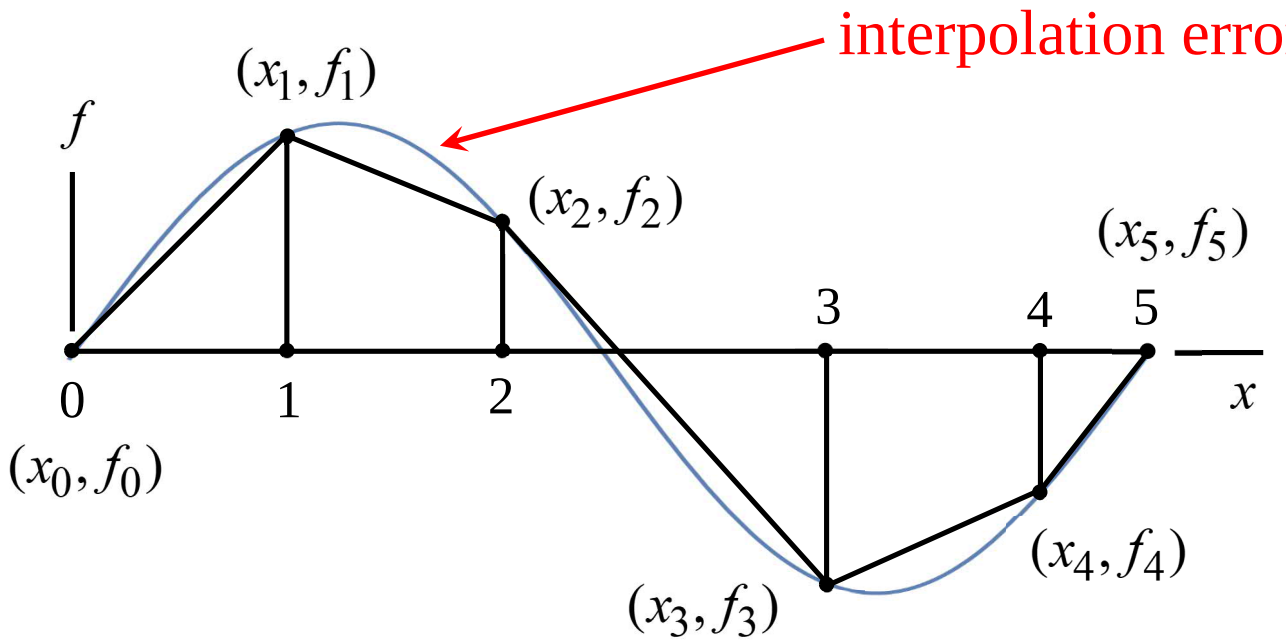
4 FINITE ELEMENT METHOD

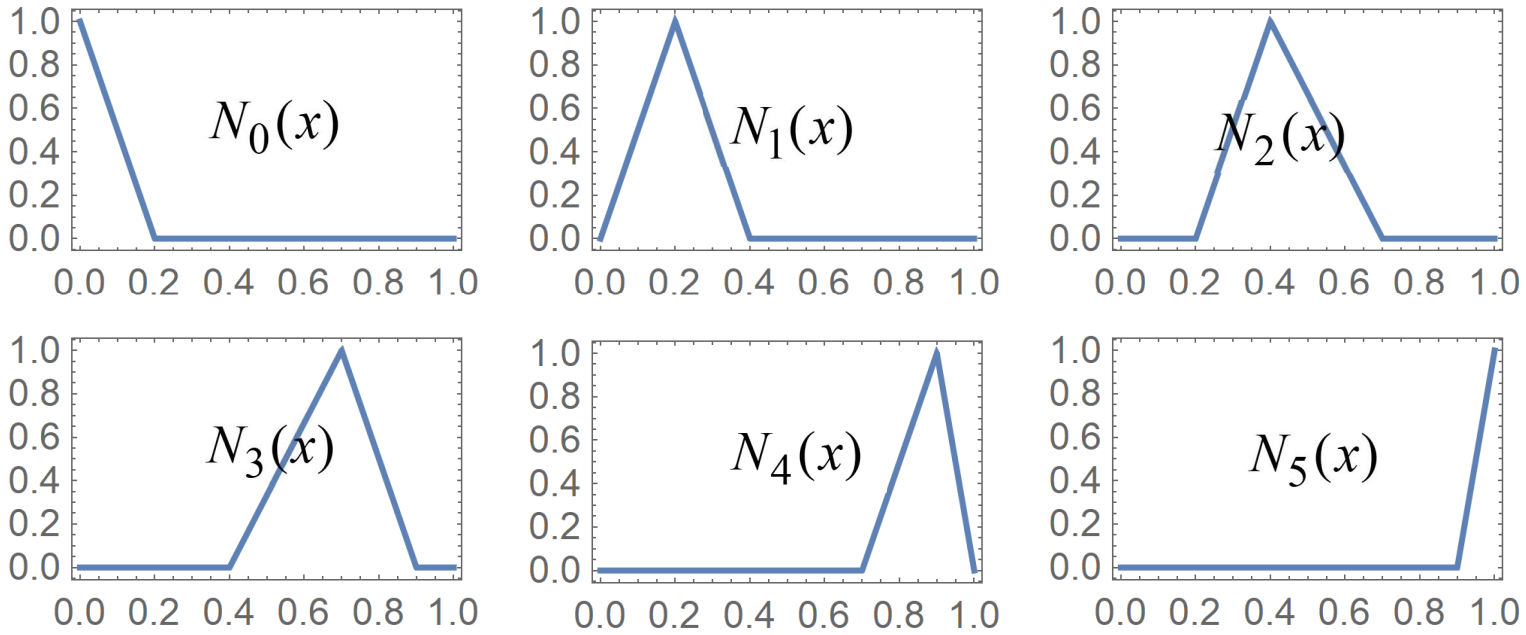
4.1 INTERPOLANT AND APPROXIMATION

Piecewise linear interpolant $p(x)$ to dataset $\{(x_0, f_0),(x_1, f_1),\ldots,(x_n, f_n)\}$ gives the simplest continuous polynomial representation. Assuming that the dataset is sampling of a continuous $f(x)$, $p(x)$ can also be considered as an approximation to $f(x)$.



