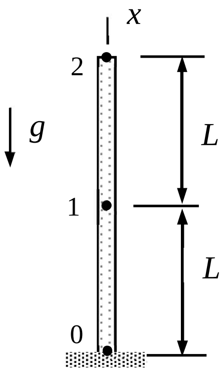
COE-C3005 Finite Element and Finite difference methods

1. Find the five point difference approximation to $f_i^{(4)}$ (fourth derivative) using the dataset $\{(-2\Delta x, f_{i-2}), (-\Delta x, f_{i-1}), (0, f_i), (\Delta x, f_{i+1}), (2\Delta x, f_{i+2})\}$. Use the Lagrange interpolation polynomial $p(x)$ to the dataset and calculate the derivative approximation using $f_i^{(4)} = p^{(4)}(0)$ (Mathematica may be useful).

   Answer $f_i^{(4)} = \dfrac{1}{\Delta x^4}(f_{i-2} - 4f_{i-1} + 6f_i - 4f_{i+1} + f_{i+2})$

2. Derive the Crank-Nicolson time integration iteration using (1) Taylor series of displacement $a(t)$ and velocity $\dot{a}(t)$ with respect to time and the mean value approximation to the remainder containing the second time derivative, and (2) differential equation $m\ddot{a} + ka = 0$ written at the end points of the time interval of length Δt.

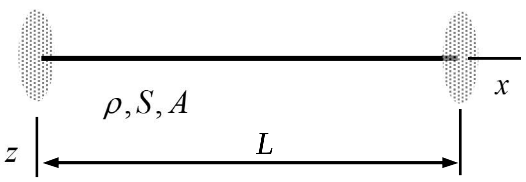
   Answer $\begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_i = \dfrac{1}{4+\alpha^2} \begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_{i-1}$ where $\alpha = \sqrt{\dfrac{k}{m}}\Delta t$



3. The bar shown is loaded by its own weight. Determine the displacements at the grid points 1 and 2 using the Finite Difference Method. Cross-sectional area A, Young's modulus E, and density ρ of the material are constants.

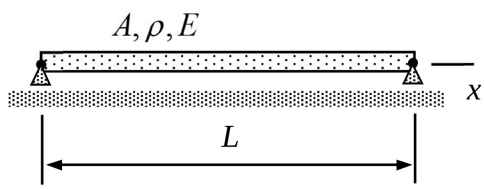
   Answer $u_0 = 0$, $u_1 = u_2 = -\dfrac{\rho g L^2}{E}$

4. Consider the string of tightening S and mass per unit length ρA shown. Use the Finite Difference Method with second order accurate central differences on a regular grid $i \in \{0,1,\ldots,n\}$ to find the angular velocities ω_k of the free vibrations using the solution trial $w_i = a(t)\sin(k\pi i / n)$ $k \in \{1,2\ldots,n-1\}$.



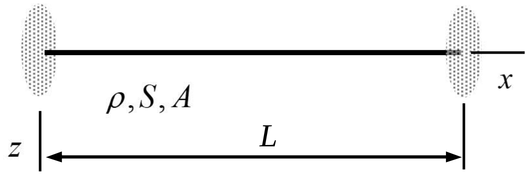
   Answer $\omega_k = \dfrac{2}{L}\sqrt{\dfrac{S}{A\rho}}n\sin(\dfrac{k\pi}{2n})$ $k \in \{1,2\ldots,n-1\}$ (exact $\omega_k = \dfrac{k\pi}{L}\sqrt{\dfrac{S}{A\rho}}$)

5. A bar is free to move in the horizontal direction as shown. Write the equation system $\mathbf{Ka} + \mathbf{M\ddot{a}} = 0$ given by the Finite Difference Method on a regular grid with $i \in \{0,1,2\}$. Also, determine the angular velocities and modes of the free vibrations. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Answer $2\frac{EA}{L^2}\begin{bmatrix} L & -L & 0 \\ -2 & 4 & -2 \\ 0 & -L & L \end{bmatrix}\begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix} + \rho A \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \ddot{u}_0 \\ \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = 0$ and $(\omega, \mathbf{A})_1 = (0, \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix})$

6. Consider the string of tightening S and mass per unit length ρA shown. First, use the Finite Difference Method with the second order accurate central differences on a regular grid $i \in \{0,1,2\}$ to find the equations of motion of the form $ka + m\ddot{a} = 0$. Second, write the iteration equation for a typical time-step of size Δt according to Crank-Nicolson method giving the values of displacement and velocity on the temporal grid.



