5 FDM FOR MEMBRANE MODEL

ENGINEERINGS MODELS

BAR is a body which is thin in two dimensions and has straight initial geometry. Displacement has only the axial component. Internal force is aligned with the axis.

STRING is a body which is thin in two dimensions and has straight initial geometry. Displacement has only the transverse component. Internal force is aligned with the tangent of the mid-curve at the initial and deformed geometries.

THIN SLAB is a body which is thin in one dimension and has planar initial geometry. Displacement has only the mid-plane components. Internal force does not have transverse component.

MEMBRANE is a body which is very thin in one dimension and has planar initial geometry. Displacement has only the transverse component. Internal force is aligned with the tangent of the mid-plane at the initial and deformed geometries.

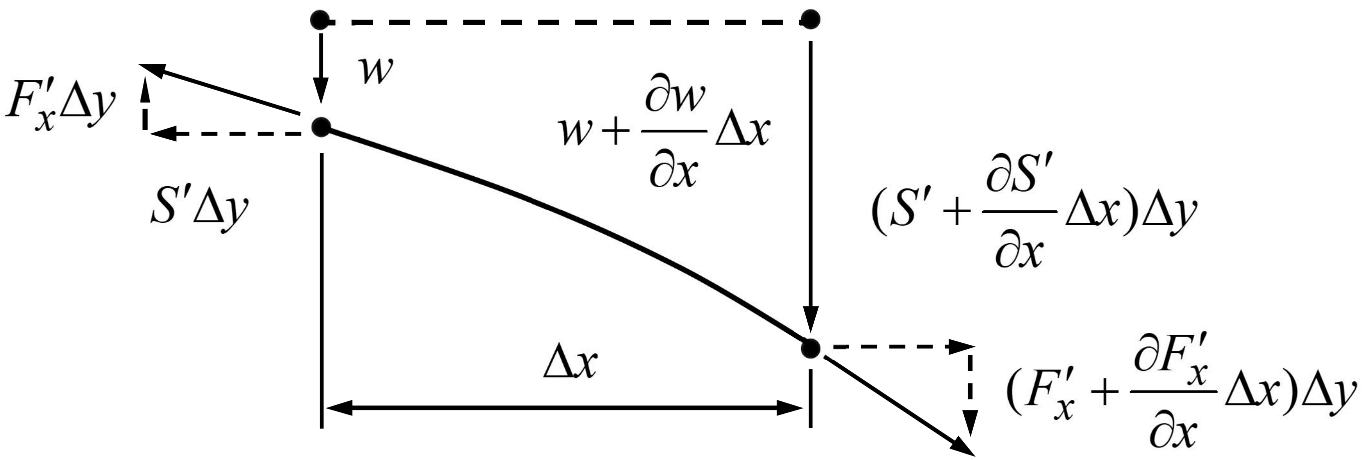
MATHEMATICAL PREREQUISITES

In an analytical solution method, solution trial is used to transform a partial differential differentia equation into an ordinary differential equation, another solution trial is used to transform the ordinary differential equation into an algebraic equation etc.

Equation	Solution trial
$k'(\frac{\partial^2 a}{\partial x^2}+\frac{\partial^2 a}{\partial y^2})-m'\frac{\partial^2 a}{\partial t^2}=0$	$a(x,y,t)=A(t)e^{\mathrm{i}(\lambda_x x+\lambda_y y)}$
$A(t)k'(\lambda_x^2+\lambda_y^2)+m'\ddot{A}(t)=0$	$A(t)=\alpha e^{\mathrm{i}\omega t}$

$$a(x,y,t)=\sum (\alpha_t \sin\omega t+\beta_t\cos\omega t)(\alpha_x\sin\lambda_x x+\beta_x\cos\lambda_x x)(\alpha_y\sin\lambda_y y+\beta_y\cos\lambda_y y)$$

5.1 MEMBRANE MODEL

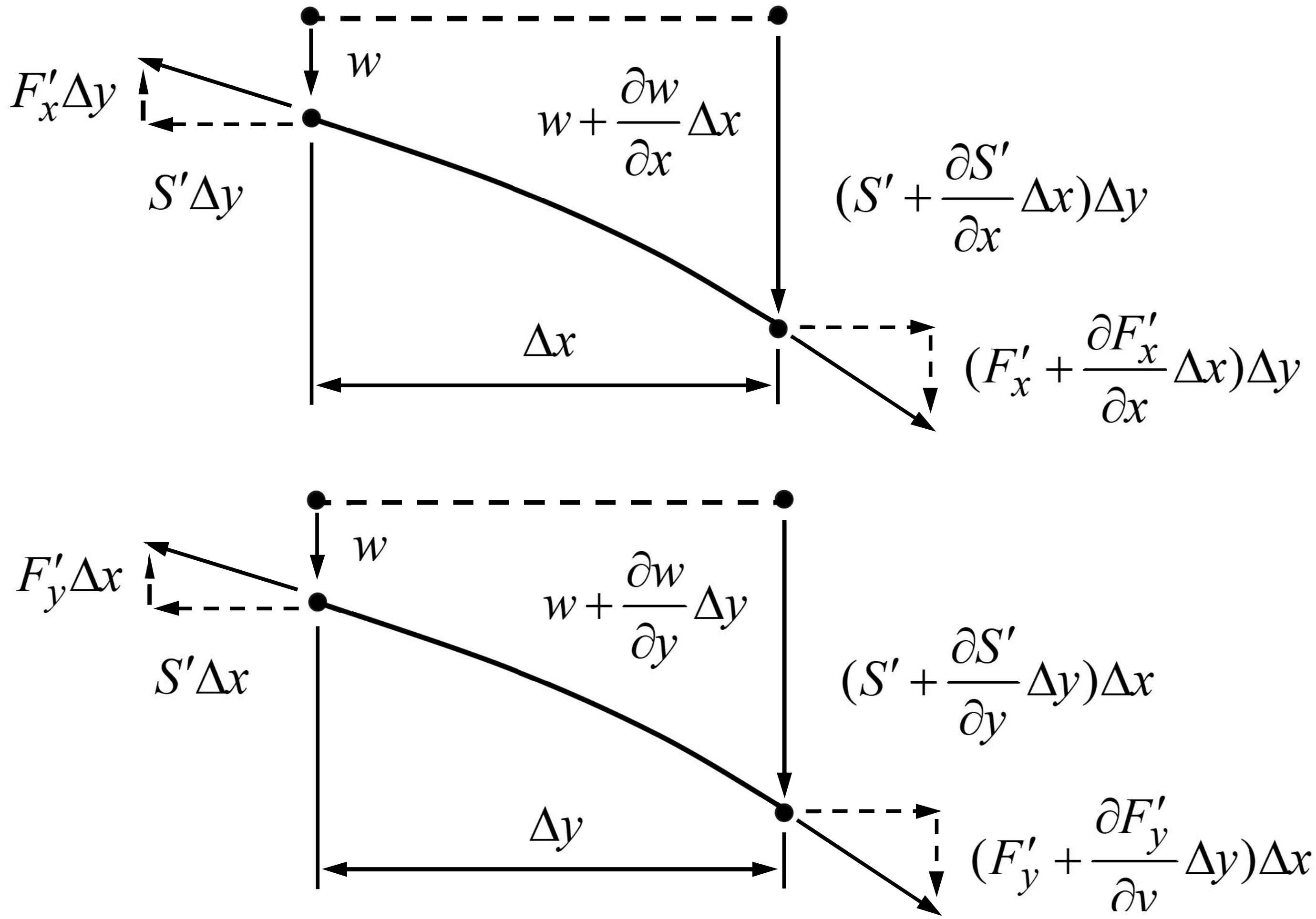


Equation of motion $S'(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}) + f' = m'\frac{\partial^2 w}{\partial t^2} \quad (x,y) \in \Omega \quad t > 0,$