SHAPE FUNCTIONS

In the Finite Element Method, linear interpolants to datasets $\{(x_0,\delta_{0i}),(x_1,\delta_{1i}),\ldots,(x_n,\delta_{ni})\}$, where $i\in\{0,1,\ldots,n\}$ and δ_{ji} is the Kronecker delta defined as $\delta_{ji}=1$ when $i=j$ and $\delta_{ji}=0\ \ i\neq j$, are called as the linear shape functions $N_i(x)\ \ i\in\{0,1,\ldots,n\}$. As an example, the shape functions on the irregular grid $\{0,3,4,7,9,10\}/10$ of 6 nodes on $\Omega=[0,1]$ are:



The datasets $\{(x_0,\delta_{0i}),(x_1,\delta_{1i}),\ldots,(x_n,\delta_{ni})\}$ $i\in\{0,1,\ldots,n\}$ correspond to value 1 for grid point i the remaining being zeros. The $n+1$ interpolants to the $n+1$ dataset are denoted $N_i(x)$ and called as the shape functions. With this concept the linear interpolant to dataset $\{(x_0,f_0),(x_1,f_1),\ldots,(x_n,f_n)\}$ is given by

$$p(x)=\sum_{i\in\{0,1,\ldots,n\}} f_i N_i(x)$$

where f_i $i\in\{0,1,\ldots,n\}$ are the values at the grid points. In a typical line segment $[x_i,x_j]$, only the shape functions of grid points i and j are non-zeros the shape function and the interpolant expressions being ($x\in[x_i,x_j]$)

$$\begin{Bmatrix} N_i(x) \\ N_j(x) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_i & x_j \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} \text{ and } p(x)=\begin{Bmatrix} f_i \\ f_j \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} N_i(x) \\ N_j(x) \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ x_i & x_j \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix}.$$

4.2 WEIGHTED RESIDUAL APPROXIMATION

Finding an approximation $g(x)$ to function $f(x)$ is one the basic tasks in numerical mathematics. In the Least Squares Method and Weighted Residual Methods, the grid point values g_i of approximation $g(x) = \sum g_i N_i(x) = \mathbf{N}^{\mathbf{T}}\mathbf{g}$ follow from the steps

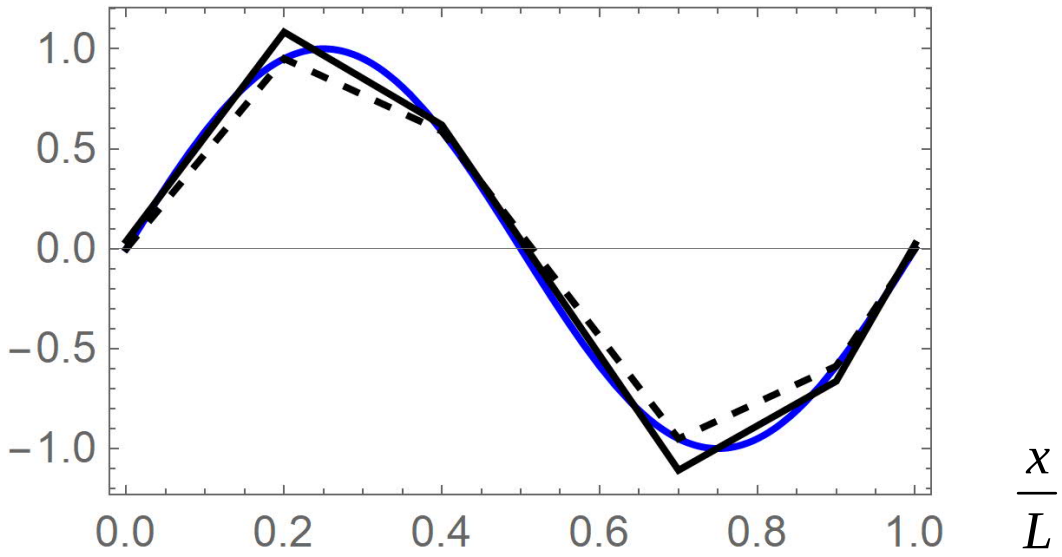
Distance: $\Pi(\mathbf{g}) = \frac{1}{2}\int_0^L (g - f)^2 dx = \frac{1}{2}\int_0^L (\mathbf{N}^{\mathrm{T}}\mathbf{g} - f)^2 dx,$

Minimizer: $\mathbf{Kg} - \mathbf{F} = \mathbf{0}$ where $\mathbf{K} = \int_0^L \mathbf{N}\mathbf{N}^{\mathrm{T}} dx$ and $\mathbf{F} = \int_0^L \mathbf{N} f dx,$

Nodal values: $\mathbf{g} = \mathbf{K}^{-1}\mathbf{F}.$

In practice, the nodal values $\mathbf{g}$ are solved from the linear equation system without matrix inversion (to avoid excess computational work). The method works in the same manner irrespective of the series approximation used.

Interpolant $p(x)$ to function $f(x)$ is accurate on the grid points but the interpolation error at the other points is not under control. Least squares method considers all points of the domain and control the error everywhere as well as possible.



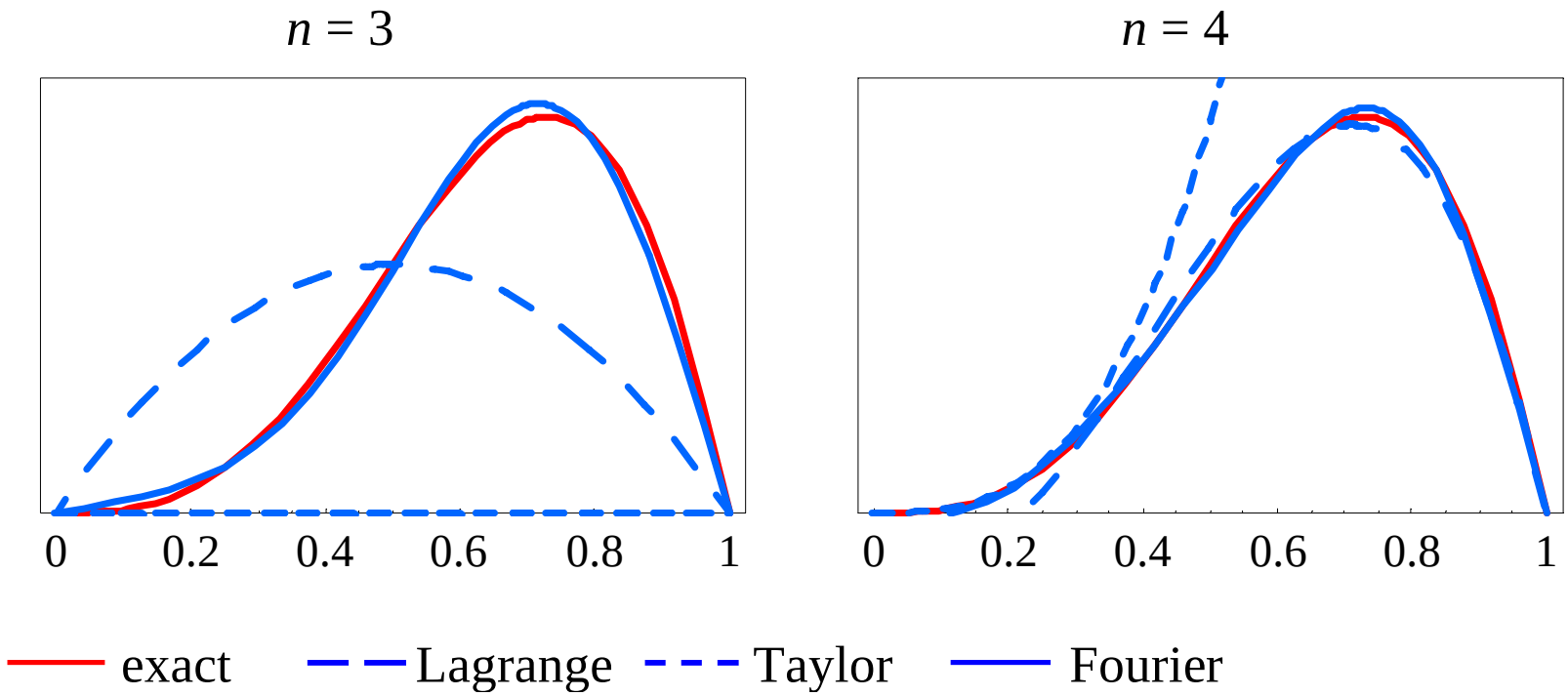
The figure above compares the interpolation (broken-black) and approximation (solid-black) of $f(x) = \sin(2\pi x / L)$ (solid-blue) on irregular grid $x / L \in \{0, 2, 4, 7, 9, 10\} / 10$ of 6 nodes on $\Omega = [0, L]$. Judging from the figure, approximation $g(x)$ gives a better fit to $f(x)$ than interpolant $p(x)$.

