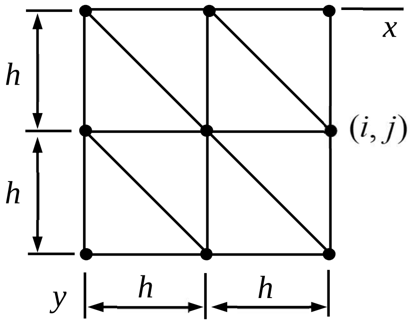
COE-C3005 Finite Element and Finite difference methods

1. Derive the linear interpolant expression for line and triangle elements

$$w(x)=\begin{Bmatrix} w_i \\ w_j \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ x_i & x_j \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} \text{ and } w(x,y)=\begin{Bmatrix} w_i \\ w_j \\ w_k \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}.$$

2. Derive the difference equation for boundary point (i, j) of a regular grid shown. Use piecewise linear approximation w and the weighted residual expression $R_{(i,j)}^{\text{int}} + R_{(i,j)}^{\text{ext}} = 0$, where ($S'$ and f' are constants)

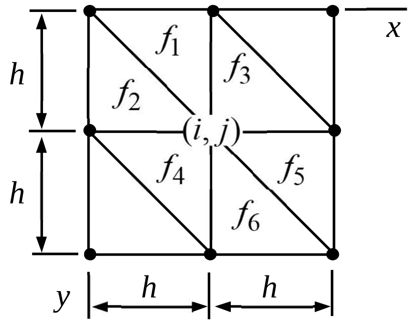


$$R_{(i,j)}^{\text{int}} = -\int_{\Omega} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA,$$

$$R_{(i,j)}^{\text{ext}} = \int_{\Omega} (N_{(i,j)} f')dA.$$

Answer $\frac{1}{2}S[2w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i,j+1)}] + f'\frac{h^2}{2} = 0$

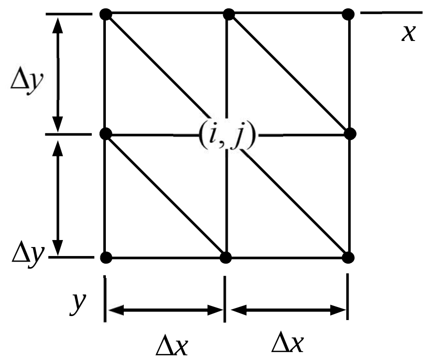
3. Find the weighted residual expression $R_{(i,j)}^{\text{ext}}$ for an external distributed force f' which is piecewise constant. Use the notation in the figure for the constant values in the triangle elements and expression



$$R_{(i,j)}^{\text{ext}} = \int_{\Omega} (N_{(i,j)} f')dA.$$

Answer $R_{(i,j)}^{\text{ext}} = \frac{h^2}{6}(f_1 + f_2 + \ldots + f_6)$

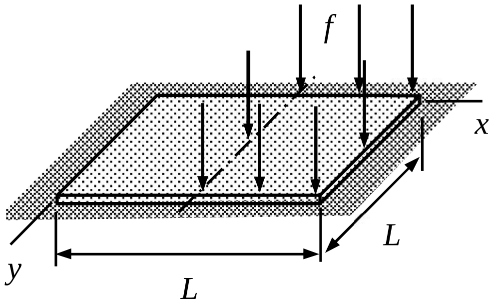
4. Find the weighted residual expression $R_{(i,j)}^{\text{int}}$ for internal forces when the spacing of regular grid differs in the $x-$ and $y-$directions. Use the expression



$$R_{(i,j)}^{\text{int}} = -\int_{\Omega} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA.$$

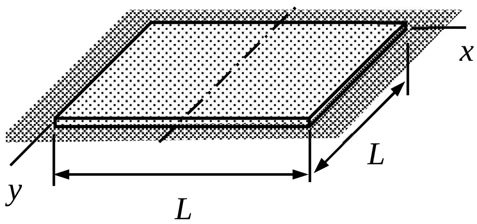
Answer $R_{(i,j)}^{\text{int}} = -S'\frac{\Delta x}{\Delta y}[-w_{(i,j-1)} + 2w_{(i,j)} - w_{(i,j+1)}] - S'\frac{\Delta y}{\Delta x}[-w_{(i-1,j)} + 2w_{(i,j)} - w_{(i+1,j)}]$

5. A rectangular membrane of side length L and tightening S' is loaded by a constant distributed force f acting on half of the membrane as shown. If the edges are fixed, find the transverse displacement using the Finite Element Method on a regular grid $(i,j) \in \{0,1,2\} \times \{0,1,2\}$.



