ELEC-E4130

Lecture 1: House keeping + mathematical review 1

Aalto University
School of Electrical
Engineering

ELEC-E4130 / Taylor

Sept. 13, 2021

Instructors

Prof. Zachary Taylor

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- Research interests
  - *THz and submillimeter wave techniques*
  - *Medical imaging*

Dr. Henrik Wallen

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- Research interests
  - *Computational electromagnetics*
  - *Electromagnetic theory*

Course Assistants

Pouyan Rezapoor

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- Research interests
  - *THz and submillimeter wave techniques*
  - ***Surgical flap viability via submillimeter wave metamaterials***

Joel Lamburg

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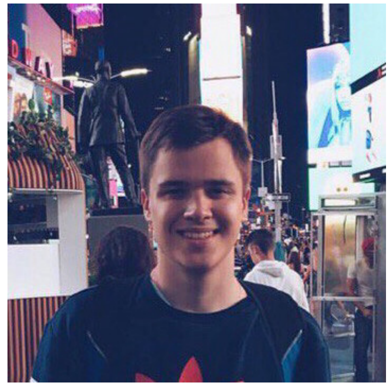
- Research interests
  - *THz and submillimeter wave techniques*
  - ***Fourier optics for corneal sensing***

Maxim Masyukov



- maxim.masyukov@aalto.fi
- Research interests
  - *THz and submillimeter wave techniques*
  - ***Submillimeter wave on-wafer and quasioptical calibration***

How about you?

Poll 1

- **Where are you now?**
  - *Near campus*
  - *In Finland but not near campus*
  - *Outside of Finland*

Poll 2

- **Will it be possible for you to come to campus starting closer to period II (Middle of October) to have in person exercise sessions?**
  - *Yes*
  - *No*

We may try to organize in person exercise sessions if the situation permits

Poll 3

- **How comfortable are you with vector calculus?**
  - *Comfortable*
  - *Uncomfortable*
  - *I don't know anything about it*

ELEC-E4130 Learning outcomes

- Understand electromagnetic theory behind guided waves, waves in free space, and the coupling between these modes.
- Explain the link between Maxwell’s equations and plane wave propagation in arbitrary media.
- Explain the connection between plane wave theory and radio wave propagation.
- Calculate the effects of interference on reflection and transmission.
- Understand the connection between guided waves and transmission line models.
- Derive fields and modes inside waveguides, calculate wave propagation parameters, and compare them to free space propagation.
- Understands how current distributions generate radiated fields
- Calculate antenna parameters from radiated fields.
- **Have a solid ground for further studies in antennas, microwave engineering, and related topics in applied electromagnetics.**

ELEC-E4130 details

Prerequisites

- Working knowledge of engineering mathematics (vector calculus, complex numbers and integrals)
- Basic knowledge of electrical circuits and undergraduate electromagnetics (electric and magnetic fields and forces, electromagnetic induction).

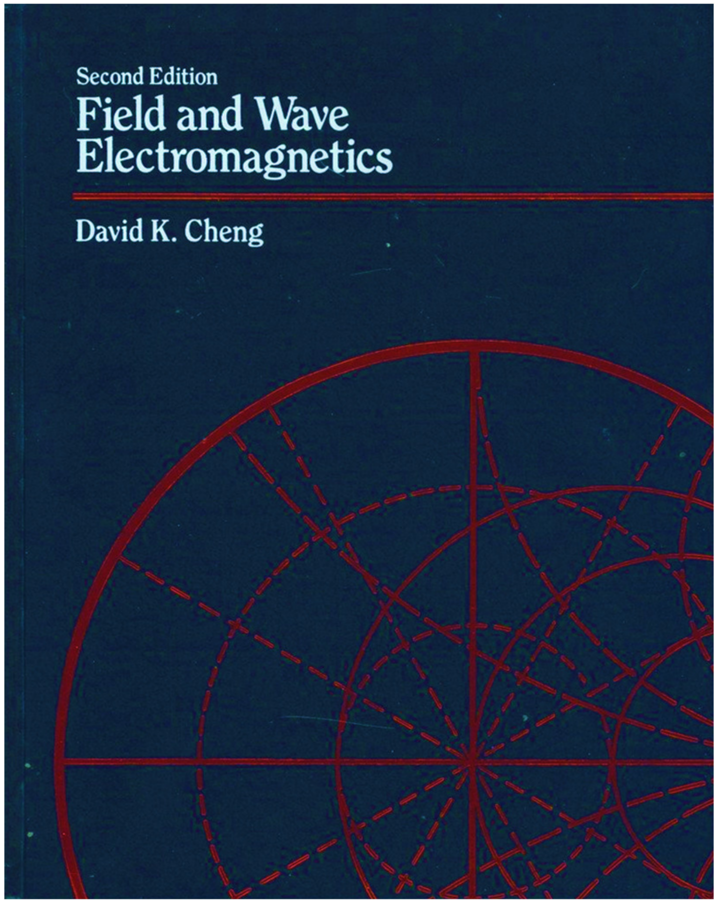
Assessment Methods and Criteria:

- Exercises, midterm exams.

Student workload:

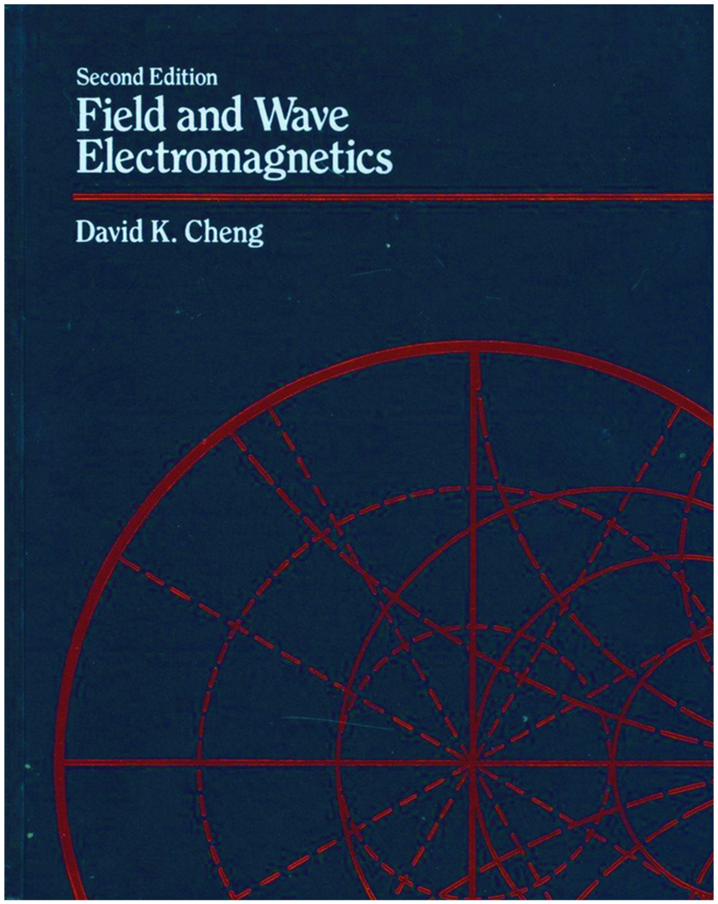
- Lectures and exercises 48 h (4 h per week)
- Midterm exams 4 h
- Independent work (exercises) 80 h

Materials



- "Field and Wave Electromagnetics" Cheng
  - ***Chapter 2**: Vector Analysis (Taylor)*
  - ***Chapters 7 & 8**: Time-Varying Fields and Maxwell's Equations & Plane Electromagnetic Waves (Wallén)*
  - ***Chapter 9**: Theory and Applications of Transmission Lines (Taylor)*
  - ***Extra Material:** Multilayer calculations (Taylor)*
  - ***Chapter 10**: Waveguides and Cavity Resonators (Taylor)*
  - ***Chapter 11**: Antennas and Radiating Systems (Wallén)*
- Any version of the 2nd edition is good
- 1st edition is reasonably similar to the 2nd
- Some additional materials will be used for the stratified media lectures