Answer $8\frac{EA}{L^2}u_1 + \rho A\ddot{u}_1 = 0$, $\begin{Bmatrix} u \\ \Delta t\dot{u} \end{Bmatrix}_i = \frac{1}{4+\alpha^2}\begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix}\begin{Bmatrix} u \\ \Delta t\dot{u} \end{Bmatrix}_{i-1}$ where $\alpha = \sqrt{8\frac{E}{\rho}}\frac{\Delta t}{L}$

Find the five point difference approximation to $f_i^{(4)}$ (fourth derivative) using the dataset $\{(-2\Delta x, f_{i-2}), (-\Delta x, f_{i-1}), (0, f_i), (\Delta x, f_{i+1}), (2\Delta x, f_{i+2})\}$. Use the Lagrange interpolation polynomial $p(x)$ to the dataset and calculate the derivative approximation using $f_i^{(4)} = p^{(4)}(0)$ (Mathematica may be useful).

Solution

Let us start with the Lagrange interpolation polynomials taking the value one at grid points and vanishing at all the other grid points of the dataset. The fourth derivatives of the fourth order polynomial are given by denominators multiplied by $4\times3\times2\times1=24$ (fourth derivative of the nominator):

$$p_{i-2}(x)=\frac{(x+\Delta x)(x)(x-\Delta x)(x-2\Delta x)}{(-2\Delta x+\Delta x)(-2\Delta x)(-2\Delta x-\Delta x)(-2\Delta x-2\Delta x)} \quad\Rightarrow\quad p_{i-2}^{(4)}(0)=\frac{1}{\Delta x^4},$$

$$p_{i-1}(x)=\frac{(x+2\Delta x)(x)(x-\Delta x)(x-2\Delta x)}{(-\Delta x+2\Delta x)(-\Delta x)(-\Delta x-\Delta x)(-\Delta x-2\Delta x)} \quad\Rightarrow\quad p_{i-1}^{(4)}(0)=-\frac{4}{\Delta x^4},$$

$$p_{i}(x)=\frac{(x+2\Delta x)(x+\Delta x)(x-\Delta x)(x-2\Delta x)}{(0+2\Delta x)(0+\Delta x)(0-\Delta x)(0-2\Delta x)} \quad\Rightarrow\quad p_{i}^{(4)}(0)=\frac{6}{\Delta x^4},$$

$$p_{i+1}(x)=\frac{(x+2\Delta x)(x+\Delta x)(x)(x-2\Delta x)}{(\Delta x+2\Delta x)(\Delta x+\Delta x)(\Delta x)(\Delta x-2\Delta x)} \quad\Rightarrow\quad p_{i+1}^{(4)}(0)=-\frac{4}{\Delta x^4},$$

$$p_{i+2}(x)=\frac{(x+2\Delta x)(x+\Delta x)(x)(x-\Delta x)}{(2\Delta x+2\Delta x)(2\Delta x+\Delta x)(2\Delta x)(2\Delta x-\Delta x)} \quad\Rightarrow\quad p_{i+2}^{(4)}(0)=\frac{1}{\Delta x^4}.$$

The fourth derivative of the Lagrange interpolation polynomial $p(x)$ gives an approximation to derivative of the assumed continuous $f(x)$ (evaluated at the grid points to get the dataset) at $x_i=0$

$$f_i^{(4)} = p^{(4)}(0) = p_{i-2}^{(4)}(0)\, f_{i-2} + p_{i-1}^{(4)}(0)\, f_{i-1} + p_{i}^{(4)}(0)\, f_{i} + p_{i+1}^{(4)}(0)\, f_{i+1} + p_{i+2}^{(4)}(0)\, f_{i+2} \quad\Rightarrow$$

$$f_i^{(4)} = \frac{1}{\Delta x^4}(f_{i-2} - 4 f_{i-1} + 6 f_i - 4 f_{i+1} + f_{i+2}).$$ ←

Derive the Crank-Nicolson time integration iteration using (1) Taylor series of displacement $a(t)$ and velocity $\dot{a}(t)$ with respect to time and the mean value approximation to the remainder containing the second time derivative, and (2) differential equation $m\ddot{a}+ka=0$ written at the end points of the time interval of length Δt.