Answer $w_{(1,1)} = \dfrac{h^2 f}{8S'}$

6. Consider a rectangular membrane of side length L, ρ, thickness t, and tightening S'. If the edges are fixed, find angular velocities of the modes, that are reflection symmetric with respect to the line through the center point shown using the Finite Element Method. Use a regular grid $(i,j) \in \{0,1,2,3,4\} \times \{0,1,2\}$ of different spacings in the coordinate directions.



Answer $\omega = \dfrac{4}{L}\sqrt{3(2 \pm \sqrt{2})\dfrac{S'}{\rho t}}$

Derive the linear interpolant expression for line and triangle elements

$$w(x)=\begin{Bmatrix} w_i \\ w_j \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ x_i & x_j \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} \quad \text{and} \quad w(x,y)=\begin{Bmatrix} w_i \\ w_j \\ w_k \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}.$$

Solution

Let us start with the line element. By definition, interpolant $w(x)$ is linear in x

$$w(x)=a+bx=\begin{Bmatrix} a \\ b \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

and takes the prescribed values at the end points so $w(x_i)=a+bx_i=w_i$ and $w(x_j)=a+bx_j=w_j$. Using the matrix notation

$$\begin{bmatrix} 1 & x_i \\ 1 & x_j \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} w_i \\ w_j \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{bmatrix} 1 & x_i \\ 1 & x_j \end{bmatrix}^{-1} \begin{Bmatrix} w_i \\ w_j \end{Bmatrix}.$$

Therefore, in terms of the end point values

$$w(x)=\begin{Bmatrix} a \\ b \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \begin{Bmatrix} w_i \\ w_j \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ x_i & x_j \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix}.$$

Then using the same steps in case of a triangle element: By definition, interpolant $w(x,y)$ is linear in x and y

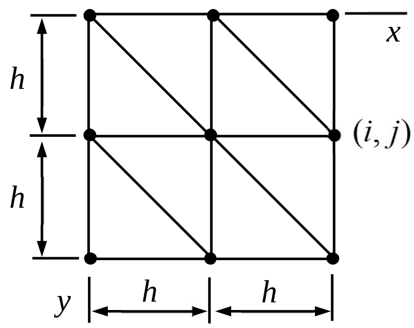
$$w(x,y)=a+bx+cy=\begin{Bmatrix} a \\ b \\ c \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

and takes the prescribed values at the end points so $w(x_\alpha, y_\alpha)=a+bx_\alpha+cy_\alpha=w_\alpha$ where $\alpha \in \{i,j,k\}$. Using the matrix notation

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} w_i \\ w_j \\ w_k \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}^{-1} \begin{Bmatrix} w_i \\ w_j \\ w_k \end{Bmatrix}.$$

Therefore, in terms of the end point values

$$w(x,y)=a+bx+cy=\begin{Bmatrix} w_i \\ w_j \\ w_k \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}.$$



Derive the difference equation for boundary point (i, j) of a regular grid shown. Use piecewise linear approximation w and the weighted residual expression $R^{\text{int}}_{(i,j)} + R^{\text{ext}}_{(i,j)} = 0$, where ($S'$ and f' are constants)

$$R^{\text{int}}_{(i,j)} = -\int_{\Omega} S'\left(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y}\right) dA,$$

$$R^{\text{ext}}_{(i,j)} = \int_{\Omega} (N_{(i,j)} f') dA.$$

Solution

The integrals of the residuals are calculated element-by-element using the linear expression inside the elements using Mathematica or deducing the expression based on the geometrical picture about plane defined by the values of w at the vertex points and the piecewise linear shape function taking the value one at point (i, j) and vanishing at all other grid points. It is enough to consider only the elements having (i, j) as one of the vertex point as $N_{(i,j)}$ vanishes elsewhere. Let us start with $R^{\text{int}}_{(i,j)}$

