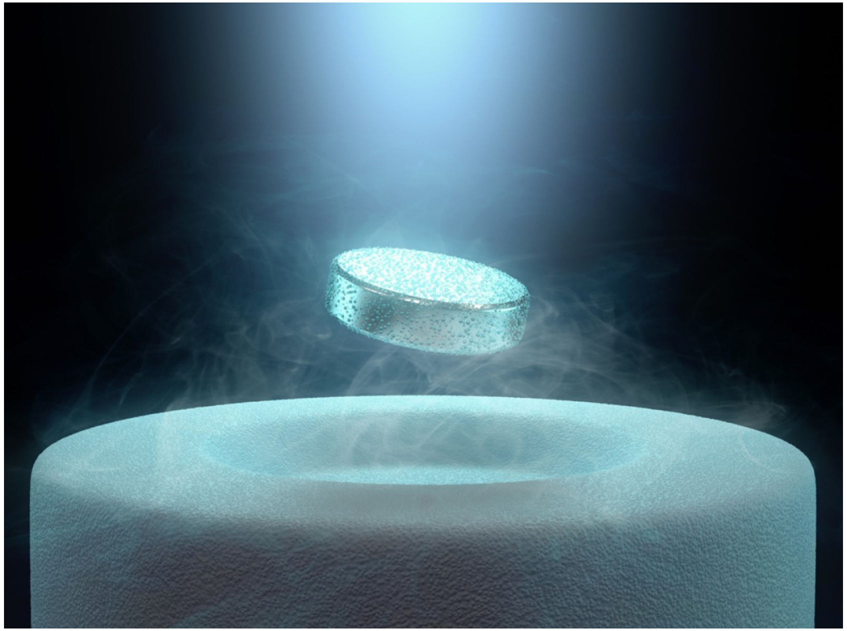
$$\sigma_{xy} = \nu \frac{e^2}{h}$$

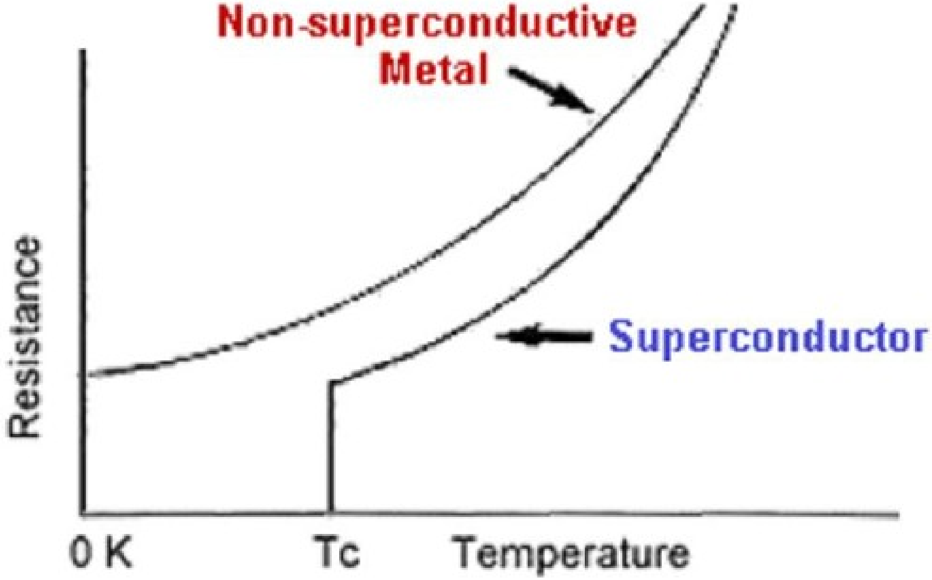
Single-particle state

Quantization of conductance

Superconductivity

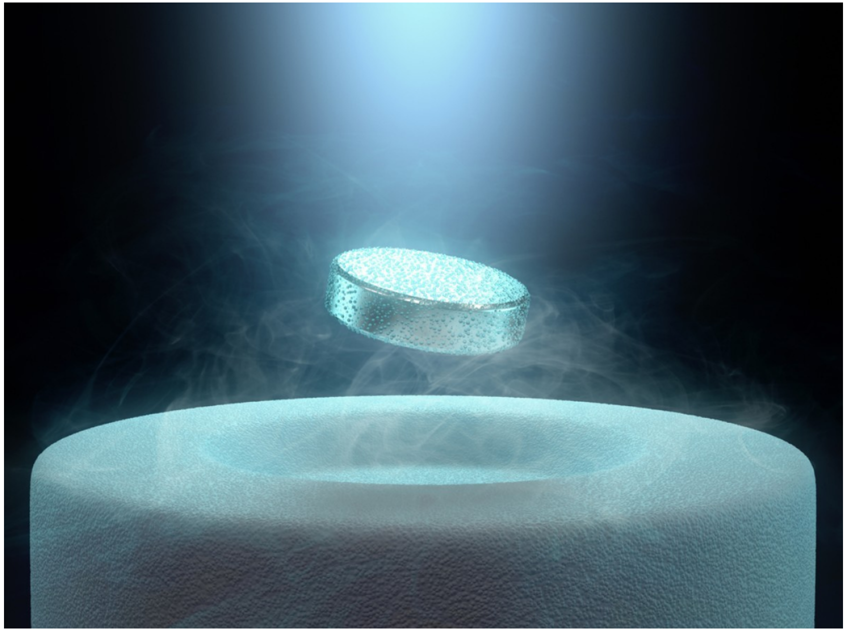
What is a superconductor?