Boundary conditions $w = \underline{w}$ or $S'(n_x \frac{\partial w}{\partial x} + n_y \frac{\partial w}{\partial y}) = F' \quad (x,y) \in \partial\Omega \ t > 0,$

Initial conditions $w = g$ and $\frac{\partial w}{\partial t} = h \quad (x,y) \in \Omega \quad t = 0.$

Derivation starting from the first principles goes in the same manner as for the string model but uses a rectangular material element of side lengths Δx and Δy:



The tightening S'_x and S'_y (force per unit length) in the $x-$ and $y-$directions may differ, in principle. Let us consider the isotropic case $S'_x = S'_y = S'$. As material elements are assumed to move only in the transverse direction, equations of motions in the horizontal directions become

$$\text{Momentum balance } (x) \;:\quad (S' + \frac{\partial S'}{\partial x}\Delta x)\Delta y - S'\Delta y = 0 \quad \Rightarrow \quad \frac{\partial S'}{\partial x} = 0\ ,$$

$$\text{Momentum balance } (y) \;:\quad (S' + \frac{\partial S'}{\partial y}\Delta y)\Delta x - S'\Delta x = 0 \quad \Rightarrow \quad \frac{\partial S'}{\partial y} = 0.$$

Hence, S' (force per unit length) needs to be constant. Using the deformed geometry of the material element and the assumption that the internal forces are aligned with the tangent of the mid-plane, gives the representations

$$F'_x = S'\frac{\partial w}{\partial x} \quad \text{and} \quad F'_y = S'\frac{\partial w}{\partial y}$$

so the equation of motion in the transverse direction takes the form

$$\frac{\partial F_x'}{\partial x}+\frac{\partial F_y'}{\partial y}+f'=S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2}. \quad \Leftarrow$$

where $m'=\rho t$ is the mass per unit area.

NOTICE: The string and membrane equations as defined in this course do not follow, e.g., from the principle of virtual work for linear elasticity theory in the same manner as, e.g., the well-known beam, plate etc. models but requires the use of large-displacement theory with the kinetic assumption that tightening of the initial flat geometry is constant and not affected by the transverse displacement.

FOURIER SERIES

The Fourier series (various forms exist) can be used to represent a function as the sum of harmonic terms. For example, the sine-transformation pair for a function $a(x)$ $x \in [0,L]$ with vanishing values at the end points is given by

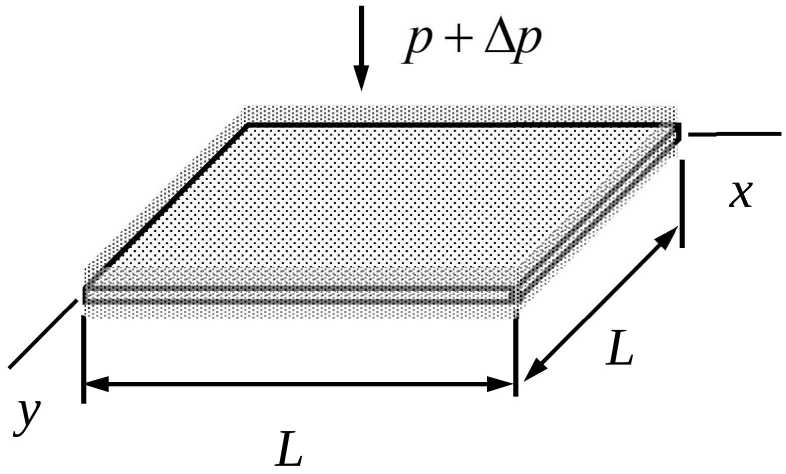
$$\alpha_j = \frac{2}{L}\int_0^L \sin(j\pi\frac{x}{L})a(x)dx \quad j \in \{1,2,\ldots\} \quad \Leftrightarrow \quad a(x) = \sum_{j\in\{1,2,\ldots\}} \alpha_j \sin(j\pi\frac{x}{L}).$$

The transformation pair is based on the orthogonality of the modes

$$\int_0^L \sin(j\pi\frac{x}{L})\sin(l\pi\frac{x}{L})dx = \frac{L}{2}\delta_{jl} \text{ (Kronecker delta).}$$

The transformation (with respect to time) can be used to analyze frequency contents of data, filtering, to find the combination of the terms of the generic series solution for bar, string and membrane models satisfying the initial conditions, etc.

EXAMPLE A rectangular membrane of fixed edges and constant tightening S' (force per unit length) is loaded by pressures $p+\Delta p$ acting on the upper surface and p acting on the lower surface. Find the transverse displacement by using the double sine series trial solution and double sine series representation of the loading.



Answer $w(x,y)=\sum_{k\in\{1,3,5\ldots\}}\sum_{l\in\{1,3,5\ldots\}}\frac{16}{\pi^4}\frac{\Delta p L^2}{S'}\frac{1}{kl(k^2+l^2)}\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L})$

According to the problem decription, displacement and the constant loading due to excess pressure on the upper surface, should be presented by double sine series ($k, l \in \{1, 2, \ldots\}$)

$$w(x,y) = \sum \sum w_{kl} \sin(k\pi \frac{x}{L}) \sin(l\pi \frac{y}{L}), \quad f(x,y) = \sum \sum f_{kl} \sin(k\pi \frac{x}{L}) \sin(l\pi \frac{y}{L}).$$

As the boundary conditions are satisfied by the trial solution, it is enough to concentrate on the differential equation. Substituting the typical terms of series representations, gives a condition for the multipliers of the sine terms:

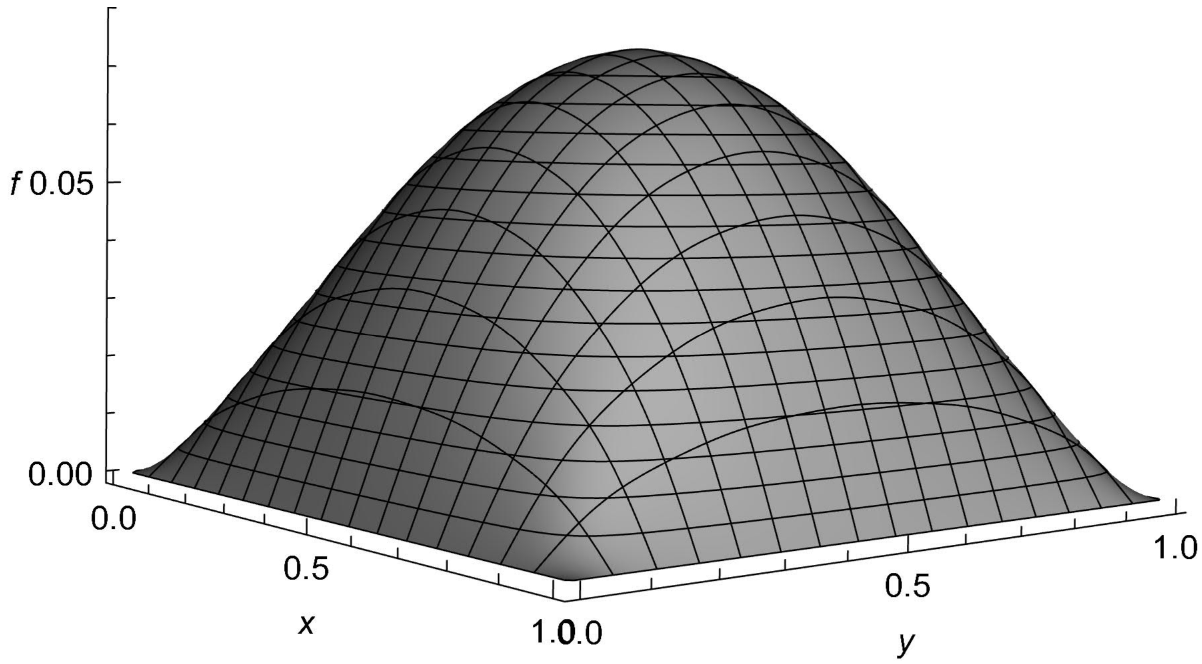
$$-w_{kl} S' [(\frac{k\pi}{L})^2 + (\frac{l\pi}{L})^2] + f_{kl} = 0 \quad \Leftrightarrow \quad w_{kl} = \frac{L^2}{S'\pi^2} \frac{1}{k^2 + l^2} f_{kl}.$$

What remains, is finding the coefficients of the double sine representation of the loading. Using the orthogonality of sines in both coordinate directions

$$\Delta p = \sum \sum f_{kl} \sin(k\pi \frac{x}{L}) \sin(l\pi \frac{y}{L}) \;\Leftrightarrow\; f_{kl} = \frac{16}{\pi^2} \frac{1}{kl} \Delta p \quad \text{where } k, l \in \{1, 3, 5, \ldots\}$$

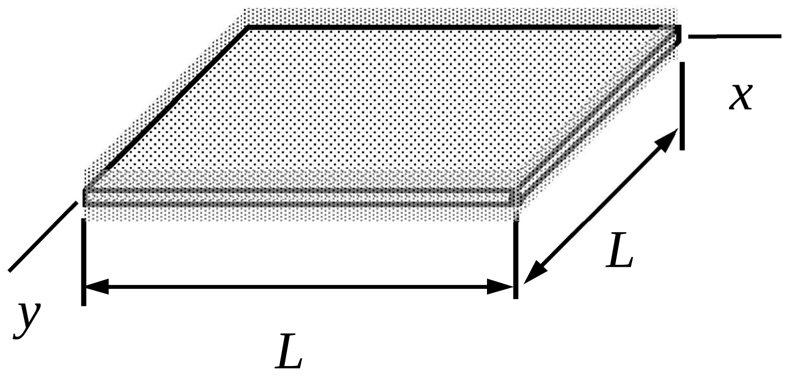
the other coefficients being zeros. Therefore, the series solution takes the form

$$w(x, y) = \sum_{k \in \{1,3,5\ldots\}} \sum_{l \in \{1,3,5\ldots\}} \frac{16}{\pi^4} \frac{\Delta p L^2}{S'} \frac{1}{kl(k^2 + l^2)} \sin(k\pi \frac{x}{L}) \sin(l\pi \frac{y}{L}).$$



EXAMPLE Consider a rectangular drumhead of fixed edges, constant tightening S' (force per unit length) and density ρt (per unit area). Find the frequencies of the free vibrations by using the double sine series trial solution

$$w(x,y,t)=\sum\sum w_{kl}(t)\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L}).$$



Answer $f_{kl}=\dfrac{\pi}{2L}\sqrt{(k^2+l^2)\dfrac{S'}{\rho t}}$

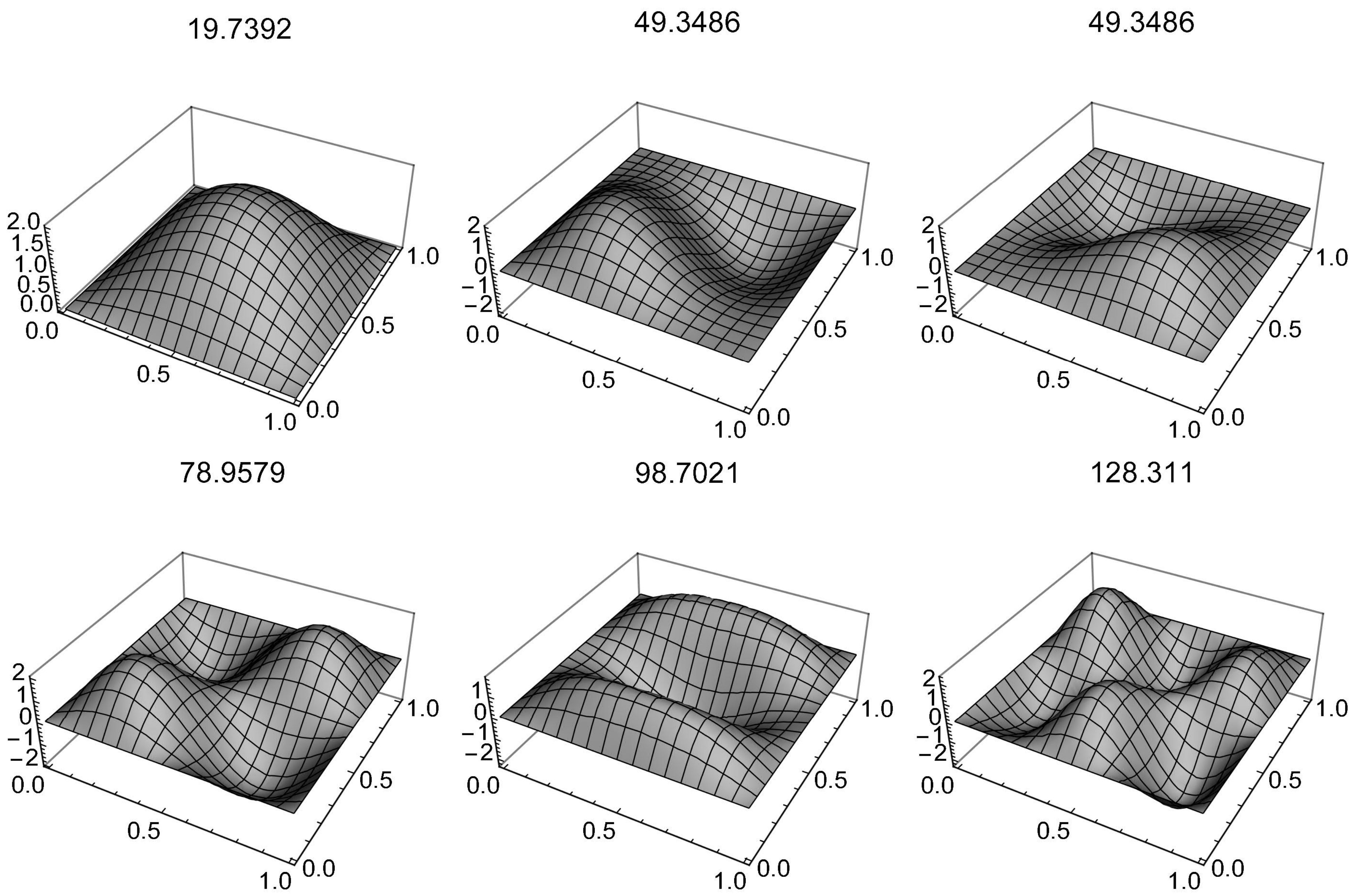
The solution trial, composed of the sines in both directions, and amplitudes depending on time is given by $k, l \in \{1, 2, \ldots\}$)

$$w(x, y, t) = \sum \sum w_{kl}(t) \sin(k\pi \frac{x}{L}) \sin(l\pi \frac{y}{L}).$$