Triangle Ω_1 of vertices (i, j), $(i, j-1)$, and $(i-1, j-1)$ (counterclockwise):

$$\frac{\partial N_{(i,j)}}{\partial x} = 0, \quad \frac{\partial N_{(i,j)}}{\partial y} = \frac{1}{h}, \quad \frac{\partial w}{\partial x} = \frac{w_{(i,j-1)} - w_{(i-1,j-1)}}{h}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i,j)} - w_{(i,j-1)}}{h} \quad \Rightarrow$$

$$-\int_{\Omega_1} S'\left(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y}\right) dA = -S'\frac{1}{2}(w_{(i,j)} - w_{(i,j-1)}).$$

Triangle Ω_2 of vertices (i, j), $(i-1, j-1)$, and $(i-1, j)$ (counterclockwise):

$$\frac{\partial N_{(i,j)}}{\partial x} = \frac{1}{h}, \quad \frac{\partial N_{(i,j)}}{\partial y} = 0, \quad \frac{\partial w}{\partial x} = \frac{w_{(i,j)} - w_{(i-1,j)}}{h}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i-1,j)} - w_{(i-1,j-1)}}{h} \quad \Rightarrow$$

$$-\int_{\Omega_2} S'\left(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y}\right) dA = -S'\frac{1}{2}(w_{(i,j)} - w_{(i-1,j)})$$

Triangle Ω_3 of vertices (i, j), $(i-1, j)$, and $(i, j+1)$ (counterclockwise):

$$\frac{\partial N_{(i,j)}}{\partial x} = \frac{1}{h}, \quad \frac{\partial N_{(i,j)}}{\partial y} = -\frac{1}{h}, \quad \frac{\partial w}{\partial x} = \frac{w_{(i,j)} - w_{(i-1,j)}}{h}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i,j+1)} - w_{(i,j)}}{h} \quad \Rightarrow$$

$$-\int_{\Omega_3} S'\left(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y}\right) dA = -S'\frac{1}{2}(-w_{(i-1,j)} + 2w_{(i,j)} - w_{(i,j+1)})$$

Therefore

$$R^{\text{int}}_{(i,j)} = -\int_{\Omega} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = \frac{1}{2}S(2w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i,j+1)}) .$$ ←

Let us consider then $R^{\text{ext}}_{(i,j)}$ in the same manner. As distributed force is constant it can be taken outside the integral. The remaining task is to find the integral of $N_{(i,j)}$ over the elements. Using the generic expression for an interpolant taking the value 1 at

Triangle Ω_1 of vertices (i,j), $(i,j-1)$, and $(i-1,j-1)$.

$$N_{(i,j)} = \begin{Bmatrix}1\\0\\0\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}1 & 1 & 1\\ 2h & 2h & h\\ h & 0 & 0\end{bmatrix}^{-1} \begin{Bmatrix}1\\x\\y\end{Bmatrix} = \frac{y}{h} \quad\Rightarrow\quad \int_h^{2h}\int_0^{x-h}\frac{y}{h}dydx = \int_h^{2h}\frac{1}{2}\frac{(x-h)^2}{h}dx = \frac{h^2}{6}$$

Triangle Ω_2 of vertices (i,j), $(i-1,j-1)$, and $(i-1,j)$:

$$N_{(i,j)} = \begin{Bmatrix}1\\0\\0\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}1 & 1 & 1\\ 2h & h & h\\ h & 0 & h\end{bmatrix}^{-1} \begin{Bmatrix}1\\x\\y\end{Bmatrix} = \frac{x}{h} - 1 \quad\Rightarrow\quad \int_h^{2h}\int_{x-h}^{h}(\frac{x}{h}-1)dydx = \frac{h^2}{6}$$