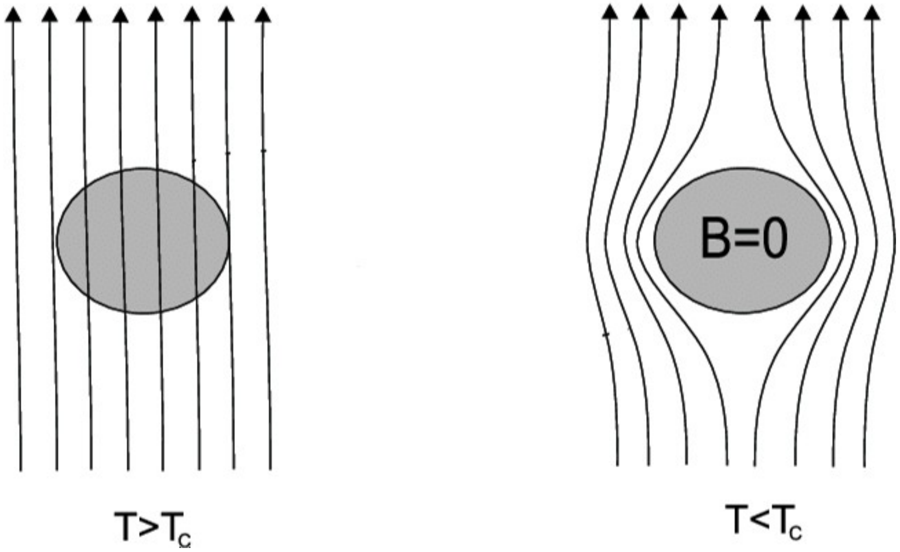




The resistance suddenly drops to 0 at a certain temperature

What is a superconductor?





The magnetic field lines get expelled in the superconducting state

Interactions and mean field

Free Hamiltonian ***Interactions***

$$H = \sum_{ij} t_{ij} c_i^\dagger c_j + \sum_{ijkl} V_{ijkl} c_i^\dagger c_j c_k^\dagger c_l$$

What are these interactions coming from?

- Electrostatic (repulsive) interactions
- Mediated by other quasiparticles (phonons, magnons, plasmons,...)

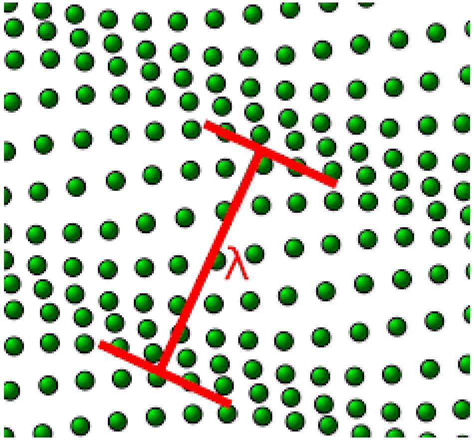
The net effective interaction can be attractive or repulsive

<u>Superconductivity requires effective attractive interactions</u>

Origin of attractive interactions

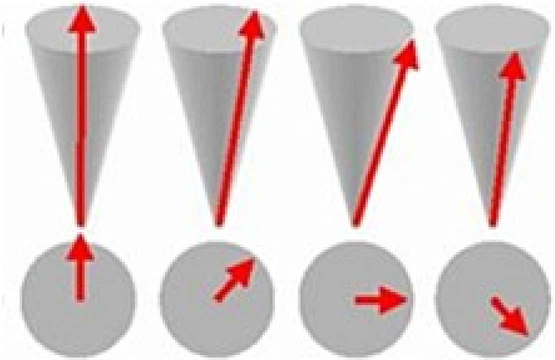
Interactions between electrons can be effectively attractive when mediated by other quasiparticles

Phonons



Conventional SC
(Al, etc)

Magnons



Unconventional SC
(cuprates, etc)

A simple interacting Hamiltonian

Free Hamiltonian

Interactions (Hubbard term)

$$H = \sum_{ij} t_{ij} [c^\dagger_{i\uparrow} c_{j\uparrow} + c^\dagger_{i\downarrow} c_{j\downarrow}] + \sum_i U c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow}$$

From now on lets consider we have a spin degree of freedom $\uparrow, \downarrow$

What is the ground state of this Hamiltonian?

