Recalling some basic notions in commutative algebra, geometry and topology, on the way toward Algebraic Geometry 1

Alexander Engström   Milo Orlich

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The official description of the course prerequisites is: *"Some advanced algebra, geometry and topology. Mathematical maturity is more important than specifics. If available, taking any course in applied or numerical algebraic geometry in advance is useful. Prospective students are encouraged to take a glance at the study material."* In this note we parse this out in greater detail, as requested by some students, and provide a list of exercises that should work as a warm-up before the course actually starts. Some of these exercises will probably be discussed during the very first exericse session at the very beginning of the course. It's not mandatory to present a solution, but it's in your best interest to know these facts. You might be able to solve part of these exercises without writing anything down, as most of them are meant to be easy.

Commutative algebra

In this course we will always assume that rings are commutative. You should have familiarity with the following notions: ring, (radical, prime, maximal) ideal, ring homomorphism, kernel of a homomorphism, quotient of a ring, R-modules, submodules, module homomorphisms, quotient of a module, operations on modules or ideals (like taking their sum, direct sum, intersection,. . .). You should be acquainted with fundamental results such as the following.

Theorem 1 (correspondence theorem for rings)**.** *Let R be a ring and let I be an ideal of R. There is a bijection between the sets*

$$\begin{aligned}\{\textit{ideals of } R \textit{ containing } I\} &\longrightarrow \{\textit{ideals of } R/I\}\\ J &\longmapsto \overline{J}.\end{aligned}$$

Theorem 2 (first isomorphism theorem for modules)**.** *Let R be a ring and let M_1 and M_2 be R-modules. Let $\varphi\colon M_1 \to M_2$ be a homomorphism of R-modules. Then there is an isomorphism*

$$R/\ker(\varphi) \cong \operatorname{im}(\varphi).$$

The following exercises should be crystal clear. Again, by "ring" we mean a commutative ring, containing a unit element 1. In the course we will *not* assume in general that $1 \neq 0$. In fields and integral domains, by definition $0 \neq 1$.

Exercise 3. Let I be an ideal of a ring R. Prove the implications

$$I \text{ is maximal} \quad \Rightarrow \quad I \text{ is prime} \quad \Rightarrow \quad I \text{ is radical.}$$

Exercise 4. Let I be an ideal of a ring R. Show that

1. I is a radical ideal iff R/I does not contain nilpotent elements other than 0;

2. I is a prime ideal iff R/I is an integral domain;

3. I is a maximal ideal iff R/I is a field.

Let k be a ring (usually it will be a field) and $S = k[x_1, \dots, x_n]$ be the polynomial ring with coefficients in k and n variables.

Exercise 5. Give examples in $\mathbb{Z}$ and in S (for some k of your choice) of ideals that are prime but not maximal, and ideals that are radical but not prime. Give examples of rings where it's not possible to find such ideals.

Exercise 6. Determine whether the following statements are true or false.

1. The map $S \to S$ that takes a polynomial and replaces every variable with the number $1 \in k$ is a ring homomorphism.

2. The same map as above is a homomorphism of S-modules.

3. Every ideal of $\mathbb{Z}$ is principal (i.e., it can be generated by a single element).

4. Every ideal of S is principal. (Does the answer depend on k or n?)

Exercise 7. Let R be a ring. Show that the following are equivalent:

- every ideal in R is finitely generated,

- any strictly increasing (with respect to inclusion) chain of ideals in R is finite.

If a ring R satisfies the equivalent conditions in the exercise above, R is called a *Noetherian ring.* Familiarity with this concept is recommended. For instance, $\mathbb{Z}$ is Noetherian, any field is Noetherian, and if R is a Noetherian ring then a polynomial ring with coefficients in R and finitely many variables is Noetherian (this last fact is *Hilbert's basis theorem*).

Rings that have exactly one maximal ideal are called *local rings.* Some familiarity with them (in particular, with localizations) could be helpful.

Classical algebraic geometry

Let k be a field. In classical algebraic geometry, one studies *algebraic varieties*, that is, the zero loci of some polynomials. For instance the unit circle

$$x^2 + y^2 = 1$$

is the zero locus of the polynomial x^2+y^2-1. For an ideal $I \subset k[x_1, \dots, x_n]$, denote

$$V(I) := \{(a_1, \dots, a_n) \in k^n \mid f(a_1, \dots, a_n) = 0 \text{ for all } f \in I\}$$

the algebraic variety associated to I. Be aware that this definition is just meant for this document. In the actual course things will be more abstract.

Exercise 8. If possible (for k and n of your choice), give examples of

- $V(I)$ equal to a finite set of points;
- $V(I)$ equal to an infinite set of isolated points;
- $V(I)$ equal to the union of two disjoint curves;
- $V(I)$ equal to a plane;
- $V(I)$ equal to the empty set.

You should know what projective spaces are.

Exercise 9. What goes wrong if one defines projective varieties (that is, in a projective space instead of k^n) in the same way as above? How should one define things in the projective setting in order to be as close as possible to the definition of $V(I)$ above?

You should have at least some familiarity with the concept of variety, in the sense that what is written above should at least ring a bell.

Definition 10 (Zariski topology, rudimentary version)**.** Let τ be the topology on k^n whose closed sets are the varieties $V(I)$, for $I \subset k[x_1, \dots, x_n]$.

Exercise 11.

1. Show that τ is a topology on k^n. (You may want to show in particular that $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.)
2. Is (k^n, τ) a Hausdorff space?
3. What is the relation between the Euclidean topology and the Zariski topology on $\mathbb{R}^n$ (that is, for $k = \mathbb{R}$)?
4. If k is a finite field, describe the Zariski topology on k^n.