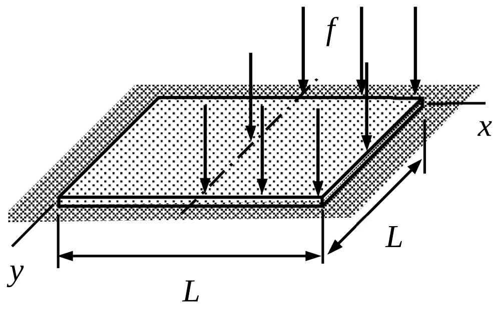
COE-C3005 Finite Element and Finite difference methods

1. A rectangular membrane of side length L, density ρ, thickness t, and tightening S' is loaded by a constant distributed force f acting on half of the membrane as shown. If the edges are fixed, find the transverse displacement by using the continuum model and double sine series representation of the displacement and force.



Answer $w(x,y)=\sum\sum \frac{f}{S'}(\frac{L}{\pi})^4 \frac{1}{kl}\frac{1}{k^2+l^2}[\cos(k\frac{\pi}{2})-\cos(k\pi)][1-\cos(l\pi)]\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L})$

2. Derive the partial difference equation to the membrane model according to the Finite Difference Method and a regular grid of different spacings Δx and Δy in the coordinate directions. Consider a generic interior point (i, j). Start with the equation of motion for the continuum model

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2} \quad (x,y)\in\Omega \quad t>0$$

where tightening per unit length S', distributed force f', and mass per unit area m' are constants.

Answer $\frac{S'}{h^2}[\alpha^2 w_{(i-1,j)}+\alpha^2 w_{(i+1,j)}-2(1+\alpha^2)w_{(i,j)}+w_{(i,j-1)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)}$

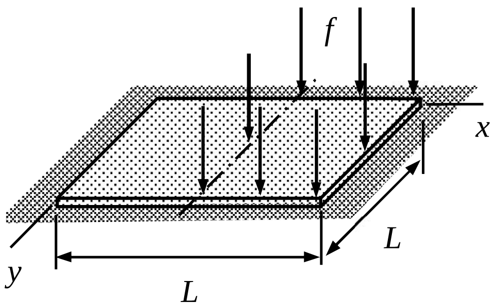
3. Derive the partial difference equation to the membrane model according to the Finite Difference Method and a regular grid of different spacings Δx and Δy in the coordinate directions. Consider a generic boundary point (i, j). Start with the equilibrium equation for the continuum model

$$S'(n_x\frac{\partial w}{\partial x}+n_y\frac{\partial w}{\partial y})=F' \quad (x,y)\in\partial\Omega \quad t>0,$$

where tightening per unit length S' and transverse external force per unit length F' are constants. Above, n_x and n_y are the components of the unit outward normal to the boundary.

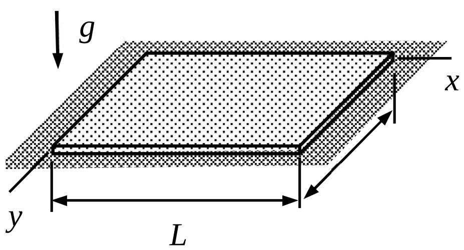
Answer When $n_x=1$ and $n_y=0$: $\frac{S'}{\Delta x}(-w_{(i-1,j)}+w_{(i,j)})=F'$ (for example)

4. A rectangular membrane of side length L and tightening S' is loaded by a constant distributed force f acting on half of the membrane as shown. If the edges are fixed, find the transverse displacement using the Finite Difference Method on a regular grid $(i,j) \in \{0,1,2,3\} \times \{0,1,2,3\}$.