Solution

Taylor series of displacement $a(t)$ and velocity $\dot{a}(t)$ with respect to time with remainders containing the second time derivative are

$$a(t+\Delta t)=a(t)+\dot{a}(t)\Delta t+\frac{1}{2}\ddot{a}(\xi)\Delta t^2 \quad \text{and} \quad \dot{a}(t+\Delta t)=\dot{a}(t)+\ddot{a}(\xi)\Delta t\,,$$

where $\xi\in[t,t+\Delta t]$ is different in all its occurrences. Denoting the end points of the time interval $[t,t+\Delta t]$ and values of $a(t)$ by indices by $i-1$ and i and the mean value approximations to the second derivatives

$$a_i=a_{i-1}+\dot{a}_{i-1}\Delta t+\frac{1}{4}(\ddot{a}_i+\ddot{a}_{i-1})\Delta t^2 \quad \text{and} \quad \dot{a}_i=\dot{a}_{i-1}+\frac{1}{2}(\ddot{a}_i+\ddot{a}_{i-1})\Delta t\,.$$

Differential equation $m\ddot{a}+ka=0$ written at the end points of the time interval and index notation for the end points give

$$m\ddot{a}_{i-1}+ka_{i-1}=0 \quad \text{and} \quad m\ddot{a}_i+ka_i=0\,.$$

Eimination of second derivatives in the Taylor's series using the differential equations gives the forms

$$ma_i=ma_{i-1}+m\dot{a}_{i-1}\Delta t+\frac{1}{4}(m\ddot{a}_i+m\ddot{a}_{i-1})\Delta t^2=ma_{i-1}+m\dot{a}_{i-1}\Delta t-\frac{1}{4}(ka_i+ka_{i-1})\Delta t^2\,,$$

$$m\dot{a}_i=m\dot{a}_{i-1}+\frac{1}{2}(m\ddot{a}_i+m\ddot{a}_{i-1})\Delta t=m\dot{a}_{i-1}-\frac{1}{2}(ka_i+ka_{i-1})\Delta t\,.$$

In matrix form, containing a_i and $\dot{a}_i$ on the left-hand side and a_{i-1} and $\dot{a}_{i-1}$ on the right-hand side, the equations are

$$\begin{bmatrix} m+\frac{1}{4}k\Delta t^2 & 0 \\ \frac{1}{2}k\Delta t & \frac{1}{\Delta t}m \end{bmatrix}\begin{Bmatrix} a_i \\ \Delta t\dot{a}_i \end{Bmatrix}=\begin{bmatrix} m-\frac{1}{4}k\Delta t^2 & m \\ -\frac{1}{2}k\Delta t & \frac{1}{\Delta t}m \end{bmatrix}\begin{Bmatrix} a_{i-1} \\ \Delta t\dot{a}_{i-1} \end{Bmatrix}, \quad \alpha=\sqrt{\frac{k}{m}}\Delta t$$

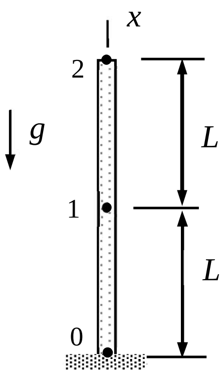
or after left multiplication by the inverse of the matrix on the left-hand side

$$\begin{Bmatrix} a_i \\ \Delta t\dot{a}_i \end{Bmatrix}=\begin{bmatrix} m+\frac{1}{4}k\Delta t^2 & 0 \\ \frac{1}{2}k\Delta t & m/\Delta t \end{bmatrix}^{-1}\begin{bmatrix} m-\frac{1}{4}k\Delta t^2 & m \\ -\frac{1}{2}k\Delta t & m/\Delta t \end{bmatrix}\begin{Bmatrix} a_{i-1} \\ \Delta t\dot{a}_{i-1} \end{Bmatrix}, \quad \alpha=\sqrt{\frac{k}{m}}\Delta t$$

which gives finally the iteration

$$\begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_i = \frac{1}{4+\alpha^2} \begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_{i-1}, \quad \alpha = \sqrt{\frac{k}{m}} \Delta t .$$

The bar shown is loaded by its own weight. Determine the displacements at the grid points 1 and 2 using the Finite Difference Method. Cross-sectional area A, Young's modulus E, and density ρ of the material are constants.