$U < 0$ Superconductivity

$U > 0$ Magnetism

The mean-field approximation

Mean field: *Approximate four fermions by two fermions times expectation values*

Four fermions
(not exactly solvable)

Two fermions
(exactly solvable)

$$U c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} \approx U \langle c^\dagger_{i\uparrow} c^\dagger_{i\downarrow} \rangle c_{i\uparrow} c_{i\downarrow} + h.c.$$

$$U c^\dagger_{i\uparrow} c_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} \approx \Delta c_{i\uparrow} c_{i\downarrow} + h.c.$$

For $U < 0$

i.e. attractive interactions

$\Delta \sim \langle c^\dagger_{i\uparrow} c^\dagger_{i\downarrow} \rangle$ is the superconducting order

A Hamiltonian for a superconductor

Free Hamiltonian ***Pairing term***

$$H = \sum_{ij} t_{ij} [c^\dagger_{i\uparrow} c_{j\uparrow} + c^\dagger_{i\downarrow} c_{j\downarrow}] + \Delta \sum_i c_{i\uparrow} c_{i\downarrow} + h.c.$$

Lets have a look to the term $\Delta c_{i\uparrow} c_{i\downarrow}$

Superconductivity and symmetries

$$H \sim \Delta c_{i\uparrow} c_{i\downarrow} + h.c.$$

$$H|GS\rangle = E_{GS}|GS\rangle$$

This term destroys two electrons

$|GS\rangle$ Ground state

The ground state can not have a well defined number of electrons

$$|GS\rangle \sim |2e\rangle + |4e\rangle + |6e\rangle + ...$$

$$|GS\rangle \sim |1e\rangle + |3e\rangle + |5e\rangle + ...$$

Gauge symmetry and superconductivity

What we know from quantum mechanics

"The phase of a wavefunction (field operator) does not have physical meaning"

This is what we know as gauge symmetry

$$c_n \to e^{i\phi} c_n$$
$$c_n^\dagger \to e^{-i\phi} c_n^\dagger$$

Terms in the Hamiltonian that do not change under this transformation

$$c_n^\dagger c_m$$

Hopping

$$c_{n,\uparrow}^\dagger c_{n,\uparrow} - c_{n,\downarrow}^\dagger c_{n,\downarrow}$$

Magnetism

$$c_{n,\uparrow}^\dagger c_{m,\downarrow}$$

Spin-orbit coupling

$$c_{n,\uparrow}^\dagger c_{m,\uparrow} c_{n,\downarrow}^\dagger c_{m,\downarrow}$$

Interactions

Superconductivity and gauge symmetry breaking

Gauge symmetry

$$c_n \to e^{i\phi} c_n \qquad c_n^\dagger \to e^{-i\phi} c_n^\dagger$$

How does the superconducting order transform under a gauge transformation?

$$\Delta = \langle c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger \rangle$$

Superconductivity and gauge symmetry breaking

Gauge symmetry

$$c_n \to e^{i\phi} c_n \qquad\qquad c_n^\dagger \to e^{-i\phi} c_n^\dagger$$

How does the superfluid density transform under a gauge transformation?

$$\Delta \to e^{-2i\phi} \Delta$$

A superconductor breaks gauge symmetry

Generic forms of superconductivity

A generic superconducting Hamiltonian

$$\widehat{H}' = \sum_{\boldsymbol{k},\sigma} \epsilon_{\boldsymbol{k}} \widehat{c}^{\dagger}_{\boldsymbol{k},\sigma} \widehat{c}_{\boldsymbol{k},\sigma} - \frac{1}{2} \sum_{\boldsymbol{k}}{}' \sum_{\sigma_1,\sigma_2} \left[\Delta_{\boldsymbol{k},\sigma_1\sigma_2} \widehat{c}^{\dagger}_{\boldsymbol{k},\sigma_1} \widehat{c}^{\dagger}_{-\boldsymbol{k},\sigma_2} + \Delta^{*}_{\boldsymbol{k},\sigma_1\sigma_2} \widehat{c}_{\boldsymbol{k},\sigma_1} \widehat{c}_{-\boldsymbol{k},\sigma_2} \right]$$

Can be characterized by a superconducting matrix

$$\Delta_{\boldsymbol{k}} = \begin{pmatrix} \Delta_{\boldsymbol{k},\uparrow\uparrow} & \Delta_{\boldsymbol{k},\uparrow\downarrow} \\ \Delta_{\boldsymbol{k},\downarrow\uparrow} & \Delta_{\boldsymbol{k},\downarrow\downarrow} \end{pmatrix}$$

The symmetry of the SC order determines the nature of the SC order

Superconducting momentum symmetries

A generic type of a superconductor is characterized by the order parameter

Real space

$$\Delta_{\uparrow\downarrow}(\mathbf{r},\mathbf{r}') \sim \langle c_{\mathbf{r}\uparrow} c_{\mathbf{r}'\downarrow} \rangle$$

Reciprocal space

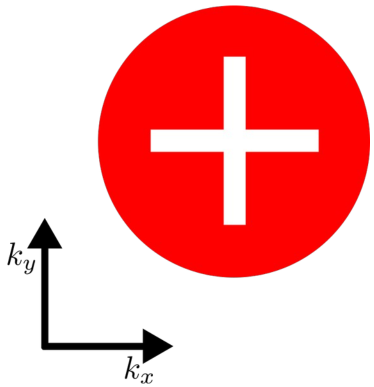
$$\Delta_{\uparrow\downarrow}(\mathbf{k}) \sim \langle c_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow} \rangle$$