Least Squares Method is useful in various tasks in numerical mathematics. One of the applications is related with the condition for the minimum of Π, which can be written in the form

$$\int_0^L N_i R dx = 0 \quad i \in \{0,1,\ldots\}$$

where $R = g(x) - f(x)$ is called as the residual. In the weighted residual interpretation of the method, linear equations giving the values of the approximation are obtained as the weighted residuals with the shape functions. The idea extends to residuals of differential equations and is one of the starting points for the Finite Element Method for bar and string model problems.

EXAMPLE Find an approximation to $f(x) = 5x^2 \sin(\pi x)$ $x \in [0,1]$ by using: (a) Lagrange interpolation polynomial (n evenly spaced points), (b) Taylor series (at $x = 0$ and n terms), and (c) Fourier sine-series (n terms). Consider the cases $n = 3$ and $n = 4$.



Answer $f(x)$ and the approximations are shown in the figure.

Lagrange interpolation polynomial is continuous and it coincides with a function on a given set of points

recipe!

$$p = \mathbf{N}^{\mathrm{T}}\mathbf{p} \quad \text{where} \quad N_i = \Pi_{j\in\{1\ldots n\}\setminus i}\frac{x_j - x}{x_j - x_i} \quad \text{and} \quad p_i = f(x_i)$$

For the given set of points $\mathbf{x} \in \{0,1,2\}/2$:

$$\mathbf{p} = \{0 \;\; 5/4 \;\; 0\}^{\mathrm{T}} \text{ and } \mathbf{N}^{\mathrm{T}} = \{\frac{(1/2-x)(1-x)}{(1/2-0)(1-0)} \quad \frac{(0-x)(1-x)}{(0-1/2)(1-1/2)} \quad \frac{(0-x)(1/2-x)}{(0-1)(1/2-1)}\}$$

$$p(x) = \mathbf{N}^{\mathrm{T}}\mathbf{p} = 0 + \frac{5}{4}\frac{(0-x)(1-x)}{(0-1/2)(1-1/2)} + 0 = 5(x - x^2).$$

The given (shape) functions can be taken as monomials like $N_i = x^{(i-1)}$ without affecting the approximation, but then the recipe is worse from the numerical viewpoint.

A truncated Taylor series is a continuous polynomial whose derivatives coincide with those of the given function up to a point.

recipe!

$$t(x) = \mathbf{N}^{\mathrm{T}}\mathbf{t} \quad \text{where} \quad N_i = \frac{1}{i!}(x - x_0)^{(i)} \quad \text{and} \quad t_i = \left(\frac{d^i f}{dx^i}\right)_{x=x_0}$$

If the number of terms is chosen to be 3 and $x_0 = 0$,

$$\mathbf{t} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{and} \quad \mathbf{N} = \begin{Bmatrix} 1 \\ x \\ x^2/2 \end{Bmatrix} \quad \text{so} \quad t(x) = \mathbf{N}^{\mathrm{T}}\mathbf{t} = 0 + 0 + 0 = 0.$$

The given functions can be chosen in some other way without affecting the approximation, but then the recipe is not as good.

Sine-series is a linear combination of sine-functions where the coefficient are given by the Least-Squares Method (or Weighted Residuals)

recipe!

$$s(x) = \mathbf{N}^{\mathrm{T}}\mathbf{s} \quad \text{where} \quad N_i = \sqrt{\frac{2}{L}}\sin(i\pi\frac{x}{L}) \quad \text{and} \quad \mathrm{s}_i = \int_0^L f N_i dx$$

If the number of terms is chosen as 3, the series becomes

$$\mathbf{s} = \frac{\sqrt{2}}{\pi^2}\{5(\frac{\pi^2}{2} - \frac{1}{4}) \quad -\frac{40}{9} \quad \frac{15}{16}\}^{\mathrm{T}} \quad \text{and} \quad \mathbf{N}^{\mathrm{T}} = \sqrt{2}\{\sin(\pi x) \quad \sin(2\pi x) \quad \sin(3\pi x)\}$$

$$s(x) = \mathbf{N}^{\mathrm{T}}\mathbf{s} = \frac{2}{\pi^2}[5(\frac{\pi^2}{2} - \frac{1}{4})\sin(\pi x) - \frac{40}{9}\sin(2\pi x) + \frac{15}{16}\sin(3\pi x)].$$

The criterion for the parameters produces a good approximation. Clearly, considering all points of the domain in the recipe may be a good idea!

APPROXIMATIONS TO DERIVATIVES

Linear interpolation and shape functions give an alternative way to find difference stencils for derivatives at the interior points $i \in \{1, 2, \ldots, n-1\}$. The weighted average of a derivative using N_i is just interpreted as an approximation to derivative at point i (multiplied by Δx)

Term	Weighted residual	Stencil
a	$\int_0^L N_i a dx$	$\frac{\Delta x}{6}(a_{i-1} + 4a_i + a_{i+1})$
$\frac{\partial a}{\partial x}$	$\int_0^L N_i \frac{\partial a}{\partial x} dx$	$\frac{1}{2}(a_{i+1} - a_{i-1})$
$\frac{\partial^2 a}{\partial x^2}$	$-\int_0^L \frac{\partial N_i}{\partial x}\frac{\partial a}{\partial x} dx$	$\frac{1}{\Delta x}(a_{i-1} - 2a_i + a_{i+1})$

The stencils by the weighted residuals with N_i and linear interpolant to a on the regular grid coincide with the 2:nd order accurate central finite differences for the first and second derivatives. For the 0:th derivative (function itself)

$$\int_0^L N_0 a dx = \int_{x_0}^{x_0+\Delta x} N_0 a dx = \frac{\Delta x}{6}(2a_0 + a_1).$$

$$\int_0^L N_i a dx = \int_{x_i-\Delta x}^{x_i+\Delta x} N_i a dx = \frac{\Delta x}{6}(a_{i-1} + 4a_i + a_{i+1}) \quad i \in \{1, 2, \ldots, n-1\},$$

$$\int_0^L N_n a dx = \int_{x_0}^{x_0+\Delta x} N_n a dx = \frac{\Delta x}{6}(a_{n-1} + 2a_n).$$