Solution

The discrete equations for the Finite Difference Method on a regular grid and stationary case are given

$$\frac{k}{\Delta x^2}(a_{i-1}-2a_i+a_{i+1})+f'=0 \quad \text{or} \quad \frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F=0,$$

$$a_0=\underline{a}_0 \quad \text{or} \quad \frac{k}{\Delta x}(a_1-a_0)+\underline{F}_0=0 \quad \text{and} \quad a_n=\underline{a}_n \quad \text{or} \quad \frac{k}{\Delta x}(a_{n-1}-a_n)+F_n=0.$$

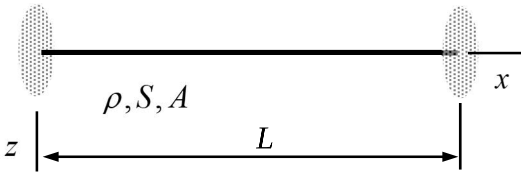
In the present case of a bar and regular grid with $i\in\{0,1,2\}$, $a=u$, $k=EA$, $f'=-\rho Ag$, and $\Delta x=L$, the equations for the grid points are

$$u_0=0, \quad \frac{EA}{\Delta x^2}(u_0-2u_1+u_2)-A\rho g=0, \quad \text{and} \quad \frac{EA}{\Delta x}(u_1-u_2)=0.$$

Solution to the displacements are ($\Delta x=L$)

$$u_0=0, \; u_1=u_2=-\frac{\rho g L^2}{E}.$$ ←

Consider the string of tightening S and mass per unit length ρA shown. Use the Finite Difference Method with second order accurate central differences on a regular grid $i \in \{0,1,\ldots,n\}$ to find the angular velocities ω_k of the free vibrations using the solution trial $w_i = a(t)\sin(k\pi i/n)$ $k \in \{1,2\ldots,n-1\}$.



Solution

The trial solution is chosen in such manner that the zero displacement conditions at the end points are satisfied 'a priori'. Let us use the difference equation for the generic point inside the domain $i \in \{1,2\ldots,n-1\}$ (no external forces)

$$\frac{S}{\Delta x^2}(w_{i-1} - 2w_i + w_{i+1}) = \rho A \ddot{w}_i$$

to deduce the expression for $a(t)$ which is the unknown of the solution trial. Substituting the trial solution and using the trigonometric identity $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ (or letting Mathematica to do the manipulation)

$$w_{i-1} - 2w_i + w_{i+1} = a(t)\sin(k\pi\frac{i-1}{n}) - 2a(t)\sin(k\pi\frac{i}{n}) + a(t)\sin(k\pi\frac{i+1}{n}) \quad \Leftrightarrow$$

$$w_{i-1} - 2w_i + w_{i+1} = a(t)\sin(k\pi\frac{i}{n})2[-1+\cos(\frac{k\pi}{n})],$$

$$\ddot{w}_i = \ddot{a}(t)\sin(k\pi\frac{i}{n}).$$

Therefore, the difference equation for the generic point takes the form

$$[\ddot{a}(t) + \omega_k^2 a(t)]\sin(k\pi\frac{i}{n}) = 0 \quad \text{where} \quad \omega_k^2 = \frac{S}{\rho A \Delta x^2} 2[1-\cos(\frac{k\pi}{n})].$$

The equation to hold, $a(t)$ should be the solution to $\ddot{a}(t) + \omega_k^2 a(t) = 0$ which is a linear combination of $\sin(\omega_k t)$ and $\cos(\omega_k t)$. As a conclusion, the angular velocity of the free vibration

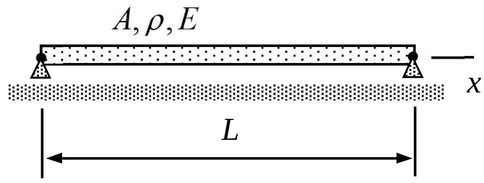
$$\omega_k = \frac{1}{L} n \sqrt{2\frac{S}{\rho A}[1-\cos(\frac{k\pi}{n})]} = \frac{2}{L}\sqrt{\frac{S}{A\rho}} n \sin(\frac{k\pi}{2n}). \quad \leftarrow$$

Let us note that, in the limit $n \to \infty$,

$$\omega_k = \frac{k\pi}{L}\sqrt{\frac{S}{A\rho}},$$

which is the angular velocity given by the continuum model.