As the boundary conditions are satisfied by the trial solution, it is enough to concentrate on the differential equation. Substitution of the typical term, gives the ordinary differential equation

$$\ddot{w}_{kl} + \omega_{kl}^2 w_{kl} = 0 \quad \text{where} \quad \omega_{kl} = 2\pi f_{kl} = \frac{\pi}{L}\sqrt{(k^2 + l^2)\frac{S'}{\rho t}} \quad \leftarrow$$

the corresponding modes being the double sine terms of the trial solution. The smallest frequency is given by selection $k = l = 1$: $f_{11} = \sqrt{S' / \rho t} / \sqrt{2}L$.


19.7392
2.0
1.5
1.0
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2
1
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78.9579
2
1
0
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1.0
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0.5
1.0
98.7021
1
0
−1
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0.5
1.0
128.311
2
1
0
−1
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0.0
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1.0
0.0
0.5
1.0

MODE SUPERPOSITION

If the initial conditions concerning position and displacement of the particles are known (quite exceptional case in practice), the outcome of the modal analysis $(\omega, A)_j$ can be used to construct a displacement solution for the given initial data starting with the series

(a) $$a(x,y,t) = \sum \sum A_{kl}(x,y)[\alpha_{kl} \frac{1}{\omega_{kl}} \sin(\omega_{kl} t) + \beta_{kl} \cos(\omega_{kl} t)].$$

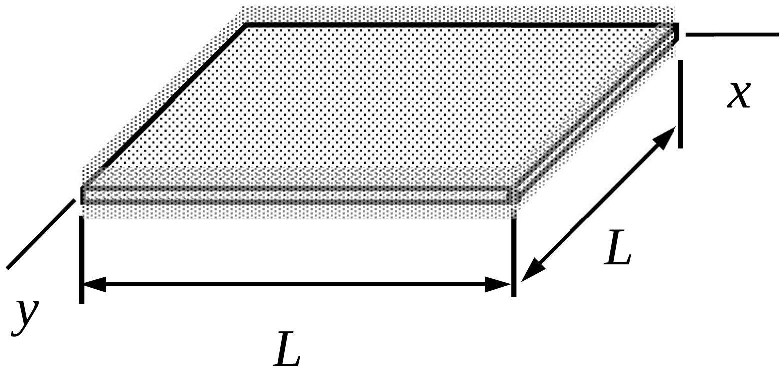
The combination of the modes giving $a = g(x,y)$ and $\partial a / \partial t = h(x,y)$ at $t = 0$ follow with the expressions

(b) $$\alpha_{kl} = \frac{1}{\mathrm{A}_{kl}^2} \int_\Omega A_{kl}(x,y) h dA, \ \ \beta_{kl} = \frac{1}{\mathrm{A}_{kl}^2} \int_\Omega A_{kl}(x,y) g dA, \ \mathrm{A}_{kl}^2 = \int_\Omega A_{kl}(x,y) A_{kl}(x,y) dA.$$

The coefficients correspond to the spatial Fourier series of the initial data obtained with the orthogonal harmonic modes from the modal analysis.

EXAMPLE Consider a rectangular drumhead of fixed edges, constant tightening S' (force per unit length) and density ρt (per unit area). Find the solution to the transverse displacement if the initial displacement $g(x,y) = W\sin(k\pi x/L)\sin(l\pi y/L)$ and initial velocity vanishes. The outcome of the modal analysis is the angular velocity-mode pairs

$$\omega_{kl} = \frac{\pi}{L}\sqrt{(k^2+l^2)\frac{S'}{\rho t}} \quad \text{and} \quad A_{kl}(x,y) = \sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L}).$$



Answer $w(x,y,t) = W\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L})\cos(\frac{\pi}{L}\sqrt{(k^2+l^2)\frac{S'}{\rho t}}t)$

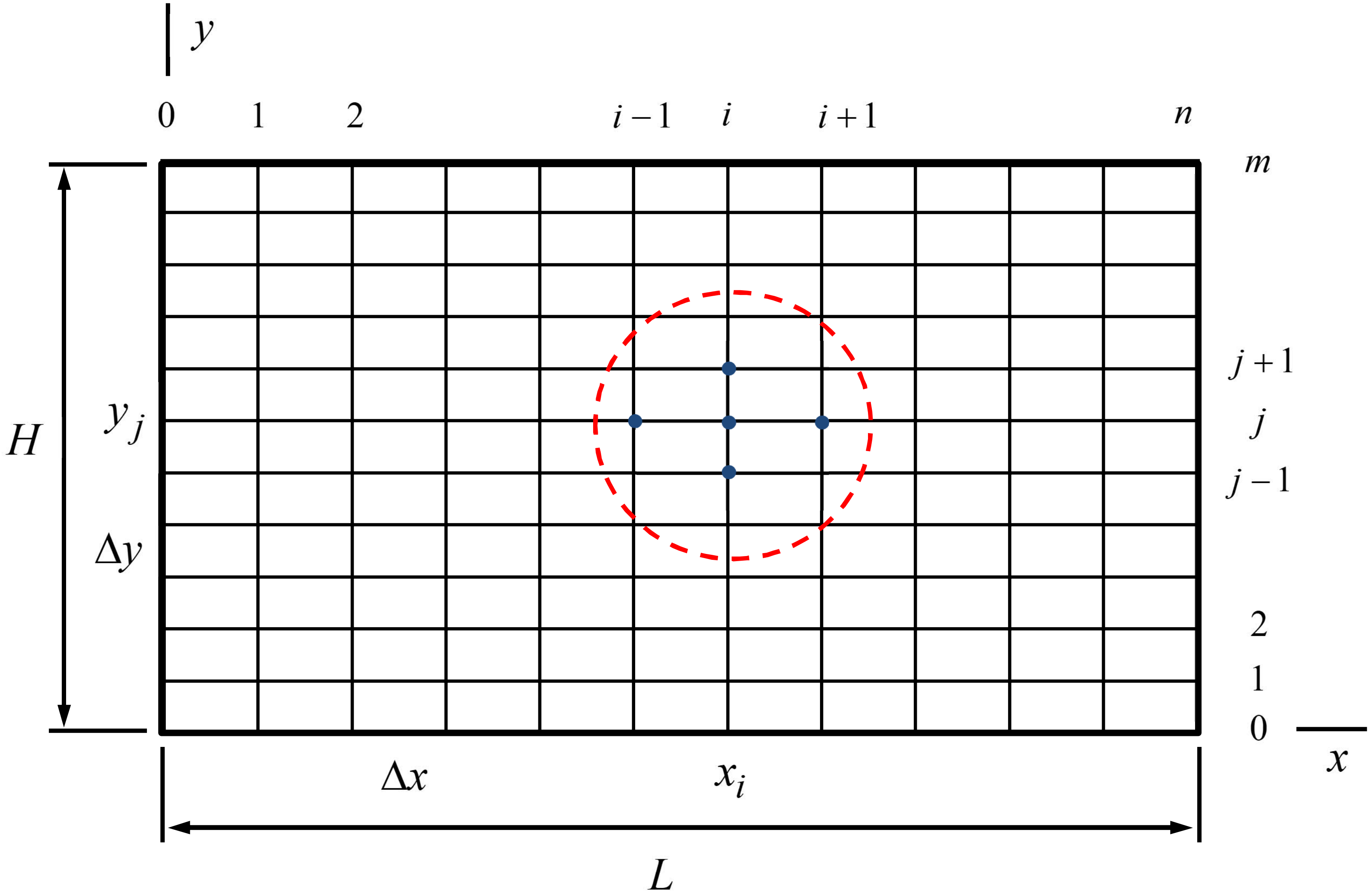
In the same manner as with the string problem, solution as the function of spatial coordinates and time is obtained by mode superposition with