Triangle Ω_3 of vertices (i,j), $(i-1,j)$, and $(i,j+1)$:

$$N_{(i,j)} = \begin{Bmatrix}1\\0\\0\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}1 & 1 & 1\\ 2h & h & 2h\\ h & h & 2h\end{bmatrix}^{-1} \begin{Bmatrix}1\\x\\y\end{Bmatrix} = \frac{x}{h} - \frac{y}{h} \quad\Rightarrow\quad \int_h^{2h}\int_h^{x}(\frac{x}{h}-\frac{y}{h})dydx = \frac{h^2}{6}$$

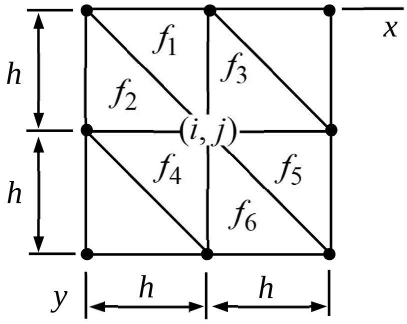
Therefore

$$R^{\text{ext}}_{(i,j)} = f'(\frac{h^2}{6} + \frac{h^2}{6} + \frac{h^2}{6}) = f'\frac{h^2}{2} .$$ ←

And the difference equation for boundary point (i,j)

$$R^{\text{int}}_{(i,j)} + R^{\text{ext}}_{(i,j)} = \frac{1}{2}S(2w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i,j+1)}) + f'\frac{h^2}{2} = 0 .$$ ←

Find the weighted residual expression $R_{(i,j)}^{\text{ext}}$ for an external distributed force f' which is piecewise constant. Use the notation in the figure for the constant values in the triangle elements and expression



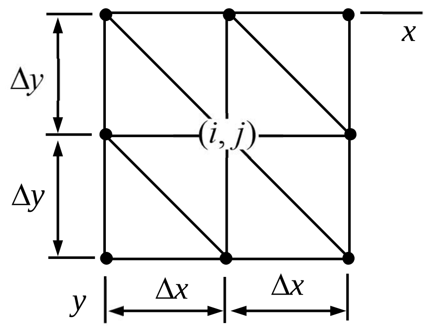
$$R_{(i,j)}^{\text{ext}} = \int_\Omega (N_{(i,j)} f')dA .$$

Solution

According to problem 2, for a regular grid of equal spacing h in the coordinate directions, integral of the shape function of point (i, j) over a tringle having (i, j) as one of its vertex points is always $h^2 / 6$. As distributed force f' constant in each element

$$R_{(i,j)}^{\text{ext}} = \int_\Omega (N_{(i,j)} f')dA = \sum\nolimits_{e\in\{1,\ldots,6\}} f_e \int_{\Omega_e} N_{(i,j)})dA = \frac{h^2}{6} \sum\nolimits_{e\in\{1,\ldots,6\}} f_e .$$ ←

Find the weighted residual expression $R_{(i,j)}^{\text{int}}$ for internal forces when the spacing of regular grid differs in the $x-$ and $y-$ directions. Use the expression



$$R_{(i,j)}^{\text{int}} = -\int_{\Omega} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA .$$

Solution

The integrals of the residuals are calculated element-by-element using the linear expression inside the elements using Mathematica or deducing the expression based on the geometrical picture about a plane defined by the values of w at the vertex points and the piecewise linear shape function taking the value one at point (i, j) and vanishing at all other grid points. It is enough to consider only the elements having (i, j) as one of the vertex point as $N_{(i,j)}$ vanishes elsewhere.

Triangle Ω_1 of vertices (i, j), $(i, j-1)$, and $(i-1, j-1)$:

$$\frac{\partial N_{(i,j)}}{\partial x} = 0, \quad \frac{\partial N_{(i,j)}}{\partial y} = \frac{1}{\Delta y}, \quad \frac{\partial w}{\partial x} = \frac{w_{(i,j-1)} - w_{(i-1,j-1)}}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i,j)} - w_{(i,j-1)}}{\Delta y} \quad \Rightarrow$$

$$-\int_{\Omega_1} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = -S'\frac{1}{2}\frac{\Delta x}{\Delta y}[w_{(i,j)} - w_{(i,j-1)}] .$$