A bar is free to move in the horizontal direction as shown. Write the equation system $\mathbf{Ka}+\mathbf{M}\ddot{\mathbf{a}}=0$ given by the Finite Difference Method on a regular grid with $i\in\{0,1,2\}$. Also, determine the angular velocities and modes of the free vibrations. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Solution

The generic equation set for the model problems and the Finite Difference Method on a regular grid with the simplest possible difference approximations to the derivatives is given by

$$\frac{k}{\Delta x^2}(a_{i-1}-2a_i+a_{i+1})+f'=m'\ddot{a}_i \quad \text{or} \quad \frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F=0 \qquad t>0$$

$$a_0=\underline{a}_0 \;\text{ or }\; \frac{k}{\Delta x}(a_1-a_0)+F_0=0 \;\text{ and }\; a_n=\underline{a}_n \;\text{ or }\; \frac{k}{\Delta x}(a_{n-1}-a_n)+F_n=0 \qquad t>0$$

$$a_i=g_i \;\text{ and }\; \dot{a}_i=h_i \qquad t=0$$

In the bar application external forces vanish, $k=EA$, $m'=\rho A$, and $\Delta x=L/2$. Initial conditions do not matter in modal analysis. Equations for $i\in\{0,1,2\}$ simplify to

$$u_1-u_0=0\,,\quad \frac{EA}{\Delta x^2}(u_0-2u_1+u_2)=\rho A\ddot{u}_1\,,\quad a_1-a_2=0$$

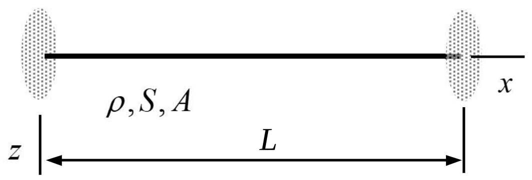
In solution methods for time dependent problem, algebraic equations are used to eliminate the displacements of the boundary points from the differential equation, which simplifies to

$$\ddot{u}_1=0\,.$$

Using the solution trial of the modal analysis $u_1=Ae^{\mathrm{i}\omega t}$ gives the angular velocity value $\omega=0$ the corresponding mode being $A=1$ (say), so trial gives $u_1=1$ (not the solution to the problem but solution to the mode) and then with use of the algebraic equations $u_0=u_1=u_2=1$ so

$$(\omega,A)=(0,\begin{Bmatrix}1\\1\\1\end{Bmatrix})\,.$$

Consider the string of tightening S and mass per unit length ρA shown. First, use the Finite Difference Method with the second order accurate central differences on a regular grid $i \in \{0,1,2\}$ to find the equations of motion of the form $ka + m\ddot{a} = 0$. Second, write the iteration equation for a typical time-step of size Δt according to Crank-Nicolson method giving the values of displacement and velocity on the temporal grid.



Solution

The generic equation set for the model problems and the Finite Difference Method on a regular grid with the simplest possible difference approximations to the derivatives is given by

$$\frac{k}{\Delta x^2}(a_{i-1} - 2a_i + a_{i+1}) + f' = m'\ddot{a}_i \quad \text{or} \quad \frac{k}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) + F = 0 \quad t > 0$$

$$a_0 = \underline{a}_0 \;\text{ or }\; \frac{k}{\Delta x}(a_1 - a_0) + F_0 = 0 \;\text{ and }\; a_n = \underline{a}_n \;\text{ or }\; \frac{k}{\Delta x}(a_{n-1} - a_n) + F_n = 0 \quad t > 0$$

$$a_i = g_i \;\text{ and }\; \dot{a}_i = h_i \quad t = 0$$

In the string application external forces vanish, $k = S$, $m' = \rho A$, and $\Delta x = L/2$. Initial conditions do not matter in modal analysis. Equations for $i \in \{0,1,2\}$ simplify to

$$w_0 = 0, \; \frac{S}{\Delta x^2}(w_0 - 2w_1 + w_2) = \rho A \ddot{w}_1, \text{ and } w_2 = 0 \quad t > 0$$

$$w_1 = g_1 \;\text{ and }\; \dot{w}_1 = h_1 \quad t = 0$$

In solution methods for time dependent problem, algebraic equations are used to eliminate the displacements of the boundary points from the differential equation, so the initial value problem simplifies to

$$2\frac{S}{\Delta x^2} w_1 + \rho A \ddot{w}_1 = 0 \quad t > 0, \; w_1 = g \;\text{ and }\; \dot{w}_1 = h \quad t = 0.$$

With definition $w_1 = a$, time integration by Crank-Nicolson method is given by iteration

$$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \frac{1}{4+\alpha^2}\begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}_0 \quad \text{where } \alpha = 2\sqrt{2\frac{S}{\rho A}}\frac{\Delta t}{L}.$$ ←