$$a(x, y, t) = \sum \sum A_{kl}(x, y)[\alpha_{kl} \frac{1}{\omega_{kl}} \sin(\omega_{kl} t) + \beta_{kl} \cos(\omega_{kl} t)].$$

As initial velocity vanishes $\alpha_{kl} = 0$. The initial displacement is one of the modes (some fixed k and l, so $\beta_{kl} = W$ the remaining being zeros (no summing now)

$$w(x, y, t) = W \sin(k\pi \frac{x}{L}) \sin(l\pi \frac{y}{L}) \cos(\frac{\pi}{L} \sqrt{(k^2 + l^2) \frac{S'}{\rho t}} t).$$ ←

5.2 APPROXIMATION TO DERIVATIVES



TAYLOR'S THEOREM

Taylor series with the remainder term is an important tool in numerics, e.g., in the finite difference method. Theorem tells how to approximate a function in some neighborhood of a point by a polynomial.

1D: $$f(x+\Delta x) = \sum_{i=0}^{n} \frac{1}{i!}(\Delta x \frac{d}{dx})^i f(x) + [\frac{1}{(n+1)!}(\Delta x \frac{d}{dx})^{n+1} f(x)]_{\xi}$$

nD: $$f(\mathbf{x}+\Delta \mathbf{x}) = \sum_{i=0}^{n} \frac{1}{i!}(\Delta \mathbf{x}^{\mathrm{T}} \nabla)^i f(\mathbf{x}) + [\frac{1}{n+1!}(\Delta \mathbf{x}^{\mathrm{T}} \nabla)^{n+1} f(\mathbf{x})]_{\boldsymbol{\xi}}$$

Theorem assumes existence of the *n*:th derivative. In the remainder term, ξ is some point to the interval and is different in each occurrence). For example, finite difference approximations to derivatives in terms of values of pointwise values of a function follow from the theorem.

The generic form simplifies to

$$f(x+\Delta x, y+\Delta y) = \sum_{i=0}^{n} \frac{1}{i!}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^i f(x,y) + R(\xi,\eta)$$

when expressions $\Delta \mathbf{x}^{\mathrm{T}} = \{\Delta x \quad \Delta y\}$, $\nabla^{\mathrm{T}} = \{\partial / \partial x \quad \partial / \partial y\}$ and $\boldsymbol{\xi} = (\xi, \eta)$ are used there. The remainder term is given by

$$R(\xi,\eta) = [\frac{1}{n+1!}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^{n+1} f(x,y)]_{(\xi,\eta)},$$

where $\xi \in [x, x+\Delta x]$ and $\eta \in [y, y+\Delta y]$.

DIFFERENCE APPROXIMATIONS

Derivative	Central difference	Order
$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(i,j)}$	$\frac{f_{(i-1,j)} - 2f_{(i,j)} + f_{(i+1,j)}}{\Delta x^2}$	2
$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(i,j)}$	$\frac{f_{(i,j-1)} - 2f_{(i,j)} + f_{(i,j+1)}}{\Delta y^2}$	2
$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(i,j)}$	$\frac{f_{(i+1,j+1)} - f_{(i+1,j-1)} - f_{(i-1,j+1)} - f_{(i-1,j-1)}}{4\Delta x \Delta y}$	2

Although the expressions follow in the same manner as in the one-dimensional case, hand calculations are a bit tedious with the method based on Taylor series.

The method using the interpolation of a dataset work also in two-dimensions although Mathematica may prove to be necessary in manipulations. Let us consider the stencil $\{i-1,i,i+1\}\times\{j-1,j,j+1\}$ of constant spacing Δx and Δy and interpolation $p(x,y)=\mathbf{N}^{\mathrm{T}}\mathbf{f}$ with (9) shape functions $\mathbf{N}(x,y)=\mathbf{N}(y)\times\mathbf{N}(x)$ and function values $\mathbf{f}$ where

$$\mathbf{N}(\xi)=\{\frac{(\xi)(\xi-\Delta\xi)}{(-\Delta\xi)(-\Delta\xi-\Delta\xi)},\frac{(\xi+\Delta\xi)(\xi-\Delta\xi)}{(0+\Delta\xi)(0-\Delta\xi)},\frac{(\xi+\Delta\xi)(\xi)}{(\Delta\xi+\Delta\xi)(\Delta\xi)}\}\quad \xi\in\{x,y\},$$

$$\mathbf{f}=\{f_{(i-1,j-1)},f_{(i,j-1)},f_{(i+1,j-1)},f_{(i-1,j)},f_{(i,j)},f_{(i+1,j)},f_{(i-1,j+1)},f_{(i,j+1)},f_{(i+1,j+1)}\}.$$

Difference approximations follow also from the Taylor's representation truncated at certain term and written for $\{i-1,i,i+1\}\times\{j-1,j,j+1\}$, adding and subtracting on both sides, rearranging, and dividing with an appropriate power of Δx. However, the proper combination depends on the derivative which makes the Taylor series method a bit tedious in several physical dimensions.