Triangle Ω_2 of vertices (i, j), $(i-1, j-1)$, and $(i-1, j)$:

$$\frac{\partial N_{(i,j)}}{\partial x} = \frac{1}{\Delta x}, \quad \frac{\partial N_{(i,j)}}{\partial y} = 0, \quad \frac{\partial w}{\partial x} = \frac{w_{(i,j)} - w_{(i-1,j)}}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i-1,j)} - w_{(i-1,j-1)}}{\Delta y} \quad \Rightarrow$$

$$-\int_{\Omega_2} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = -S'\frac{1}{2}\frac{\Delta y}{\Delta x}[w_{(i,j)} - w_{(i-1,j)}] .$$

Triangle Ω_3 of vertices (i, j), $(i-1, j)$, and $(i, j+1)$:

$$\frac{\partial N_{(i,j)}}{\partial x} = \frac{1}{\Delta x}, \quad \frac{\partial N_{(i,j)}}{\partial y} = -\frac{1}{\Delta y}, \quad \frac{\partial w}{\partial x} = \frac{w_{(i,j)} - w_{(i-1,j)}}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i,j+1)} - w_{(i,j)}}{\Delta y} \quad \Rightarrow$$

$$-\int_{\Omega_3} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = -S'\frac{1}{2}\frac{\Delta y}{\Delta x}[w_{(i,j)} - w_{(i-1,j)}] + S'\frac{1}{2}\frac{\Delta x}{\Delta y}[w_{(i,j+1)} - w_{(i,j)}] .$$

Triangle Ω_4 of vertices (i, j), $(i, j+1)$, and $(i+1, j+1)$:

$$\frac{\partial N_{(i,j)}}{\partial x} = 0, \quad \frac{\partial N_{(i,j)}}{\partial y} = -\frac{1}{\Delta y}, \quad \frac{\partial w}{\partial x} = \frac{w_{(i+1,j+1)} - w_{(i,j+1)}}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i,j+1)} - w_{(i,j)}}{\Delta y} \quad \Rightarrow$$

$$-\int_{\Omega_4} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = S'\frac{1}{2}\frac{\Delta x}{\Delta y}[w_{(i,j+1)}-w_{(i,j)}].$$

Triangle Ω_5 of vertices (i, j), $(i+1, j+1)$, and $(i+1, j)$:

$$\frac{\partial N_{(i,j)}}{\partial x} = -\frac{1}{\Delta x}, \quad \frac{\partial N_{(i,j)}}{\partial y} = 0, \quad \frac{\partial w}{\partial x} = \frac{w_{(i+1,j)}-w_{(i,j)}}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i+1,j+1)}-w_{(i+1,j)}}{\Delta y} \quad \Rightarrow$$

$$-\int_{\Omega_5} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = S'\frac{1}{2}\frac{\Delta y}{\Delta x}[w_{(i+1,j)}-w_{(i,j)}].$$

Triangle Ω_6 of vertices (i, j), $(i+1, j+1)$, and $(i+1, j)$:

$$\frac{\partial N_{(i,j)}}{\partial x} = -\frac{1}{\Delta x}, \quad \frac{\partial N_{(i,j)}}{\partial y} = \frac{1}{\Delta y}, \quad \frac{\partial w}{\partial x} = \frac{w_{(i+1,j)}-w_{(i,j)}}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i,j)}-w_{(i,j-1)}}{\Delta y} \quad \Rightarrow$$

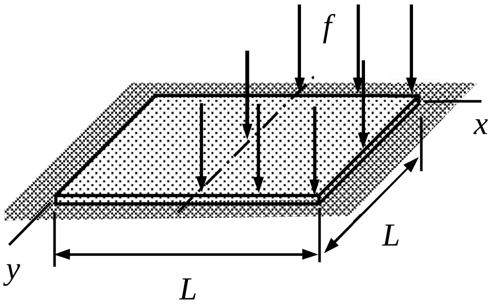
$$-\int_{\Omega_6} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = S'\frac{1}{2}\frac{\Delta y}{\Delta x}[w_{(i+1,j)}-w_{(i,j)}] - S'\frac{1}{2}\frac{\Delta x}{\Delta y}[w_{(i,j)}-w_{(i,j-1)}].$$