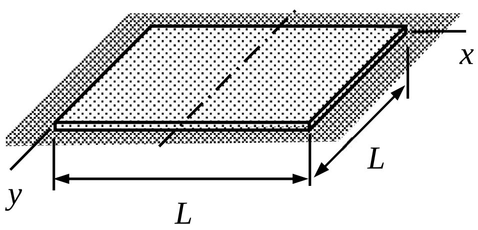
Answer $\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \frac{fL^2}{S'} \frac{1}{72} \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$

5. A rectangular membrane of side length L, density ρ, thickness t, and tightening S' is loaded by its own weight as shown. If the edges are fixed, find the transverse displacements using the Finite Difference Method on a regular grid $(i,j) \in \{0,1,2,3,4\} \times \{0,1,2\}$.



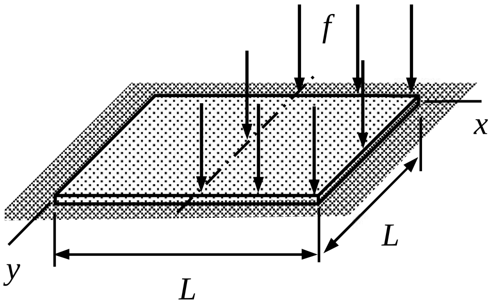
Answer $\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \frac{\rho t g L^2}{S'} \frac{1}{136} \begin{Bmatrix} 7 \\ 9 \end{Bmatrix}$

6. Consider a rectangular membrane of side length L, density ρ, thickness t, and tightening S'. If the edges are fixed, find angular velocities of the modes, that are reflection symmetric with respect to the line through the center point shown using the Finite Difference Method. Use a regular grid $(i,j) \in \{0,1,2,3,4\} \times \{0,1,2\}$ of different spacings in the coordinate directions.



Answer $\omega = \frac{2}{L} \sqrt{(10 \pm 4\sqrt{2}) \frac{S'}{\rho t}}$

A rectangular membrane of side length L, density ρ, thickness t, and tightening S' is loaded by a constant distributed force f acting on half of the membrane as shown. If the edges are fixed, find the transverse displacement by using the continuum model and double sine series representation of the displacement and force.



Solution

According to the problem, transverse displacement and force are considered to be given by double sine series

$$w(x,y)=\sum\sum w_{kl}\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L}) \quad \text{and} \quad f(x,y)=\sum\sum f_{kl}\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L}),$$

where the sums are over the sets $k\in\{1,2,\ldots\}$, $l\in\{1,2,\ldots\}$ and w_{kl}, f_{kl} should be determined by using equilibrium equation and the known distribution of the external force. Both expressions vanish on the boundaries no matter the multipliers. Let us substitute first into the equilibrium equation

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=0 \quad \Rightarrow \quad \sum\sum \sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L})\{[-S'w_{kl}(\frac{k\pi}{L})^2+(\frac{l\pi}{L})^2]+f_{kl}\}=0$$

which implies the relationship

$$w_{kl}=\frac{f_{kl}}{S'}(\frac{L}{\pi})^2\frac{1}{k^2+l^2}.$$

Orthogonality of sines according to

$$\int_0^L \sin(k\pi\frac{x}{L})\sin(i\pi\frac{x}{L})dx=\frac{L}{2}\delta_{ik} \quad \text{and} \quad \int_0^L \sin(l\pi\frac{y}{L})\sin(j\pi\frac{y}{L})dx=\frac{L}{2}\delta_{jl}$$

gives the expression

$$f_{ij}=(\frac{2}{L})^2\int_0^L\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{y}{L})f(x,y)dxdy \quad \Rightarrow$$

$$f_{ij}=f(\frac{2}{L})^2(\int_{L/2}^L \sin(i\pi\frac{x}{L})dx)(\int_0^L \sin(j\pi\frac{y}{L})dy) \quad \Rightarrow$$

$$f_{ij}=f(\frac{2}{\pi})^2\frac{1}{ij}[\cos(i\frac{\pi}{2})-\cos(i\pi)][1-\cos(j\pi)].$$