The superconducting state can be characterized by the symmetry of $\Delta_{\uparrow\downarrow}(\mathbf{k})$

Superconducting momentum symmetries

The superconducting state can be characterized by the symmetry of $\Delta(\mathbf{k})$

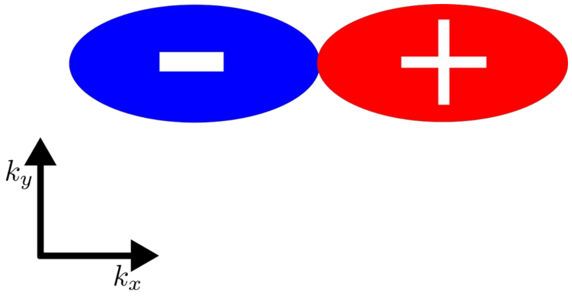
s-wave



Conventional SC
(driven by phonons)

Al

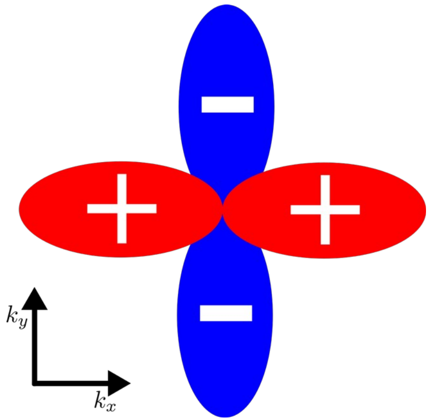
p-wave



Unconventional SC
(driven by FE magnons)

UTe$_2$

d-wave



Unconventional SC
(driven by AF magnons)

cuprates

Majorana physics in superconductors

The Nambu representation

How do we solve a Hamiltonian of the form $$H = \sum_{\mathbf{k},s} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k},s} c_{\mathbf{k},s} + \sum_{\mathbf{k}} \Delta c^\dagger_{\mathbf{k},\uparrow} c^\dagger_{-\mathbf{k},\downarrow}$$

Define a Nambu spinor $$\Psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{\mathbf{k}\downarrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \\ -c^\dagger_{-\mathbf{k}\uparrow} \end{pmatrix}$$

Electron sector

Hole sector

The Hamiltonian in the Nambu basis is quadratic and can be diagonalized

$$H = \Psi^\dagger_{\mathbf{k}} \mathcal{H} \Psi_{\mathbf{k}}$$

Majorana excitations

A very special type of fermion is a so-called Majorana fermion

$$\Psi^\dagger = \Psi$$

Which by definition it is its own antiparticle

Majorana fermion do not appear naturally in materials, as we only have electrons

Yet, mathematically, each electron can be written as two Majoranas

$$c = \Psi_\alpha + i\Psi_\beta \qquad c^\dagger = \Psi_\alpha - i\Psi_\beta \qquad \Psi_\alpha^\dagger = \Psi_\alpha \qquad \Psi_\beta^\dagger = \Psi_\beta$$

Can we isolate a single Majorana in a material?

Looking at Majorana excitations in superconductors

Excitations in superconductors are combinations of electrons and holes, for instance

$$\Psi \sim c_n + c_n^\dagger$$

But this excitation is by definition a Majorana fermion

$$\Psi^\dagger = \Psi$$

Can we have superconductors in nature that show these excitations?

The minimal model for a 1D topological superconductor

One dimensional spinless p-wave superconductor (Kitaev model)

$$H = \sum_n t c^\dagger_{n+1} c_n + \Delta c_n c_{n+1} + c.c.$$

p-wave superconductivity

$$\Delta(k) = -\Delta(-k)$$

$$c_k = \sum_n e^{ikn} c_n$$

$$H = \sum_k \epsilon_k c^\dagger_k c_k + i\Delta \left[c_{-k} c_k \sin k - c^\dagger_{-k} c^\dagger_k \sin k \right]$$

A model Hamiltonian for topological superconductivity

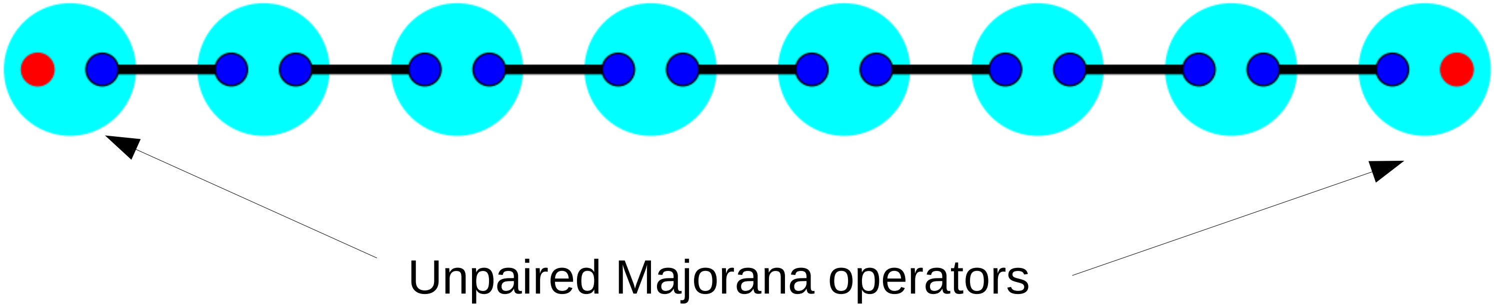
Spinless fermions in a 1D chain (Kitaev model)