For the first derivative

$$\int_0^L N_0 \frac{\partial a}{\partial x} dx = \int_{x_0}^{x_0+\Delta x} N_0 \frac{\partial a}{\partial x} dx = \frac{1}{2}(a_0 - a_1)$$

$$\int_0^L N_i \frac{\partial a}{\partial x} dx = \int_{x_i-\Delta x}^{x_i+\Delta x} N_i \frac{\partial a}{\partial x} dx = \frac{1}{2}(a_{i+1} - a_{i-1}) \quad i \in \{1, 2, \ldots, n-1\},$$

$$\int_0^L N_n \frac{\partial a}{\partial x} dx = \frac{1}{2}(a_n - a_{n-1}).$$

Weighted residual approximation to the second derivative uses an integral identity for a piecewise linear $a(x)$

$$-\int_0^L \frac{\partial N_i}{\partial x}\frac{\partial a}{\partial x} dx = -\int_{x_i-\Delta x}^{x_i+\Delta x} (\frac{\partial N_i}{\partial x}\frac{\partial a}{\partial x}) dx = \frac{1}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) \quad i \in \{1, 2, \ldots, n-1\}.$$

The derivation of the bar and string models indicate that at points where derivative is not continuous, the second derivative in the equation of motion needs to be replaced by a jump condition. Integral identity

$$\sum_{i \in \{1,2,\ldots,n-1\}} N_i [\![\frac{\partial a}{\partial x}]\!] + N_0 (\frac{\partial a}{\partial x})_0 - N_n (\frac{\partial a}{\partial x})_n = -\int_0^L \frac{\partial N_i}{\partial x}\frac{\partial a}{\partial x} dx$$

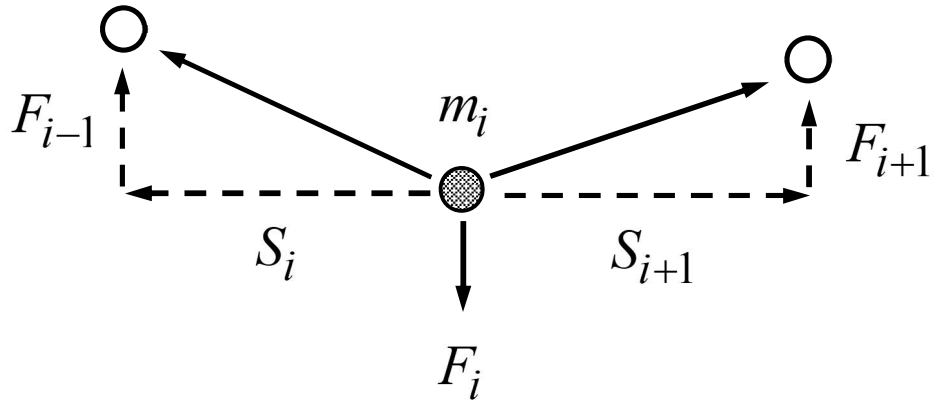
indicates that the jump conditions can be calculated as a weighted average using the derivative of the shape function as the weight. Jump bracket is a shorthand notation for the difference of the limit values $[[a(x_i)]] = \lim_{\varepsilon \to 0}[a(x_i + \varepsilon) - a(x_i - \varepsilon)]$.

4.3 PRINCIPLE OF VIRTUAL WORK

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{ine}} = 0 \quad \forall \delta a$ is an alternative representation of the equations of Particle Surrogate Method using the concept of work of forces acting on a particle. The principle holds also at the limit $n \to \infty$ and $\Delta x = L / n$:

Virtual work	**Particle**	**Continuum**
δW^{int}	$-\sum_{e \in P} \left(\frac{\Delta \delta a_e}{\Delta x} k' \frac{\Delta a_e}{\Delta x}\right) \Delta x$	$-\int_{\Omega} \left(\frac{\partial \delta a}{\partial x} k' \frac{\partial a}{\partial x}\right) dx$
δW^{ext}	$\sum_{i \in I} \left(\delta u_i \frac{F_i}{\Delta x}\right) \Delta x$	$\int_{\Omega} (\delta a f') dx + \sum_I \delta a F$
δW^{ine}	$-\sum_{i \in I} \left(\delta a_i \frac{m}{\Delta x} \ddot{a}_i\right) \Delta x$	$-\int_{\Omega} \left(\delta a m' \frac{\partial^2 a}{\partial t^2}\right) dx$

Let us start with the Particle Surrogate Method for a string with $i \in I = \{0,1,\ldots,n\}$, denote particle spacing by Δx, interacting particle pairs $p \in P \subset I \times I$, and choose $\delta w_i = 0$ whenever $w_i = \underline{w}_i$ (known), and denote $\Delta w_p = w_i - w_j$ when $p = (i, j)$.



The sum of works done by the internal, external and inertia forces on the displacement δw_i (fixed particles cannot move so $\delta w_i = 0$)

External forces: $\delta W^{\text{ext}} = \sum_{i \in I} \delta w_i (F_i + \rho A \Delta x g)$

Inertia forces: $\delta W^{\text{ine}} = -\sum_{i \in I} \delta w_i m_i \ddot{w}_i$

Internal forces: $\delta W^{\text{int}} = -\sum_{p \in P} \delta \Delta w_p \frac{S}{\Delta x} \Delta w_p$

External part is obvious as the sum of works of external forces acting on the particles. Inertia part uses the inertia force interpretation of acceleration term which is moved to the left hand side of (formally) equilibrium equations of particles. The internal part follows with some manipulations: particle i interacts with the neighbors $i-1$ and $i+1$ only. Therefore, virtual work of the internal forces (all particles accounted for as possible conditions $\delta w_i = 0$ can be applied after manipulations)

$$-\delta W^{\text{int}} = \delta w_0 F_1 + \delta w_1 (F_0 + F_3) + \delta w_2 (F_1 + F_3) + \ldots + \delta w_n F_{n-1}.$$

Substituting expressions $F_{i-1} = S(w_i - w_{i-1})/\Delta x$ and $F_{i+1} = S(w_i - w_{i+1})/\Delta x$ for the left and right neighbour interactions and rearranging

$$-\delta W^{\text{int}} \frac{\Delta x}{S} = (\delta w_1 - \delta w_0)(w_1 - w_0) + \ldots + (\delta w_n - \delta w_{n-1})(w_n - w_{n-1})$$

and, finally, using the notation $\Delta w_p = w_i - w_j$

$$\delta W^{\text{int}} = -\sum_{p \in P} \delta \Delta w_p \frac{S}{\Delta x} \Delta w_p . \quad \leftarrow$$

Principle of virtual work and the virtual work expressions give a concise reprentation of the string and bar equations of the Particle Surrogate Method. Various different boundary conditions can be included by modification of the expression using the physical work interpretation.