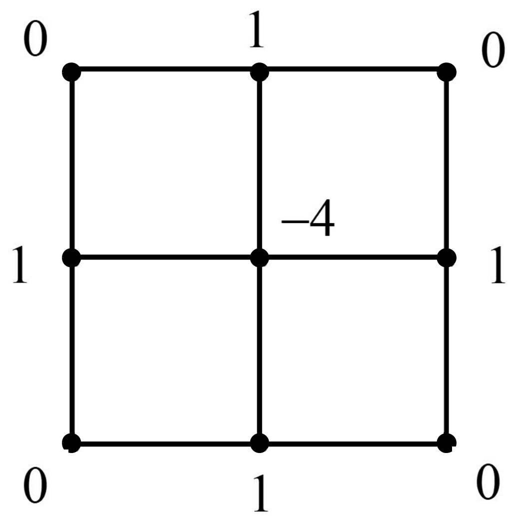
DIFFERENCE APPROXIMATIONS

Derivative	Central difference	Order
$(\frac{\partial f}{\partial x})_{(i,j)}$	$\frac{-f_{(i-1,j)} + f_{(i+1,j)}}{2\Delta x}$	2
$(\frac{\partial f}{\partial y})_{(i,j)}$	$\frac{-f_{(i,j-1)} + f_{(i,j+1)}}{2\Delta y}$	2
$(\frac{\partial f}{\partial x})_{(i,j)}$	$\frac{-f_{(i,j)} + f_{(i,j+1)}}{\Delta y}$	1
$(\frac{\partial f}{\partial y})_{(i,j)}$	$\frac{-f_{(i,j-1)} + f_{(i,j)}}{\Delta y}$	1

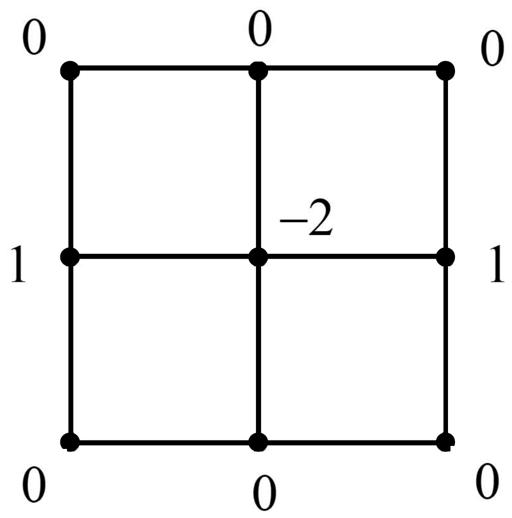
In two-dimensions, various stencils can be used in the difference approximation to derivative at point (i, j).

STENCIL

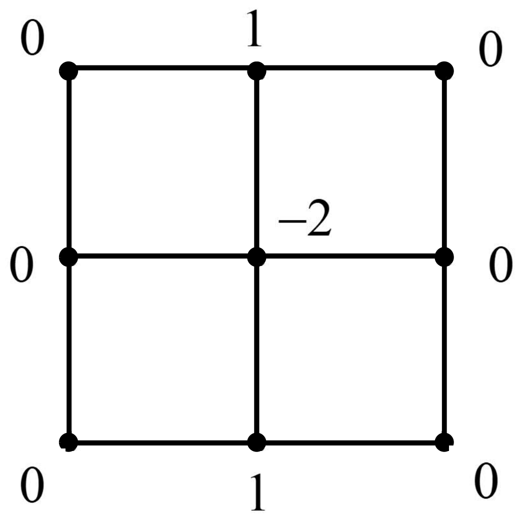
Stencil is used as a concise way to represent the difference approximations with a geometric pattern on the grid with the associated multipliers of the function values on the grid. For a regular grid $\Delta x = \Delta y$



$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$



$$\frac{\partial^2}{\partial x^2}$$



$$\frac{\partial^2}{\partial y^2}$$

5.3 FINITE DIFFERENCE METHOD

Finite Difference Method is a numerical technique for solving ordinary and partial differential equations by approximating derivatives with finite difference formulas. On a regular grid $\Delta x = \Delta y = h$, the membrane model with zero displacement boundary conditions

Interior $\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + f_i' = m_i' \ddot{w}_{(i,j)} \quad (i,j) \in I$

Boundary $w_{(i,j)} = 0 \quad (i,j) \in \partial I$

Initial conditions $w_{(i,j)} - g_{(i,j)} = 0$ and $\dot{w}_{(i,j)} - h_{(i,j)} = 0 \quad (i,j) \in I$

Where the interior grid point are denoted by I and the boundary grid points ∂I. Then, the outcome is a set of Ordinary Differential Equations which can be solved with the matrix and difference equation methods used for the bar and string models.

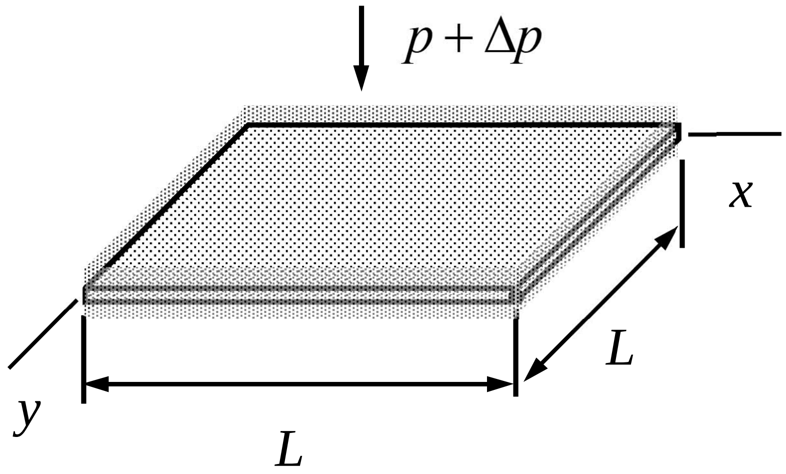
In the finite difference method, derivatives in the continuum model with respect to spatial coordinates are replaced by difference approximations, to get a set of ordinary differential equatios or a set of algebraic equations. The aim is replace a problem, which may be difficult solve as it stands, by a mathematically simpler problem. The prize one has to pay comes from the discretization error. For a membrane problem of fixed boundaries, the continuum model is given by

$$S'\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+f'=m'\frac{\partial^2 w}{\partial t^2}\quad (x,y)\in\Omega,\qquad w=0\quad (x,y)\in\partial\Omega\quad t>0,$$

$$w=g\ \text{ and }\ \frac{\partial w}{\partial t}=h\quad (x,y)\in\Omega\quad t=0.$$

Using the central difference approximation to the two partial derivatives and denoting the interior grid point by I and the boundary grid points ∂I, the equation system transforms to ordinary differential equations.

EXAMPLE A rectangular membrane of fixed edges and constant tightening s (force per unit length) is loaded by pressures $p+\Delta p$ acting on the upper surface and p acting on the lower surface. Find the solution to the transverse displacement by using the Finite Difference Method and a regular grid $(i, j) \in \{0,1,2\} \times \{0,1,2\}$.



Answer $w_{(1,1)} = \dfrac{1}{16}\dfrac{\Delta p L^2}{S'} \approx 0.0625\dfrac{\Delta p L^2}{S'}$ (exact to the model $0.0737\dfrac{\Delta p L^2}{S'}$)

In the finite difference method, derivatives in the continuum model with respect to spatial coordinates are replaced by difference approximations, to get a set of algebraic equations. In the present problem the interior and boundary grid points and the equations for the grid points are