$$H = \sum_n c_n^\dagger c_{n+1} + c_n c_{n+1} + h.c.$$

Can be transformed into

$$H = i \sum_n \gamma_{2n} \gamma_{2n+1}$$

γ **Majorana operators**



Majorana states as topological surface modes

Generalized Kitaev model

$$H = \sum_n c_n^\dagger c_{n+1} + \Delta c_n c_{n+1} + \mu c_n^\dagger c_n + h.c.$$

p-wave SC

Chemical potential

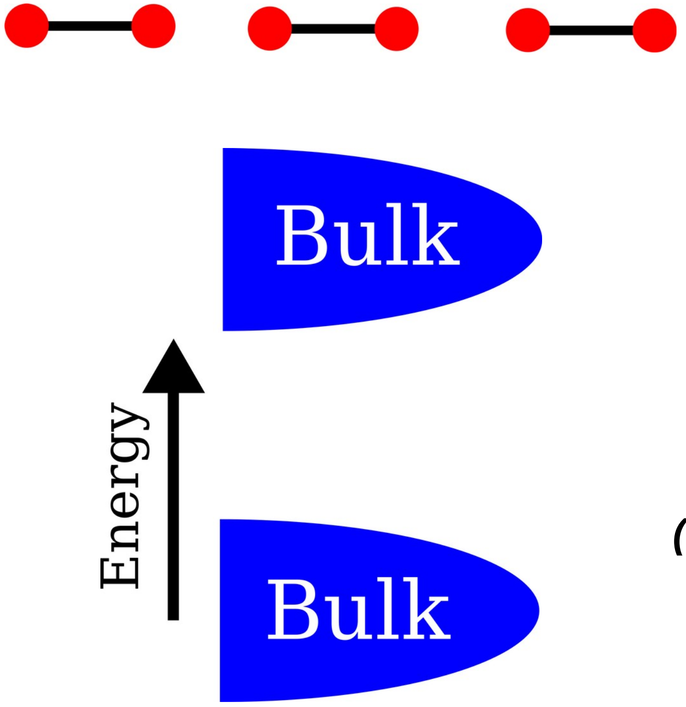
- Large chemical potential render the system filled, and topologically trivial
- p-wave SC promotes a topological phase with Majorana states

The emergence of Majorana states is associated to non-trivial topology

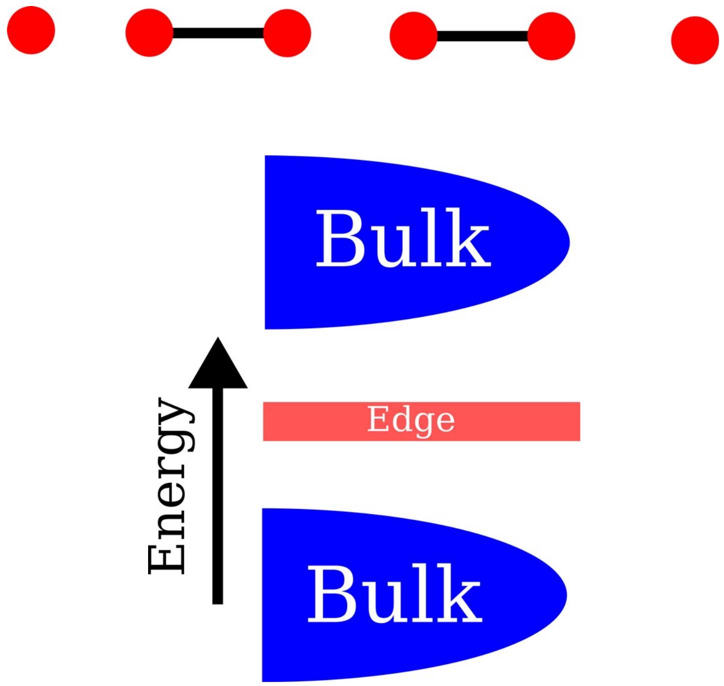
The two phases of the Kitaev model

(in the Majorana representation)