Finite Element Method uses the linit expressions $n \to \infty$ and $\Delta x = L / n$ corresponding to the continuum model. There, linear shape functions are used for a piecewise linear approximation and the weight function is chosen to be the shape functions of the free points get the equation of the motion. At the fixed points, equation $w_i - \underline{w}_i = 0$ is used instead.

4.4 FINITE ELEMENT METHOD

Finite Element Method (FEM) is a numerical technique for solving differential equations. If applied to the model problems on a regular grid of points on the spatial domain

Interior $\frac{k}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) + F_i + f'\Delta x = m'\frac{\Delta x}{6}(\ddot{a}_{i-1} + 4\ddot{a}_i + \ddot{a}_{i+1}) \quad i \in \{1, 2, \ldots, n-1\}$

Boundary $\frac{k}{\Delta x}(a_1 - a_0) + F_0 + \frac{\Delta x}{2} f' - m'\frac{\Delta x}{6}(2\ddot{a}_0 + \ddot{a}_1) = 0$ or $a_0 = \underline{a}_0,$

Boundary $\frac{k}{\Delta x}(a_{n-1} - a_n) + F_n + \frac{\Delta x}{2} f' - m'\frac{\Delta x}{6}(2\ddot{a}_n + \ddot{a}_{n-1}) = 0$ or $a_n = \underline{a}_n,$

Initial $a_i - g_i = 0$ and $\dot{a}_i - h_i = 0$ (for ODE:s of the set).

Then, the outcome is a set of ordinary differential equations of the same type as by the Particle Surrogate Method.

In FEM, the starting point is the weighted residual expression implied by the principle of virtual work. Using a linear piecewise approximation to the transverse displacement in spatial grid, selecting $\delta a = N_i$, considering the displacement values $w_i(t)$ as functions of time, and assuming constant properties, for $i \in \{0,1,\ldots,n\}$

$$-\int_\Omega \left(\frac{\partial N_i}{\partial x} S \frac{\partial w}{\partial x}\right) dx + \sum_I N_i F - \int_\Omega N_i \rho A \left(g - \frac{\partial^2 w}{\partial t^2}\right) dx = 0 \quad \text{or} \quad w_i - \underline{w} = 0 \quad t > 0,$$

$$\int_\Omega N_i (w - g) dx = 0 \quad \text{and} \quad \int_\Omega N_i \left(\frac{\partial w}{\partial t} - h\right) dx = 0 \quad t = 0.$$

The integral equation and the corresponding initial conditions are used at all points where displacement is not known. Considering a regular grid, a piecewise linear approximation to w in terms of the nodal displacements, and assuming that g and h of initial conditions are of the same form as the approximation (tacitly accepting the possible interpolation error due to the simplification).

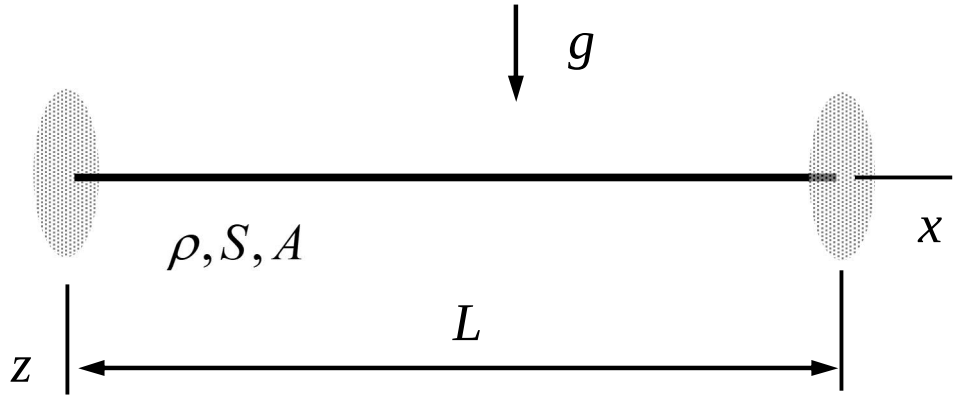
$$\frac{S}{\Delta x}(w_{i-1} - 2w_i + w_{i+1}) + \rho A g \Delta x = \rho A \frac{\Delta x}{6}(\ddot{w}_{i-1} + 4\ddot{w}_i + \ddot{w}_{i+1}) \quad i \in \{1, 2, \ldots, n-1\},$$

$$S(\frac{w_1 - w_0}{\Delta x}) + F_0 + \frac{\Delta x}{2}\rho A g - \rho A \frac{\Delta x}{6}(2\ddot{w}_0 + \ddot{w}_1) = 0 \quad \text{or} \quad w_0 = \underline{w}_0,$$

$$S(\frac{w_{n-1} - w_n}{\Delta x}) + F_n + \frac{\Delta x}{2}\rho A g - \rho A \frac{\Delta x}{6}(2\ddot{w}_n + \ddot{w}_{n-1}) = 0 \quad \text{or} \quad w_n = \underline{w}_n,$$

$w_i - g_i = 0$ and $\dot{w}_i - h_i = 0$ (for Ordinary Differential Equations).

EXAMPLE A string of length L, tightening S, cross section area A, and density ρ, and is loaded by its own weigh. If the ends are fixed and the initial geometry without loading is straight, find the solution to the transverse displacement by using the Finite Element Method and a regular grid of points $i \in \{0,1,\ldots,n\}$.



Answer $w_i = \dfrac{\rho A g L^2}{2S} \dfrac{i(n-i)}{n^2}$ (limit solution $w(x) = \dfrac{\rho A g}{2S} x(L-x)$)

The algebraic equations according to the Finite Element Method, n elements of the same size $\Delta x = L/n$, and regular node numbering $i = \{0,1,\ldots,n\}$ are given by

$$\frac{S}{\Delta x}(w_{i-1} - 2w_i + w_{i+1}) + \rho A g \Delta x = 0 \quad i = \{1,2\ldots,n-1\} \quad \text{and} \quad w_i = 0 \quad i = \{0,n\}.$$

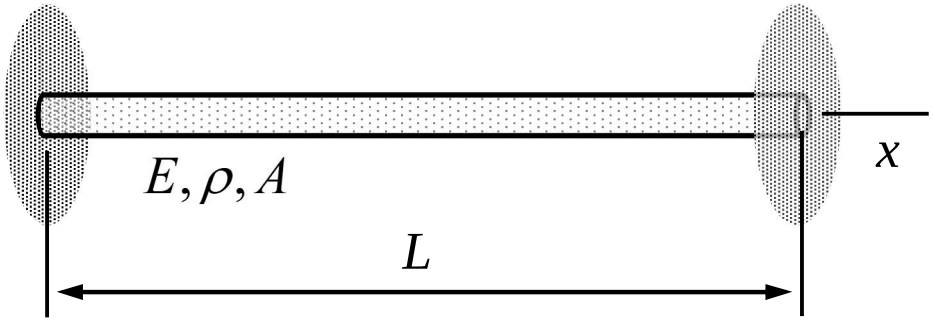
The generic solution to the difference equations consists of the generic solution to the homogeneous equation (trial solution $w_i = ar^i$) and a particular solution (trial solution $w_i = ai^2$):

$$w_i = a + bi - \frac{\rho A g \Delta x^2}{2S} i^2.$$

Using the equations of the boundary points $w_0 = w_n = 0$ for the constants a and b

$$w_i = \frac{\rho A g L^2}{2S} \frac{i(n-i)}{n^2}.$$

EXAMPLE Write the equations of motion for the free vibrations of the bar shown by using the Finite Element Method and determine the frequencies and the corresponding modes of free vibrations. Use the matrix formulation on a regular grid $i \in \{0,1,2,3\}$. Material properties E, ρ and cross-sectional area A are constants.