Therefore

$$R^{\text{int}}_{(i,j)} = -\int_{\Omega} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA \quad \Rightarrow$$

$$R^{\text{int}}_{(i,j)} = -S'\frac{\Delta x}{\Delta y}[-w_{(i,j-1)}+2w_{(i,j)}-w_{(i,j+1)}] - S'\frac{\Delta y}{\Delta x}[-w_{(i-1,j)}+2w_{(i,j)}-w_{(i+1,j)}]. \quad \leftarrow$$

A rectangular membrane of side length L and tightening S' is loaded by a constant distributed force f acting on half of the membrane as shown. If the edges are fixed, find the transverse displacement using the Finite Element Method on a regular grid $(i,j)\in\{0,1,2\}\times\{0,1,2\}$.



Solution

In stationary problem, the generic equations for the membrane model with fixed boundaries as given by the Finite Element Method on a regular grid are

$$S'[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]+\frac{h^2}{6}\sum f=0 \quad (i,j)\in I\,,$$

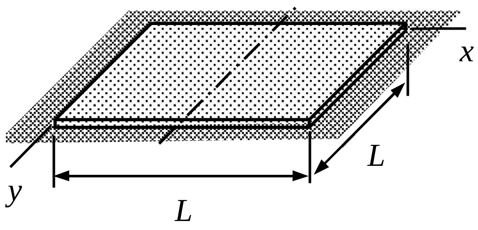
$$w_{(i,j)}=0 \quad (i,j)\in\partial I\,,$$

where the distributed loading is assumed to be piecewise constant (constant in each element) and the sum is over the constant loads in the elements having the point (i,j) as one of the vertices. In the present problem, time derivatives vanish, initial conditions are not needed. Equilibrium equation of the only interior point $(1,1)$

$$S'[w_{(0,1)}+w_{(1,0)}-4w_{(1,1)}+w_{(2,1)}+w_{(1,2)}]+\frac{h^2}{6}(f+f+f)=0 \quad \Rightarrow$$

$$S'[-4w_{(1,1)}]+\frac{h^2 f}{2}=0 \quad \Leftrightarrow \quad w_{(1,1)}=\frac{h^2 f}{8S'}.$$

Consider a rectangular membrane of side length L, density ρ, thickness t, and tightening S'. If the edges are fixed, find angular velocities of the modes, that are reflection symmetric with respect to the line through the center point shown using the Finite Element Method. Use a regular grid $(i,j) \in \{0,1,2,3,4\} \times \{0,1,2\}$ of different spacings in the coordinate directions.



Solution

Assuming regular grid of points and regular triangle division, the generic equations for the membrane model with fixed boundaries as given by the Finite Element Method are (the left hand side is given by the solution to problem 4, the right hand side can be obtained just by replacing h^2 in the expression of the formulae collection by $\Delta x \Delta y$)

$$-S'\frac{\Delta x}{\Delta y}[-w_{(i,j-1)}+2w_{(i,j)}-w_{(i,j+1)}]-S'\frac{\Delta y}{\Delta x}[-w_{(i-1,j)}+2w_{(i,j)}-w_{(i+1,j)}]=$$

$$m'\Delta x\Delta y\frac{1}{12}[\ddot{w}_{(i-1,j-1)}+\ddot{w}_{(i-1,j)}+\ddot{w}_{(i,j-1)}+6\ddot{w}_{(i,j)}+\ddot{w}_{(i+1,j)}+\ddot{w}_{(i,j+1)}+\ddot{w}_{(i+1,j+1)}] \quad (i,j)\in I \quad t>0$$

$$w_{(i,j)}=0 \quad (i,j)\in\partial I \quad t>0,$$

As the mode is assumed to be reflection symmetric with respect to lines through the center point and aligned with the coordinate axes, transverse displacements at the grid points satisfy

$$w_{(1,1)}=w_{(3,1)}=w_1 \text{ and } w_{(2,1)}=w_2,$$

the remaining displacements at the boundary points being zeros. Using $m'=\rho t$, $\Delta x = L/4$, $\Delta y = L/2$ in equations for points $(1,1)$, $(2,1)$, and $(3,1)$