Trivial



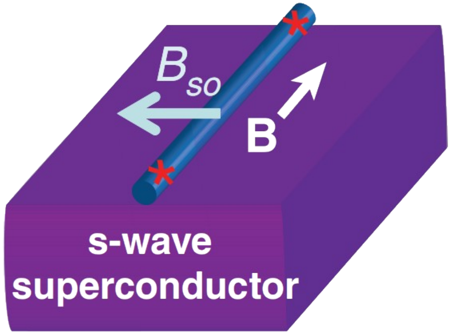
Topological



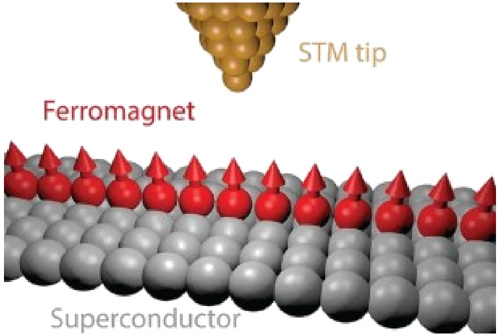
(this is like the SSH model)

Platforms for Majorana physics

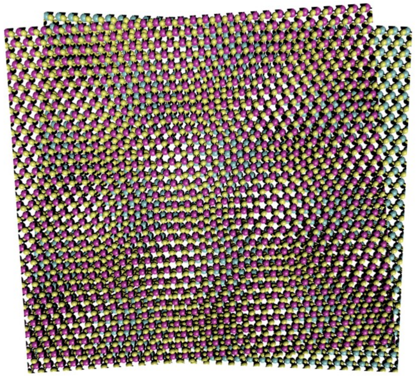
Semiconductors



Ferromagnetic atomic chains



Two-dimensional materials

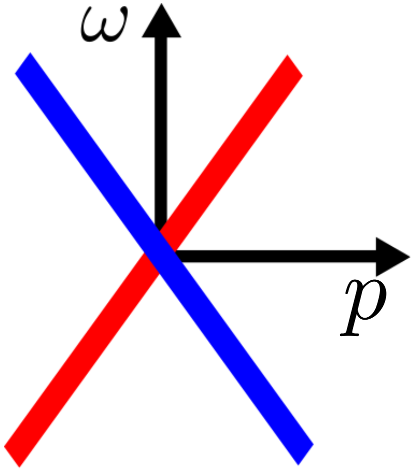


Majorana states can be engineered combining ferromagnets, superconductivity and spin-orbit coupling effects

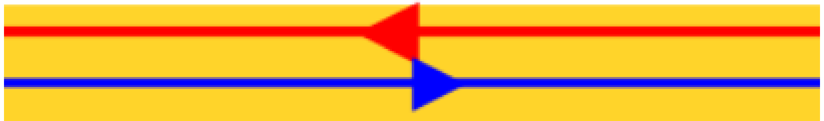
How to build your own topological superconductor

Ingredients

- s-wave pairing
- Helical states



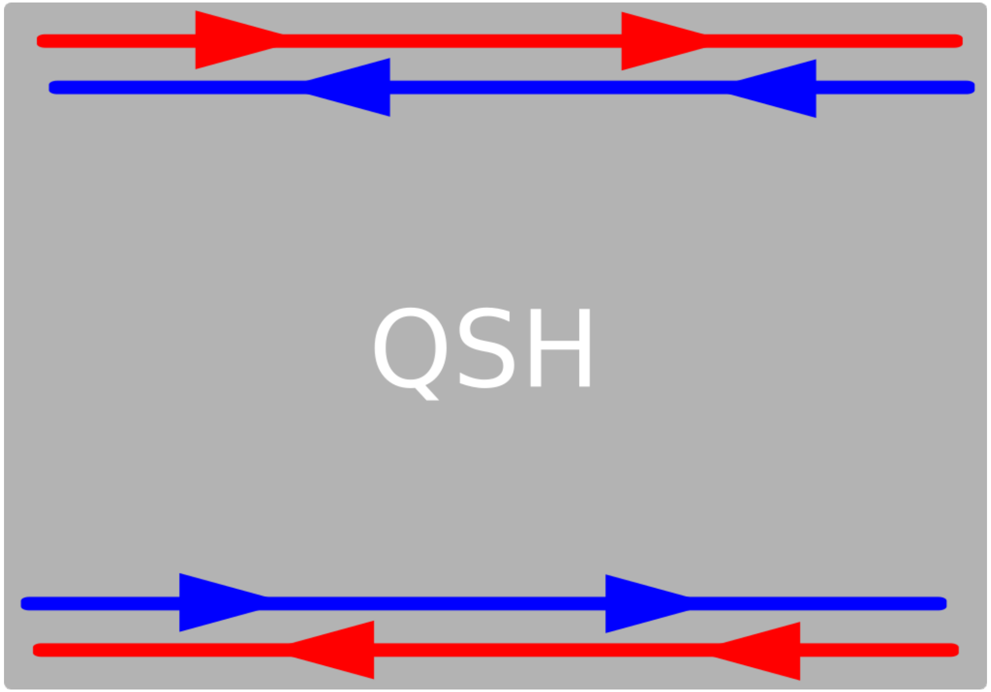
Why helical states?