Combining the results

$$w(x,y)=\sum\sum \frac{f}{S'}(\frac{L}{\pi})^4\frac{1}{kl}\frac{1}{k^2+l^2}[\cos(k\frac{\pi}{2})-\cos(k\pi)][1-\cos(l\pi)]\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L}).$$ ←

Derive the partial difference equation to the membrane model according to the Finite Difference Method and a regular grid of different spacings Δx and Δy in the coordinate directions. Consider a generic interior point (i, j). Start with the equation of motion for the continuum model

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2} \quad (x,y)\in\Omega \quad t>0$$

where tightening per unit length S', distributed force f', and mass per unit area m' are constants.

Solution

In the Finite Difference Method, the derivatives of the differential equation with respect to the spatial coordinates are replaced by difference approximations. Using the second order accurate central difference approximations for derivatives with respect to x and y at point (i, j) and evaluating the second time derivative at that point:

$$\frac{\partial^2 w}{\partial x^2}=\frac{1}{\Delta x^2}(w_{(i-1,j)}-2w_{(i,j)}+w_{(i+1,j)}) \text{ and } \frac{\partial^2 w}{\partial y^2}=\frac{1}{\Delta y^2}(w_{(i,j-1)}-2w_{(i,j)}+w_{(i,j+1)}),$$

$$\frac{\partial^2 w}{\partial t^2}=\ddot{w}_{(i,j)}.$$

Differential equation

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2}$$

gives the difference-differential equation

$$S'\frac{1}{\Delta x^2}[w_{(i-1,j)}-2w_{(i,j)}+w_{(i+1,j)}]+S'\frac{1}{\Delta y^2}[w_{(i,j-1)}-2w_{(i,j)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)} \quad \Leftrightarrow$$

$$\frac{1}{\Delta y^2}\frac{1}{\Delta x^2}[\alpha^2 w_{(i-1,j)}-2\alpha^2 w_{(i,j)}+\alpha^2 w_{(i+1,j)}]+[w_{(i,j-1)}-2w_{(i,j)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)} \quad \Leftrightarrow$$

$$\frac{S'}{h^2}[\alpha^2 w_{(i-1,j)}+\alpha^2 w_{(i+1,j)}-2(1+\alpha^2)w_{(i,j)}+w_{(i,j-1)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)}, \quad \leftarrow$$

where $\Delta y=h$ and $\alpha=\Delta y/\Delta x$.

Derive the partial difference equation to the membrane model according to the Finite Difference Method and a regular grid of different spacings Δx and Δy in the coordinate directions. Consider a generic boundary point (i, j). Start with the equilibrium equation for the continuum model

$$S'(n_x \frac{\partial w}{\partial x} + n_y \frac{\partial w}{\partial y}) = F' \quad (x, y) \in \partial\Omega \quad t > 0,$$

where tightening per unit length S' and transverse external force per unit length F' are constants. Above, n_x and n_y are the components of the unit outward normal to the boundary.

Solution

In the Finite Difference Method, the derivatives of the differential equation with respect to the spatial coordinates are replaced by difference approximations. The equation used and the type of difference approximation depend on the location of the point. The equilibrium equation (in stationary and non-stationary cases) for boundary point (i, j)

$$S'[n_x (\frac{\partial w}{\partial x})_{(i,j)} + n_y (\frac{\partial w}{\partial y})_{(i,j)}] = F'$$

depends on the components n_x and n_y of the unit outward normal vector to the domain. Let us consider a rectangle domain, assume that the edges are aligned with the coordinate axes, omit the corner points, and use first order accurate approximations at point (i, j) for derivatives with respect to x and y. Then, $n_x = \pm 1$ and $n_y = 0$ or $n_y = \pm 1$ and $n_x = 0$ and the backward and forward difference approximations to be used depend on n_x and n_y:

$$n_x = 1: \quad (\frac{\partial w}{\partial x})_{(i,j)} = \frac{1}{\Delta x}(-w_{(i-1,j)} + w_{(i,j)}),$$

$$n_x = -1: \quad (\frac{\partial w}{\partial x})_{(i,j)} = \frac{1}{\Delta x}(w_{(i+1,j)} - w_{(i,j)}),$$