Answer $(f_1, \mathbf{A}_1) = (\frac{1}{2\pi}\sqrt{\frac{6}{5}\frac{k}{m}}, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix})$ and $(f_2, \mathbf{A}_2) = (\frac{1}{2\pi}\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix})$

In bar problem $k = EA$, $m' = \rho A$, and external forces vanish. Equations for the points $i \in \{0,1,2,3\}$ are $u_0 = 0$ and $u_3 = 0$

$$\frac{EA}{\Delta x}(u_0 - 2u_1 + u_2) - \rho A \Delta x \frac{1}{6}(4\ddot{u}_0 + 4\ddot{u}_1 + \ddot{u}_2) = 0 \ ,$$

$$\frac{EA}{\Delta x}(u_1 - 2u_2 + u_3) - \rho A \Delta x \frac{1}{6}(4\ddot{u}_1 + 4\ddot{u}_2 + \ddot{u}_3) = 0$$

In matrix notation, when the known displacements at the boundary points $i \in \{0,3\}$ are used to simplify equations of points $i \in \{1,2\}$

$$\frac{EA}{\Delta x}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \rho A \Delta x \frac{1}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = 0 .$$

With the trial solution $\mathbf{u} = \mathbf{A}e^{\mathrm{i}\omega t}$, the algebraic equations for the angular velocities ω and the corresponding modes $\mathbf{A}$ takes the form

$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\right)\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{where} \quad \lambda = \omega^2 \frac{\rho \Delta x^2}{6E}.$$

The possible values of λ are given by

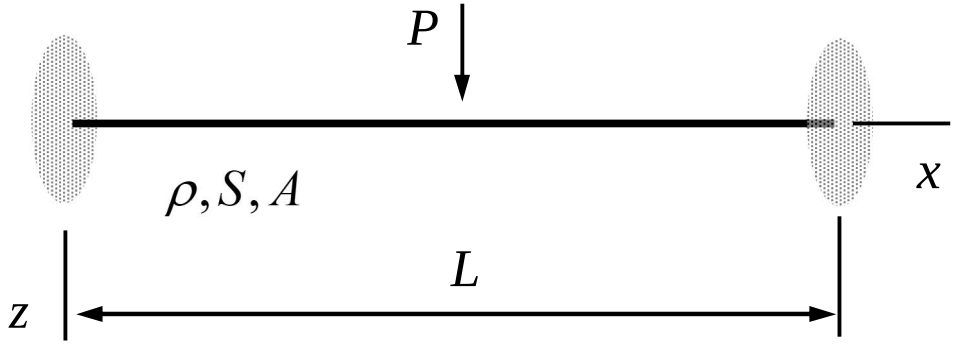
$$\det\begin{bmatrix} 2-4\lambda & -1-\lambda \\ -1-\lambda & 2-4\lambda \end{bmatrix} = (2-4\lambda)^2 - (1+\lambda)^2 = 0 \quad \Leftrightarrow \quad \lambda = \frac{1}{5} \quad \text{or} \quad \lambda = 1.$$

The corresponding modes follow from the linear equations when the values of λ are substituted there one at a time

$$\lambda_1 = \frac{1}{5}: \left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\right)\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \frac{6}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{A}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\lambda_2 = 1: \left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 1\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\right)\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 2\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{A}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}.$$

EXAMPLE A string of length L, tightening S, cross-sectional area A, and density ρ, is loaded by a point force P at its center point. If the ends are fixed and the initial geometry without loading is straight, find the solution to the transverse displacement as function of x using the Finite Element Method on a regular grid of three points $i \in \{0,1,2\}$.



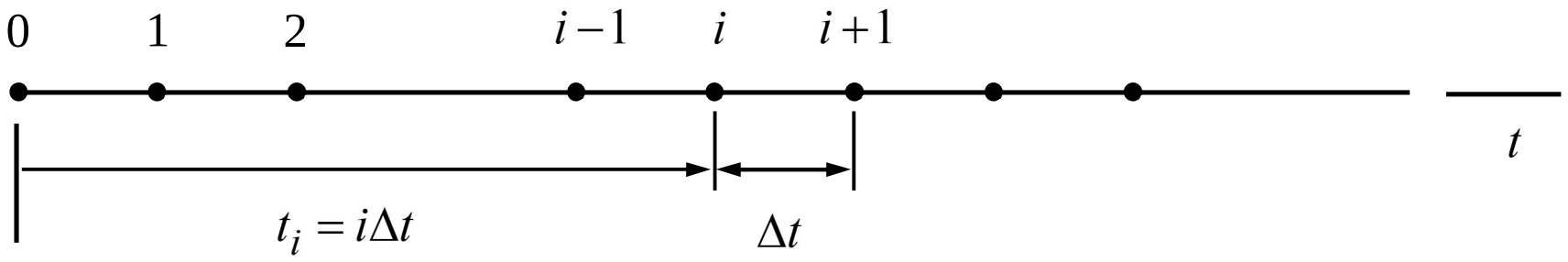
Answer $w_1 = \dfrac{PL}{4S}$

In the string problem $a_i = w_i$, $k = S$, $F_1 = P$, and $\Delta x = L/2$. The equations by the Finite Element Method considers also the possibility of points forces. Therefore

$$\frac{S}{\Delta x}(w_0 - 2w_1 + w_2) + P = 0, \quad w_0 = 0, \text{ and } w_2 = 0 \quad \Rightarrow \quad w_1 = \frac{PL}{4S}.$$

4.5 TIME INTEGRATION (DG)

In the one-step DG (Discontinuous Galerkin) method, the solution is sought step-by-step in the same manner as with the CN (Crank-Nicolson) method. Derivation of the method is, however, based on a polynomial approximation and the weighted residuals for the differential equations.



As the temporal domain for an initial value problem does not have an upper bound (strictly speaking). Also, the length of the intervals can be chosen to match the behavior of the solution (small steps for the rapid changes).

TIME INTEGRATION

Method	Iteration $i \in \{1,2,\ldots\}$	Initial $i = 0$

EX

$$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \begin{bmatrix} \cos\alpha & \alpha^{-1}\sin\alpha \\ -\alpha\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1} \qquad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta th \end{Bmatrix}$$

CN

$$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \frac{1}{4+\alpha^2} \begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1} \qquad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta th \end{Bmatrix}$$

DG

$$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \frac{2}{12+\alpha^4} \begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1} \qquad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta th \end{Bmatrix}$$

The methods coincide at the limit of vanishing step-size when $\alpha = \sqrt{\frac{k}{m}}\Delta t \to 0$.