→ We would like a single channel
→ We need paring between +k and -k
→ s-wave only couples opposite spins

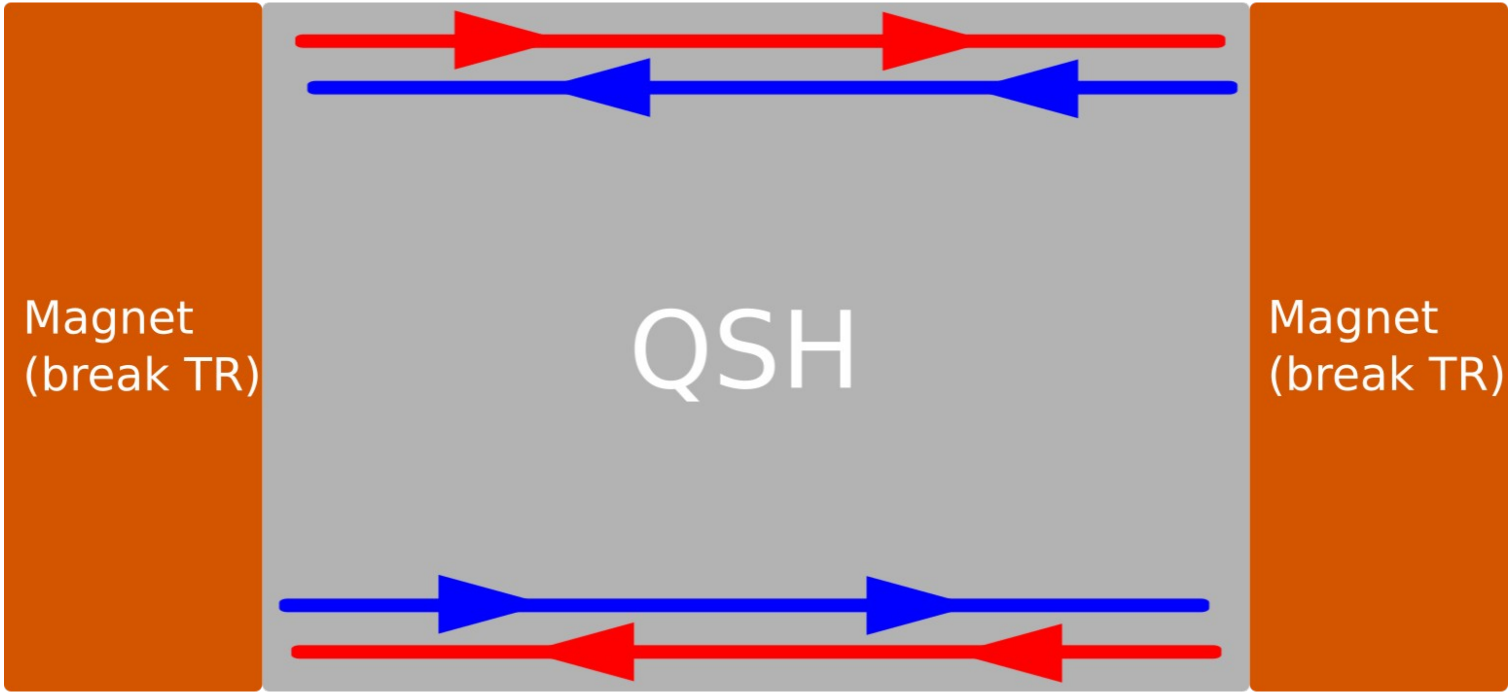
A simple way of building a topological superconductor with Majorana modes