$$-S'\frac{1}{2}[-w_{(1,0)}+2w_{(1,1)}-w_{(1,2)}]-S'\frac{2}{1}[-w_{(0,1)}+2w_{(1,1)}-w_{(2,1)}]=$$

$$m'\frac{L^2}{8}\frac{1}{12}[\ddot{w}_{(0,0)}+\ddot{w}_{(0,1)}+\ddot{w}_{(1,0)}+6\ddot{w}_{(1,1)}+\ddot{w}_{(2,1)}+\ddot{w}_{(1,2)}+\ddot{w}_{(2,2)}] \quad \Rightarrow$$

$$-S'(10w_1-4w_2)=\frac{\rho t L^2}{48}(6\ddot{w}_1+\ddot{w}_2).$$

$$-S'\frac{1}{2}[-w_{(2,0)}+2w_{(2,1)}-w_{(2,2)}]-S'\frac{2}{1}[-w_{(1,1)}+2w_{(2,1)}-w_{(3,1)}]=$$

$$m'\frac{L^2}{8}\frac{1}{12}[\ddot{w}_{(1,0)}+\ddot{w}_{(1,1)}+\ddot{w}_{(2,0)}+6\ddot{w}_{(2,1)}+\ddot{w}_{(3,1)}+\ddot{w}_{(2,2)}+\ddot{w}_{(3,2)}] \quad \Rightarrow$$

$$-S'(-8w_1+10w_2)=\rho t\frac{L^2}{48}(2\ddot{w}_1+6\ddot{w}_2)\,.$$

$$-S'\frac{1}{2}[-w_{(3,0)}+2w_{(3,1)}-w_{(3,2)}]-S'\frac{2}{1}[-w_{(2,1)}+2w_{(3,1)}-w_{(4,1)}]=$$

$$m'\frac{L^2}{8}\frac{1}{12}[\ddot{w}_{(2,0)}+\ddot{w}_{(2,1)}+\ddot{w}_{(3,0)}+6\ddot{w}_{(3,1)}+\ddot{w}_{(4,1)}+\ddot{w}_{(3,2)}+\ddot{w}_{(4,2)}] \quad \Rightarrow$$

$$-S'(-4w_2+10w_1)=\rho t\frac{L^2}{48}(\ddot{w}_2+6\ddot{w}_1)\,.$$

According to the principle of virtual work, the equation for a constrained displacement is the sum of equations for the constrained points. Here the equations coincide so one may use. e.g., equation for point $(1,1)$. In matrix notation

$$\begin{bmatrix}10 & -4\\ -8 & 10\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\end{Bmatrix}+\frac{1}{48}\frac{\rho t L^2}{S'}\begin{bmatrix}6 & 1\\ 2 & 6\end{bmatrix}\begin{Bmatrix}\ddot{w}_1\\ \ddot{w}_2\end{Bmatrix}=0\,.$$

Solution to the angular velocities and the corresponding modes follow with the trial solution $\mathbf{a}=\mathbf{A}e^{i\omega t}$:

$$(\begin{bmatrix}10 & -4\\ -8 & 10\end{bmatrix}-\lambda\begin{bmatrix}6 & 1\\ 2 & 6\end{bmatrix})\begin{Bmatrix}A_1\\ A_2\end{Bmatrix}=0 \text{ where } \lambda=\omega^2\frac{1}{48}\frac{\rho t L^2}{S'} \quad \Leftrightarrow \quad \omega=\frac{4}{L}\sqrt{3\lambda\frac{S'}{\rho t}}\,.$$

A homogeneous linear equation system can yield a non-zero solution to the mode only if the matrix is singular, i.e., its determinant vanishes. The condition can be used to find the possible values of λ

$$\det\begin{bmatrix}10-6\lambda & -4-\lambda\\ -8-2\lambda & 10-6\lambda\end{bmatrix}=(10-6\lambda)^2-2(4+\lambda)^2=0 \quad \Rightarrow \quad \lambda=2\pm\sqrt{2}\,.$$

Knowing the possible values of parameter λ, the angular velocities follow from the relationship between λ and ω:

$$\omega=\frac{4}{L}\sqrt{3(2\pm\sqrt{2})\frac{S'}{\rho t}}\,. \quad \leftarrow$$