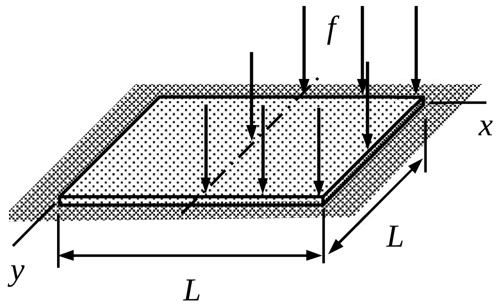
$$n_y = 1: \quad (\frac{\partial w}{\partial y})_{(i,j)} = \frac{1}{\Delta y}(-w_{(i,j-1)} + w_{(i,j)}),$$

$$n_y = -1: \quad (\frac{\partial w}{\partial y})_{(i,j)} = \frac{1}{\Delta y}(w_{(i,j+1)} - w_{(i,j)}).$$

For example, when the boundary is defined by $n_x = 1$ and $n_y = 0$:

$$\frac{S'}{\Delta x}(-w_{(i-1,j)} + w_{(i,j)}) = F'. \quad \leftarrow$$

A rectangular membrane of side length L and tightening S' is loaded by a constant distributed force f acting on half of the membrane as shown. If the edges are fixed, find the transverse displacement using the Finite Difference Method on a regular grid $(i,j) \in \{0,1,2,3\} \times \{0,1,2,3\}$.



Solution

The generic equations for the membrane model with fixed boundaries, as given by the Finite Difference Method on a regular grid, are

$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + f' = m'\ddot{w}_{(i,j)} \quad (i,j) \in I ,$$

$$w_{(i,j)} = 0 \quad (i,j) \in \partial I ,$$

$$w_{(i,j)} - g_{(i,j)} = 0 \quad \text{and} \quad \dot{w}_{(i,j)} - h_{(i,j)} = 0 \quad (i,j) \in I .$$

In the present problem, time derivatives vanish, initial conditions are not needed, and solution is reflection symmetric with respect to lines through the center point and aligned with the coordinate axes. Therefore, transverse displacements at the grid points satisfy

$$w_{(1,1)} = w_{(1,2)} = w_1 ,$$

$$w_{(2,1)} = w_{(2,2)} = w_2 .$$

As the equations by the Finite Difference Method for points $(1,1)$, $(1,2)$ and $(2,1)$, $(2,2)$ do not differ, it is enough consider $(1,1)$ and $(1,2)$ (say) with the displacement constraints. Here $h = L/3$:

$$9\frac{S'}{L^2}[w_{(0,1)} + w_{(1,0)} - 4w_{(1,1)} + w_{(2,1)} + w_{(1,2)}] = 0 \quad \Rightarrow \quad 9\frac{S'}{L^2}(-3w_1 + w_2) = 0 ,$$

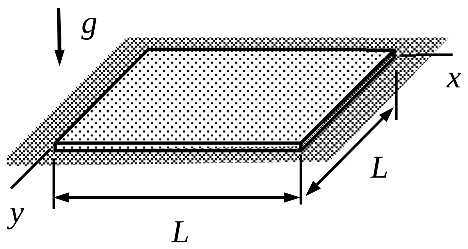
$$9\frac{S'}{L^2}[w_{(1,1)} + w_{(2,0)} - 4w_{(2,1)} + w_{(3,1)} + w_{(2,2)}] + \rho t g = 0 \quad \Rightarrow \quad 9\frac{S'}{L^2}(w_1 - 3w_2) + f = 0 .$$

Using the matrix represntation

$$-9\frac{S'}{L^2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} + f\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \frac{1}{9}\frac{fL^2}{S'}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \frac{1}{9}\frac{fL^2}{S'}\frac{1}{8}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \frac{fL^2}{S'}\frac{1}{72}\begin{Bmatrix} 1 \\ 3 \end{Bmatrix} .$$

A rectangular membrane of side length L, density ρ, thickness t, and tightening S' is loaded by its own weight as shown. If the edges are fixed, find the transverse displacements using the Finite Difference Method on a regular grid $(i,j)\in\{0,1,2,3,4\}\times\{0,1,2\}$.