Taking the surface states of a quantum spin-Hall (QSH) insulator

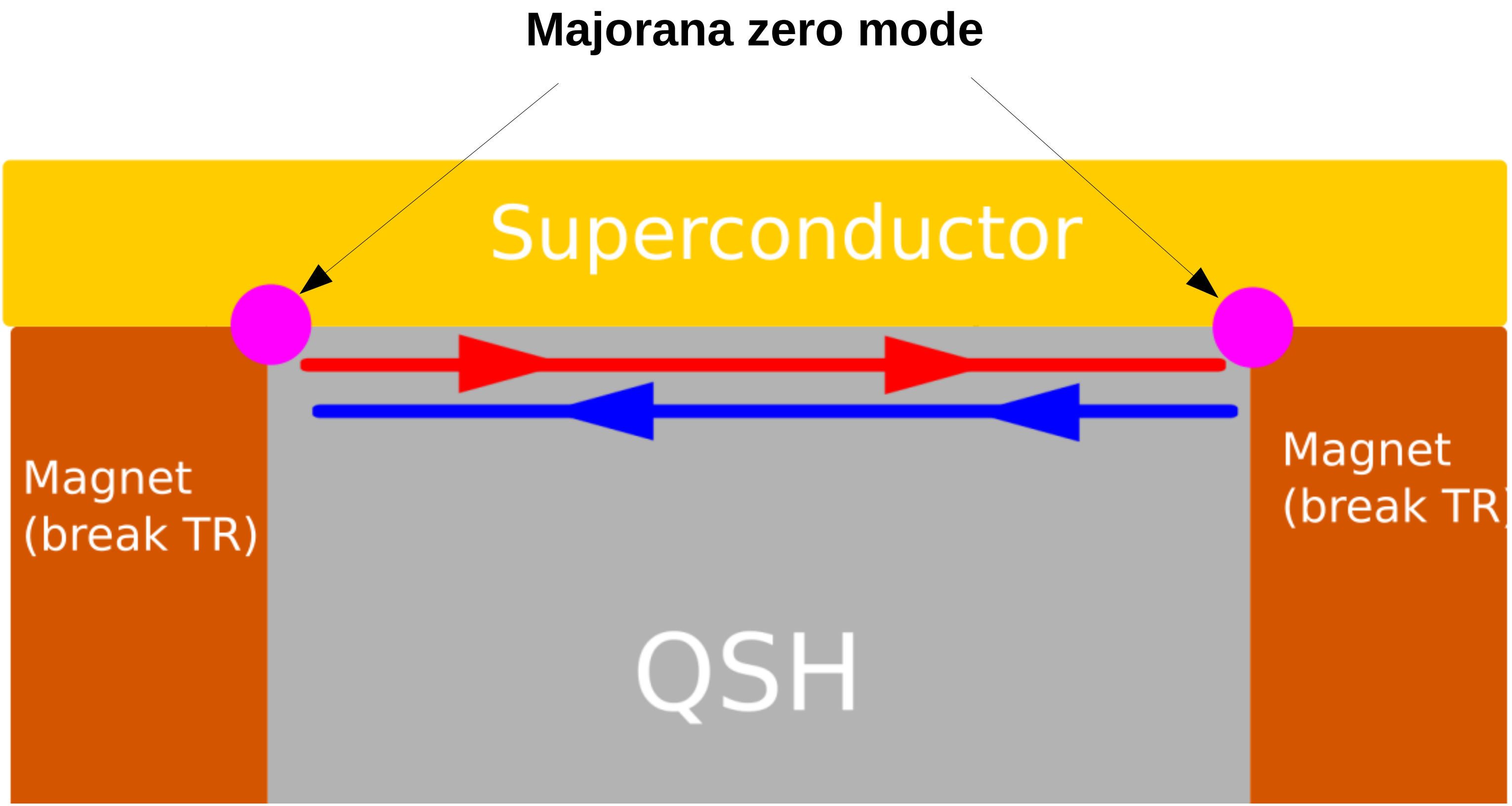


A simple way of building a topological superconductor with Majorana modes

Gap some of the helical modes with a magnet

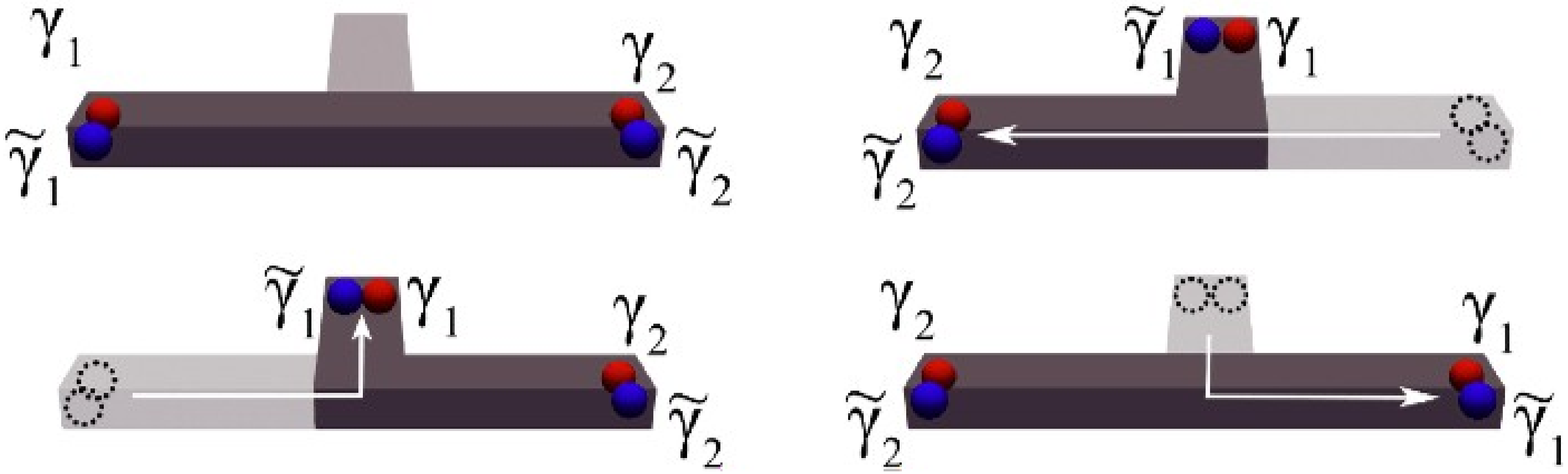


A simple way of building a topological superconductor with Majorana modes



Topological quantum computing with Majoranas

Moving Majorana excitations around each other allows to perform quantum computations



Take home

- Superconductivity arises from attractive interactions
- Superconducting states can be captured by effective single-particle mean-field models
- Unconventional superconductors can show Majorana excitations

Reading material

- Pages 78-85 from Bruus & Flensberg
- Pages 107-117 from Titus Neupert's notes
- Pages 193-201 from Bernevig & Hughes