MATRIX REPRESENTATION

Representing the unknown displacements by column matrix $\mathbf{a}(t)$, coefficients of $\ddot{\mathbf{a}}(t)$ by square mass matrix $\mathbf{M}$, coefficients of $\mathbf{a}(t)$ by square stiffness matrix $\mathbf{K}$, and the external loading by column matrix $\mathbf{F}$, the second order initial value problems by Particle Surrogate Method, Finite Difference Method, and the Finite Element Method are given by

$$-\mathbf{Ka}+\mathbf{F}=\mathbf{M}\ddot{\mathbf{a}} \quad t>0 \quad \mathbf{a}=\mathbf{g} \text{ and } \dot{\mathbf{a}}=\mathbf{h} \quad t=0.$$

The column matrices $\mathbf{g}$ and $\mathbf{h}$ represent the initial positions and velocities of the free particles. Matrix representation is the concise starting point for

(1) mode analysis for frequencies and modes of free vibrations

(2) displacement solutions based on the frequencies and modes

(3) step-by-step time integration methods on temporal grid of time instants

DERIVATION OF THE METHOD

Polynomial approximation $\mathbf{a}(t) = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 t + \ldots$ for the typical time-interval $t \in [t_{i-1}, t_i]$ where $t_i = t_{i-1} + \Delta t$. The lowest order method is given by a linear approximation

$$\mathbf{a}(t) = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 t .$$

Weighted residual method for the second order ordinary differential equations with the shape functions $p = 1$ and $p = t$ of the approximation as the weights in

$$\int_{t_{i-1}}^{t_i} p(\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} - \mathbf{F})dt + [p\mathbf{M}(\dot{\mathbf{a}} - \mathbf{h})]_{i-1} - [\dot{p}\mathbf{M}(\mathbf{a} - \mathbf{g})]_{i-1} = 0 .$$

Solving the equations for $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$ and use of $\mathbf{a}_i = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 \Delta t$ and $\dot{\mathbf{a}}_{i+1} = \dot{\mathbf{a}}_i$ gives the typical step of the Discontinuous Galerkin method for a second order initial values problem. The same recipe applies with more terms.

According to the recipe for the simplest DG time integration step to get $\{\mathbf{a},\dot{\mathbf{a}}\}_i$ in terms of solution to the previous step evaluated at its end point $\{\mathbf{a},\dot{\mathbf{a}}\}_{i-1} = \{\mathbf{g},\mathbf{h}\}$

$$\int_{t_{i-1}}^{t_i} (\mathbf{K}\mathbf{a}-\mathbf{F})dt + \mathbf{M}(\dot{\mathbf{a}}_{i-1}-\mathbf{h}) = \mathbf{K}(\boldsymbol{\alpha}_0\Delta t + \boldsymbol{\alpha}_1\frac{1}{2}\Delta t^2) - \mathbf{F}\Delta t + \mathbf{M}(\boldsymbol{\alpha}_1-\mathbf{h}) = 0,$$

$$\int_{t_{i-1}}^{t_i} t(\mathbf{K}\mathbf{a}-\mathbf{F})dt - \mathbf{M}(\mathbf{a}_{i-1}-\mathbf{g}) = \mathbf{K}(\frac{1}{2}\Delta t^2\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1\frac{1}{3}\Delta t^3) - \mathbf{F}\frac{1}{2}\Delta t^2 - \mathbf{M}(\boldsymbol{\alpha}_0-\mathbf{g}) = 0.$$

Rearranging the equations

$$\Delta t\mathbf{K}\boldsymbol{\alpha}_0 + (\frac{1}{2}\Delta t^2\mathbf{K}+\mathbf{M})\boldsymbol{\alpha}_1 = \mathbf{M}\mathbf{h} + \mathbf{F}\Delta t,$$

$$(\frac{1}{2}\Delta t^2\mathbf{K}-\mathbf{M})\boldsymbol{\alpha}_0 + \frac{1}{3}\Delta t^3\mathbf{K}\boldsymbol{\alpha}_1 = -\mathbf{M}\mathbf{g} + \mathbf{F}\frac{1}{2}\Delta t^2$$

and using the writing the equations in the recursive form of iteration with

$$\begin{Bmatrix}\mathbf{a}\\ \dot{\mathbf{a}}\end{Bmatrix}_i = \begin{bmatrix}\mathbf{I} & \mathbf{I}\Delta t\\ 0 & \mathbf{I}\end{bmatrix}\begin{Bmatrix}\boldsymbol{\alpha}_0\\ \boldsymbol{\alpha}_1\end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix}\boldsymbol{\alpha}_0\\ \boldsymbol{\alpha}_1\end{Bmatrix} = \begin{bmatrix}\mathbf{I} & -\mathbf{I}\Delta t\\ 0 & \mathbf{I}\end{bmatrix}\begin{Bmatrix}\mathbf{a}\\ \dot{\mathbf{a}}\end{Bmatrix}_i$$

gives

$$\begin{bmatrix}\Delta t\mathbf{K} & \frac{1}{2}\Delta t^2\mathbf{K}+\mathbf{M}\\ \frac{1}{2}\Delta t^2\mathbf{K}-\mathbf{M} & \frac{1}{3}\Delta t^3\mathbf{K}\end{bmatrix}\begin{bmatrix}\mathbf{I} & -\mathbf{I}\Delta t\\ 0 & \mathbf{I}\end{bmatrix}\begin{Bmatrix}\mathbf{a}\\ \dot{\mathbf{a}}\end{Bmatrix}_{i+1} = \begin{bmatrix}0 & \mathbf{M}\\ -\mathbf{M} & 0\end{bmatrix}\begin{Bmatrix}\mathbf{a}\\ \dot{\mathbf{a}}\end{Bmatrix}_i + \mathbf{F}\begin{Bmatrix}\Delta t\\ \frac{1}{2}\Delta t^2\end{Bmatrix}$$

and finally

$$\begin{bmatrix}\Delta t\mathbf{K} & -\frac{1}{2}\Delta t^2\mathbf{K}+\mathbf{M}\\ \frac{1}{2}\Delta t^2\mathbf{K}-\mathbf{M} & \Delta t\mathbf{M}-\frac{1}{6}\Delta t^3\mathbf{K}\end{bmatrix}\begin{Bmatrix}\mathbf{a}\\ \dot{\mathbf{a}}\end{Bmatrix}_i = \begin{bmatrix}0 & \mathbf{M}\\ -\mathbf{M} & 0\end{bmatrix}\begin{Bmatrix}\mathbf{a}\\ \dot{\mathbf{a}}\end{Bmatrix}_{i-1} + \mathbf{F}\begin{Bmatrix}\Delta t\\ \frac{1}{2}\Delta t^2\end{Bmatrix}.$$