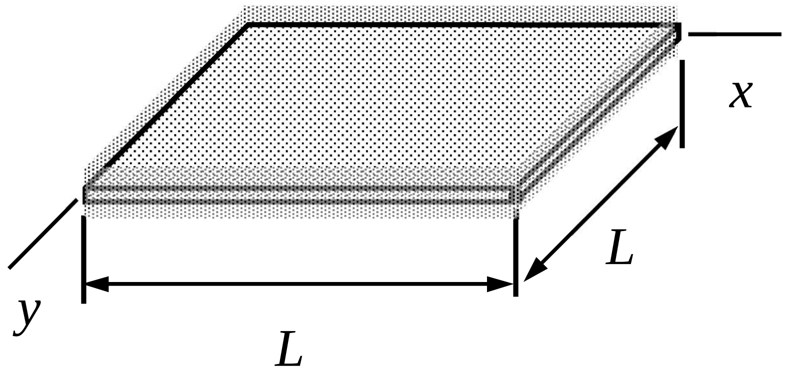
$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + \Delta p = 0 \quad (i,j) \in I = \{(1,1)\}$$

$$w_{(i,j)} = 0 \quad (i,j) \in \partial I = \{0,1,2\} \times \{0,1,2\} \setminus \{(1,1)\} \quad \text{(interior point excluded)}$$

where $\Delta x = \Delta x = h = L/2$. Eliminating the displacements of the boundary points from the equation for the interior point

$$4\frac{S'}{L^2}[-4w_{(1,1)}] + \Delta p = 0 \quad \Rightarrow \quad w_{(1,1)} = \frac{1}{16}\frac{\Delta p L^2}{S'}.$$

EXAMPLE Consider a rectangular drumhead of fixed edges, constant tightening S' (force per unit length) and density ρt (per unit area). Find the frequency of the free vibrations by using the Finite Difference Method and a regular grid $(i, j) \in \{0,1,2\} \times \{0,1,2\}$.



Answer $f = \frac{4}{2\pi} \frac{1}{L} \sqrt{\frac{S'}{\rho t}} \approx 0.64 \frac{1}{L} \sqrt{\frac{S'}{\rho t}}$ (exact to the model $\approx 0.71 \frac{1}{L} \sqrt{\frac{S'}{\rho t}}$)

In the finite difference method, derivatives in the continuum model with respect to spatial coordinates are replaced by difference approximations to get a set of ordinary differential equations. In the present problem the interior and boundary grid points and the equations for the grid points are

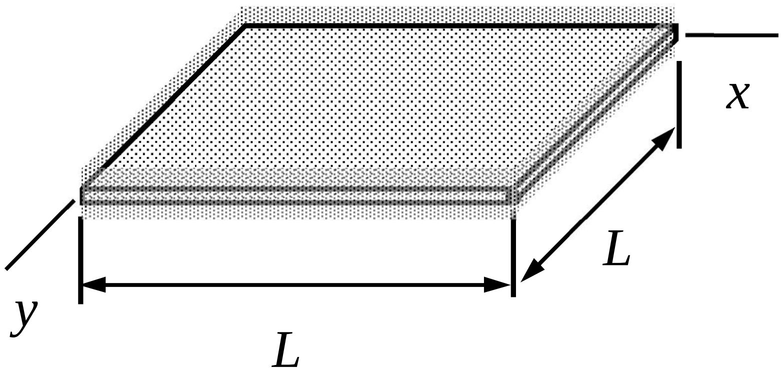
$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] = \rho t \ddot{w}_{(i,j)} \quad (i,j) \in I = \{(1,1)\}$$

$$w_{(i,j)} = 0 \quad (i,j) \in \partial I = \{0,1,2\} \times \{0,1,2\} \setminus \{(1,1)\} \quad \text{(interior point excluded)}$$

where $\Delta x = \Delta x = h = L/2$. Eliminating the displacements of the boundary points from the equation for the interior point

$$\ddot{w}_{(1,1)} + \omega^2 w_{(1,1)} = 0 \quad \text{where} \quad \omega = 2\pi f = \frac{4}{L}\sqrt{\frac{S'}{\rho t}} \quad \text{so} \quad f = \frac{4}{2\pi}\frac{1}{L}\sqrt{\frac{S'}{\rho t}}\,. \;\leftarrow$$

EXAMPLE Consider a rectangular drumhead of fixed edges, constant tightening S' (force per unit length) and density ρt (per unit area). Find the angular velocities of the free vibrations and the corresponding modes as predicted by the Finite Difference Method and a regular grid $(i, j) \in \{0,1,2,\ldots,n\} \times \{0,1,2,\ldots,n\}$. Use the trial solution for the typical mode $w_{(i,j)}(t) = a(t)\sin(k\pi i / n)\sin(l\pi j / n)$.



Answer $\omega_{kl} = \frac{1}{L}\sqrt{\frac{S'}{\rho t}\frac{2}{n^2}(2-\cos\frac{k\pi}{n}-\cos\frac{l\pi}{n})}$

The equations for the boundary points are satisfied by the trial solution so it is enough to consider the ordinary differential equations for the interior points

$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] = \rho t \ddot{w}_{(i,j)}$$

When substituted into the difference expression on the left-hand side, the trial solution $w_{(i,j)} = a(t)\sin(k\pi i/n)\sin(l\pi j/n)$ and identity $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ give expression

$$w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)} = -2(2 - \cos\frac{k\pi}{n} - \cos\frac{l\pi}{n})w_{(i,j)}$$

So the partial-difference and ordinary-differential equation simplifies to an ordinary-differential equation

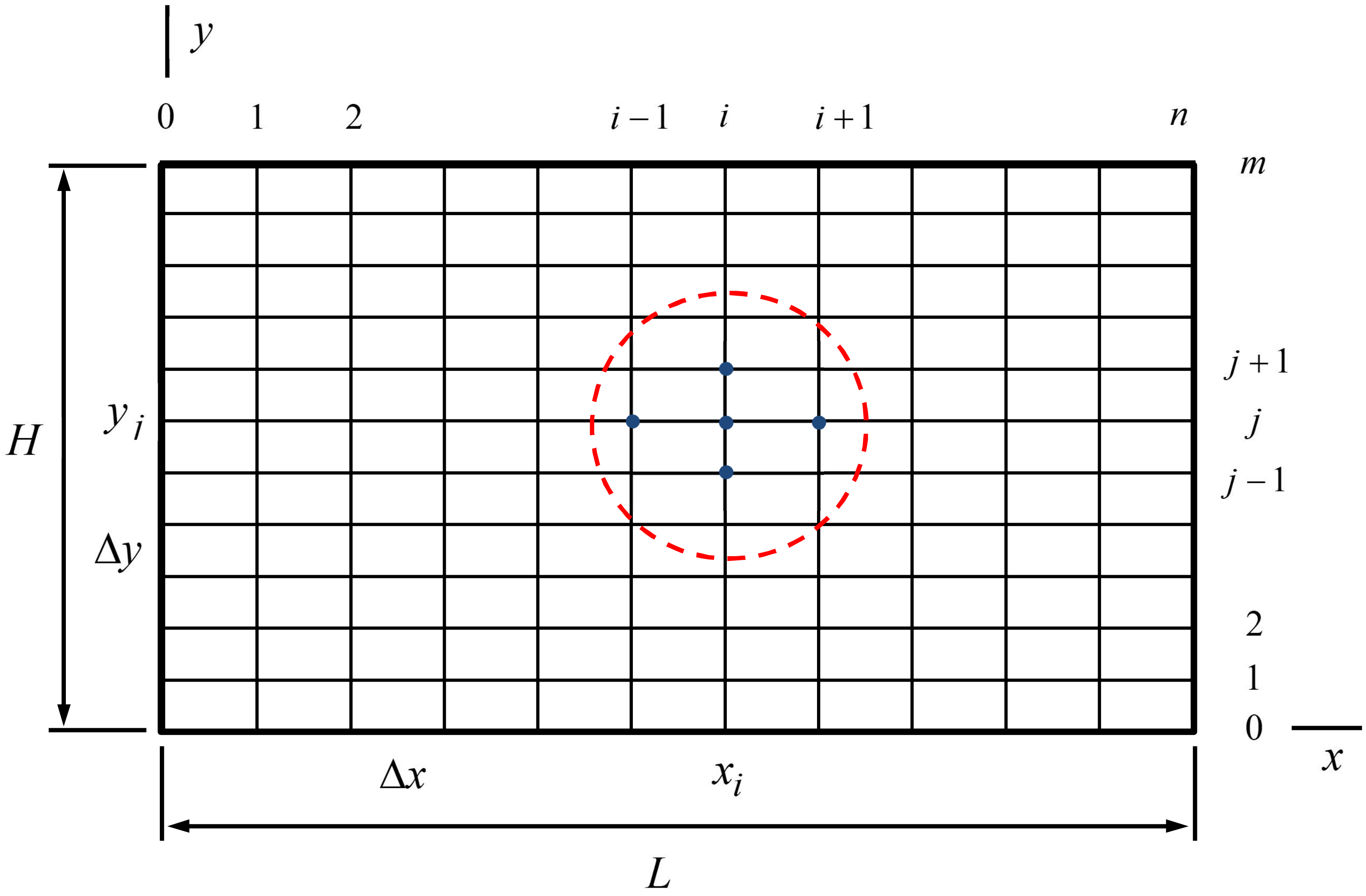
$$\ddot{w}_{(i,j)} + \omega_{kl}^2 w_{(i,j)} = 0 \quad \text{where} \quad \omega_{kl} = \frac{1}{L}\sqrt{\frac{S'}{\rho t}\frac{2}{n^2}(2 - \cos\frac{k\pi}{n} - \cos\frac{l\pi}{n})} \ .$$