Solution

The difference equations for regular grid but different spacings of the grid points in the coordinate directions follow from the continuum model boundary value problem when the difference approximations are substituted there. Equations

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=0 \quad (x,y)\in\Omega \quad \text{and} \quad w=0 \quad (x,y)\in\partial\Omega,$$

$$(\frac{\partial^2 w}{\partial x^2})_{(i,j)}=\frac{1}{\Delta x^2}(w_{(i-1,j)}-2w_{(i,j)}+w_{(i+1,j)}), \quad (\frac{\partial^2 w}{\partial y^2})_{(i,j)}=\frac{1}{\Delta y^2}(w_{(i,j-1)}-2w_{(i,j)}+w_{(i,j+1)})$$

give

$$\frac{S'}{h^2}[\alpha^2 w_{(i-1,j)}+\alpha^2 w_{(i+1,j)}-2(1+\alpha^2)w_{(i,j)}+w_{(i,j-1)}+w_{(i,j+1)}]+f'=0 \quad (i,j)\in I,$$

$$w_{(i,j)}=0 \quad (i,j)\in\partial I,$$

where $h=\Delta y=L/2$, $\alpha=\Delta y/\Delta x=2$, I is the index set for the interior points, and ∂I that for the boundary points. In the present problem, $I=\{(1,1),(2,1),(3,1)\}$ and one may use symmetry by defining

$$w_{(1,1)}=w_{(3,1)}=w_1 \quad \text{and} \quad w_{(2,1)}=w_2 .$$

As the equations given by the (present) Finite Difference Method for the constrained points do not differ, it is enough to write the equations for $(i,j)=(1,1)$ and $(i,j)=(2,1)$

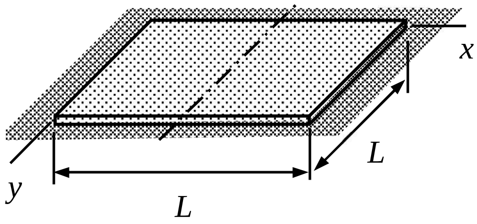
$$4\frac{S'}{L^2}[4w_{(0,1)}+4w_{(2,1)}-10w_{(1,1)}+w_{(1,0)}+w_{(1,2)}]+\rho tg=0 \quad \Rightarrow \quad 4\frac{S'}{L^2}(4w_2-10w_1)+\rho tg=0,$$

$$4\frac{S'}{L^2}[4w_{(1,1)}+4w_{(3,1)}-10w_{(2,1)}+w_{(2,0)}+w_{(2,2)}]+\rho gt=0 \quad \Rightarrow \quad 4\frac{S'}{L^2}(8w_1-10w_2)+\rho tg=0 .$$

In matrix notation, the equations are

$$-4\frac{S'}{L^2}\begin{bmatrix}10 & -4\\ -8 & 10\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\end{Bmatrix}+\rho tg\begin{Bmatrix}1\\ 1\end{Bmatrix}=0 \quad \Leftrightarrow \quad \begin{Bmatrix}w_1\\ w_2\end{Bmatrix}=\rho tg\frac{h^2}{S'}\begin{bmatrix}10 & -4\\ -8 & 10\end{bmatrix}^{-1}\begin{Bmatrix}1\\ 1\end{Bmatrix}=\frac{\rho tgL^2}{S'}\frac{1}{136}\begin{Bmatrix}7\\ 9\end{Bmatrix}. \leftarrow$$

Consider a rectangular membrane of side length L, density ρ, thickness t, and tightening S'. If the edges are fixed, find angular velocities of the modes, that are reflection symmetric with respect to the line through the center point shown using the Finite Difference Method. Use a regular grid $(i,j)\in\{0,1,2,3,4\}\times\{0,1,2\}$ of different spacings in the coordinate directions.