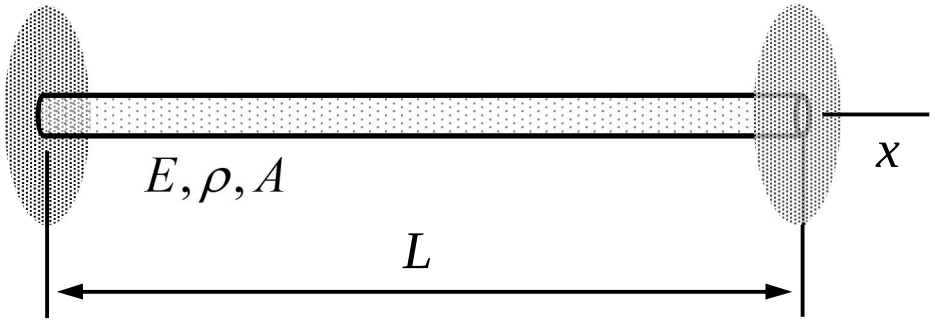
ONE-STEP METHODS FOR EQUATION SYSTEM

DG:
$$\begin{bmatrix} \Delta t^2\mathbf{K} & -\frac{1}{2}\Delta t^2\mathbf{K}+\mathbf{M} \\ \frac{1}{2}\Delta t^2\mathbf{K}-\mathbf{M} & \mathbf{M}-\frac{1}{6}\Delta t^2\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}$$

CN:
$$\begin{bmatrix} \mathbf{I} & -\frac{1}{2}\mathbf{I} \\ \frac{\Delta t}{2}\mathbf{K} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{I} & \frac{1}{2}\mathbf{I} \\ -\frac{\Delta t}{2}\mathbf{K} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}$$

The proper step-size Δt depends on the largest eigenvalue of parameter $\alpha = \mathbf{M}^{-1}\mathbf{K}\Delta t^2$. The numerical damping of DG exceeds that of CN whereas the phase error of CN exceeds that of the DG method.

EXAMPLE Finite Element Method is applied to the bar problem shown using a regular grid with $i \in \{0,1,2,3\}$. Thereafter, Discontinuous-Galerkin method is applied to find the solution at the temporal grid $t_j = j\Delta t \quad j \in \{0,1,\ldots\}$. Derive the iteration formula giving the displacements and velocities of points of the spatial discretization for initial displacement and velocities given by **g** and **h**. Material properties E and ρ and cross-sectional area A are constants.



Answer $$3\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \frac{L}{3}\rho A\frac{1}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ u_2 \end{Bmatrix} = 0$$

Using the Finite Element Method on a regular grid of points $i \in \{0,1,2,3\}$ gives the equations set: $u_0 = 0$ and $u_3 = 0$

$$\frac{EA}{\Delta x}(u_0 - 2u_1 + u_2) - \rho A \Delta x \frac{1}{6}(4\ddot{u}_0 + 4\ddot{u}_1 + \ddot{u}_2) = 0,$$

$$\frac{EA}{\Delta x}(u_1 - 2u_2 + u_3) - \rho A \Delta x \frac{1}{6}(4\ddot{u}_1 + 4\ddot{u}_2 + \ddot{u}_3) = 0.$$

Or written in the matrix form by taking into account only the points of unknown displacements and $\Delta x = L/3$

$$3\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \frac{L}{3}\rho A \frac{1}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ u_2 \end{Bmatrix} = 0.$$

With notation

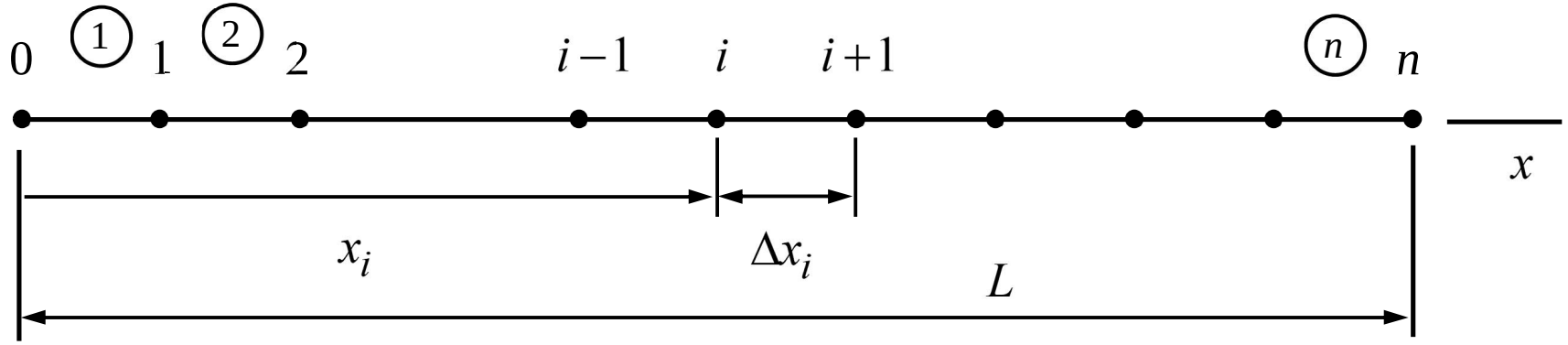
$$\mathbf{M} = \frac{L}{3}\rho A \frac{1}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix},\ \mathbf{K} = 3\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},\ \mathbf{a} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},\ \dot{\mathbf{a}} = \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}$$

the time-integration according to the Discontinuous-Galerkin method is given by

$$\begin{bmatrix} \Delta t^2\mathbf{K} & -\frac{1}{2}\Delta t^2\mathbf{K} + \mathbf{M} \\ \frac{1}{2}\Delta t^2\mathbf{K} - \mathbf{M} & \mathbf{M} - \frac{1}{6}\Delta t^2\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}.$$

4.5 ELEMENTS AND NODES

In Finite Element Method, grid points and the line segments between them are called as the nodes and elements, respectively. The representation of geometry or dataset by elements and separate lists of coordinates and function values contains the regular grid representations used in Particle Surrogate Method and Finite Difference Method as a particular case.



The element-node representation is more flexible than the regular grid and fits particularly well in the Finite Element Method also with several spatial dimensions.

Example of the representation of dataset $\{(x_0, f_0),(x_1, f_1),\ldots,(x_n, f_n)\}$ with the element concept is

$$\{(0,1),(1,2),\ldots,(n-1,n)\},\ \{x_0, x_1,\ldots, x_n\},\ \{f_0, f_1,\ldots, f_n\}$$

The element description consist of the indices of the end points and coordinates and function values are given in separate lists. In implementation of the numerical method, the numbering of the nodes starts from 1 due to the usual referencing convention to elements of the lists and tables so the representation of the dataset would likely be

$$\{(1,2),(2,3),\ldots,(n,n+1)\},\ \{x_0, x_1,\ldots, x_n\},\ \{f_0, f_1,\ldots, f_n\}$$

or even more likely

$$\{(1,2),(2,3),\ldots,(n,n+1)\},\ \{x_1, x_2,\ldots, x_{n+1}\},\ \{f_1, f_2,\ldots, f_{n+1}\}.$$

In hand calculations, the convention used does not matter.