In verification that the solution to the limit problem $n \to \infty$ coincides with the exact solution (a desirable property of a numerical method)

$$\omega_{kl} = \frac{\pi}{L}\sqrt{\frac{S'}{\rho t}(k^2 + l^2)}$$

one may assume that k, l are bounded and use $\cos\alpha \approx 1 - \alpha^2 / 2$ when $\alpha \ll 1$.

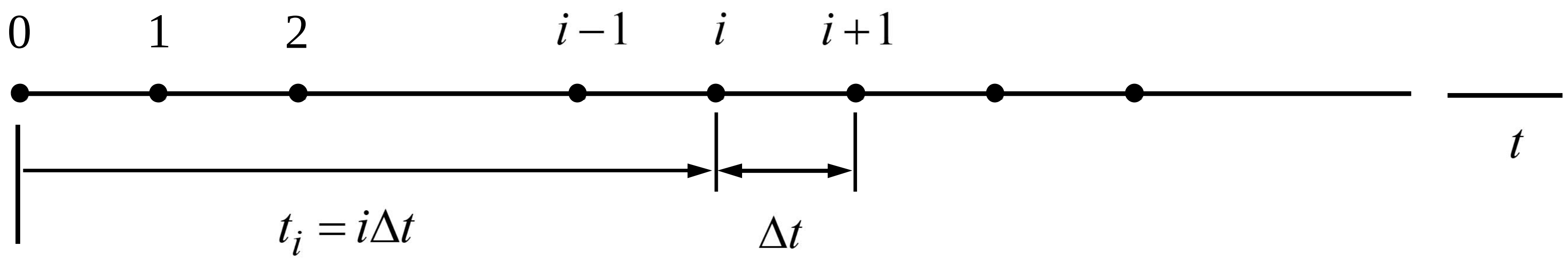
ONE-INDEX NUMBERING



In practice, a regular two-index numbering is used mainly with rectangular solution domains. The one index labelling of the grid points in any order works well in calculations with the matrix representation of the equilibrium equations and equations of motion as the order of the equations or labelling does not matter (if the number of algebraic operations needed to find the solution is not considered).

5.5 TIME INTEGRATION

The one-step DG (Discontinuous Galerkin) and CN (Crank-Nicolson) methods can be applied to the membrane problem in the same manner as for the bar, and string problems to find the solution on a grid of the temporal domain.



As the temporal domain for an initial value problem does not have an upper bound (strictly speaking). Also, the length of the intervals can be chosen to match the behavior of the solution (small steps for the rapid changes).

TIME INTEGRATION

Method	Iteration $i \in \{1, 2, \ldots\}$	Initial $i = 0$
EX	$\begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_i = \begin{bmatrix} \cos\alpha & \alpha^{-1}\sin\alpha \\ -\alpha\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}$
CN	$\begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_i = \dfrac{1}{4+\alpha^2} \begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}$
DG	$\begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_i = \dfrac{2}{12+\alpha^4} \begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}$

The methods coincide at the limit of vanishing step-size when $\alpha = \sqrt{\dfrac{k}{m}}\Delta t \to 0$.

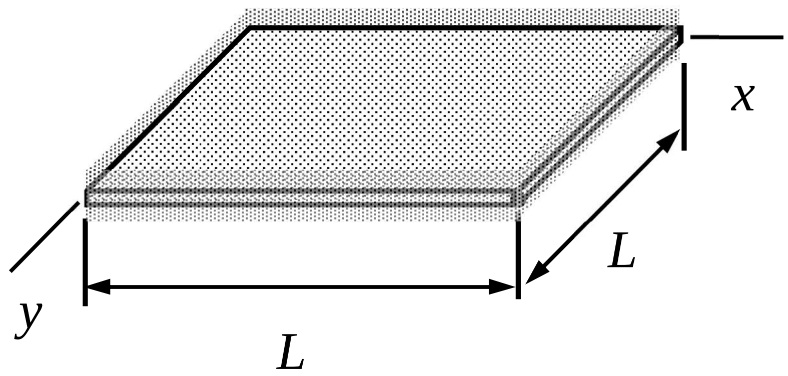
ONE-STEP METHODS FOR EQUATION SYSTEM

DG:
$$\begin{bmatrix} \Delta t^2 \mathbf{K} & -\frac{1}{2}\Delta t^2 \mathbf{K} + \mathbf{M} \\ \frac{1}{2}\Delta t^2 \mathbf{K} - \mathbf{M} & \mathbf{M} - \frac{1}{6}\Delta t^2 \mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}$$

CN:
$$\begin{bmatrix} \mathbf{I} & -\frac{1}{2}\mathbf{I} \\ \frac{\Delta t}{2}\mathbf{K} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{I} & \frac{1}{2}\mathbf{I} \\ -\frac{\Delta t}{2}\mathbf{K} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}$$

The proper step-size Δt depends on the largest eigenvalue of parameter $\boldsymbol{\alpha} = \mathbf{M}^{-1}\mathbf{K}\Delta t^2$. The numerical damping of DG exceeds that of CN whereas the phase error of CN exceeds that of the DG method.

EXAMPLE Finite Difference Method using a regular grid $(i, j) \in \{0,1,2\} \times \{0,1,2\}$ is applied to discretize the equations for the rectangular drum head shown. Thereafter, Crank-Nicolson method is applied to find the solution at the temporal grid $t_k = k\Delta t \;\; k \in \{0,1,\ldots\}$. Derive the iteration formula giving the displacements and velocities of points of the spatial discretization starting from the known initial displacement and velocities. Membrane tightening S' and density per unit area ρt are constants.



Answer Discussed during the lectures of week 20