Solution

The difference equations for a regular grid but different point spacing in the coordinate directions follow from the continuum model when the difference approximations are substituted there. Equations

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2} \quad (x,y)\in\Omega \quad \text{and} \quad w=0 \quad (x,y)\in\partial\Omega,$$

$$(\frac{\partial^2 w}{\partial x^2})_{(i,j)}=\frac{1}{\Delta x^2}(w_{(i-1,j)}-2w_{(i,j)}+w_{(i+1,j)}), \quad (\frac{\partial^2 w}{\partial y^2})_{(i,j)}=\frac{1}{\Delta y^2}(w_{(i,j-1)}-2w_{(i,j)}+w_{(i,j+1)})$$

give (initial conditions do not matter in modal analysis)

$$\frac{S'}{h^2}[\alpha^2 w_{(i-1,j)}+\alpha^2 w_{(i+1,j)}-2(1+\alpha^2)w_{(i,j)}+w_{(i,j-1)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)} \quad (i,j)\in I$$

$$w_{(i,j)}=0 \quad (i,j)\in\partial I$$

where $h=\Delta y=L/2$, $\alpha=\Delta y/\Delta x=2$, I is the index set for the interior points, and ∂I that for the boundary points. In the present problem, $I=\{(1,1),(2,1),(3,1)\}$ and one may use symmetry by defining

$$w_{(1,1)}=w_{(3,1)}=w_1 \quad \text{and} \quad w_{(2,1)}=w_2 .$$

As the equations given by the (present) Finite Difference Method for the constrained points do not differ, it is enough to write the equations for $(i,j)=(1,1)$ and $(i,j)=(2,1)$ to get

$$4\frac{S'}{L^2}(4w_2-10w_1)=\rho t\ddot{w}_1 \text{ and } 4\frac{S'}{L^2}(8w_1-10w_2)=\rho t\ddot{w}_2 .$$

In matrix notation, the equations can be written as

$$\begin{bmatrix} 10 & -4 \\ -8 & 10 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}+\frac{\rho t L^2}{4S'}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix}=0 .$$

Solution trial $\mathbf{w} = \mathbf{A}e^{i\omega t}$ gives an algebraic equation system for the mode $\mathbf{A}$ and the corresponding angular velocity ω

$$\left(\begin{bmatrix} 10 & -4 \\ -8 & 10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \text{, where } \lambda = \frac{\rho t L^2}{4S'}\omega^2 \quad \text{or} \quad \omega = \frac{2}{L}\sqrt{\lambda \frac{S'}{\rho t}} \text{ .}$$

Solution to the possible angular velocities follow from condition

$$\det\left(\begin{bmatrix} 10-\lambda & -4 \\ -8 & 10-\lambda \end{bmatrix}\right) = (10-\lambda)^2 - 32 = 0 \quad \Rightarrow \quad \lambda = 10 \pm 4\sqrt{2} \text{ .}$$

The corresponding angular velocities follow from the relationship between ω and λ

$$\omega_1 = \frac{2}{L}\sqrt{\lambda_1 \frac{S'}{\rho t}} = \frac{2}{L}\sqrt{(10+4\sqrt{2})\frac{S'}{\rho t}} \text{ and } \omega_2 = \frac{2}{L}\sqrt{\lambda_2 \frac{S'}{\rho t}} = \frac{2}{L}\sqrt{(10-4\sqrt{2})\frac{S'}{\rho t}} \text{ .} \quad \leftarrow$$