Materials cont'd

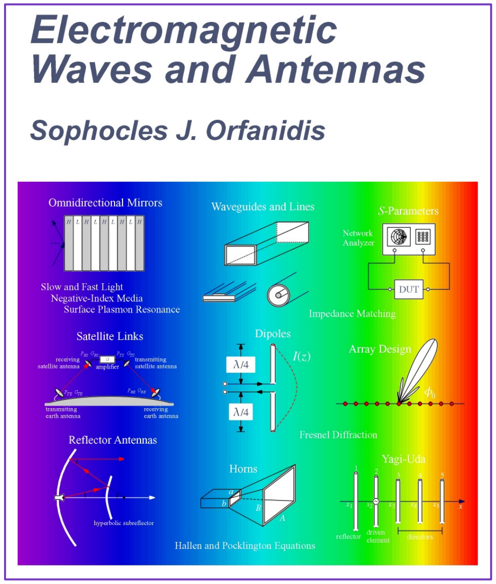


- Chapter 9: Theory and Applications of Transmission Lines (Taylor)
  - ***9-1**: Introduction*
  - ***9-2**: Transverse Electromagnetic Wave along a parallel-plate transmission line*
  - ***9-3**: General Transmission Line Equations*
  - ***9-4**: Wave Characteristics on Finite Transmission Lines*
  - ~~***9-5**: Transients on Transmission Lines*~~
  - ***9-6**: The Smith Chart*
  - ***9-7**: Transmission Line Impedance Matching*

ELEC-E4130

Book treatment is (arguably) outdated

Materials cont'd



Multilayer optical calculations

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November 6, 2018

Abstract

When light hits a multilayer planar stack, it is reflected, refracted, and absorbed in a way that can be derived from the Fresnel equations. The analysis is treated in many textbooks, and implemented in many software programs, but certain aspects of it are difficult to find explicitly and consistently worked out in the literature. Here, we derive the formulas underlying the transfer-matrix method of calculating the optical properties of these stacks, including oblique-angle incidence, absorption-vs-position profiles, and ellipsometry parameters. We discuss and explain some strange consequences of the formulas in the situation where the incident and/or final (semi-infinite) medium are absorptive, such as calculating $T > 1$ in the absence of gain. We also discuss some implementation details like complex-plane branch cuts. Finally, we derive modified formulas for including one or more "incoherent" layers, i.e. very thick layers in which interference can be neglected. This document was written in conjunction with the "tmm" Python software package, which implements these calculations.

Both sources have open source code

https://www.ece.rutgers.edu/~orfanidi/ewa/

https://arxiv.org/abs/1603.02720

- "Electromagnetic Waves and Antennas" - Orfanidis
  - ***Chapters 8**: Multilayer Film Applications (Taylor)*
- "Multilayer optical calculations" - Byrnes
  - ***Parts of PDF***

ELEC-E4130 Class structure

- Both Monday and Thursday lectures are organized as follows
  - Lecture Part 1
  - Exercise(s) 1
  - Lecture Part 2
  - Exercise(s) 2

OR

  - Lecture Part 1
  - Lecture Part 2
  - Exercise(s) 1
  - Exercise(s) 2

- Depending on the content and questions we may save the Exercises till the end of the Lecture
- Exercises consist of
  - **example** problems
  - **homework** problems
- We will introduce and give hints re: homework problems during the lectures
  - Homework discussion will take place mostly on Thursday

ELEC-E4130 Class structure cont'd

- Two midterm exams in lieu of a final exam. Midterm 2 is **NOT** cumulative
  - Midterm 1: Chapters 2, 7, and 8
  - Midterm 2: Chapters 9, 10, and ***part of*** 11 + Multilayer extra
- Homework will be posted Monday and due following Thursday at 14:00

- Course Grading
  - 50% - 59% → 1
  - 60% - 69% → 2
  - 70% - 79% → 3
  - 80% - 89% → 4
  - 90% - 100% → 5

- Exercises: 60%
- Midterm 1 Oct: 20%
- Midterm 2 Dec: 20%

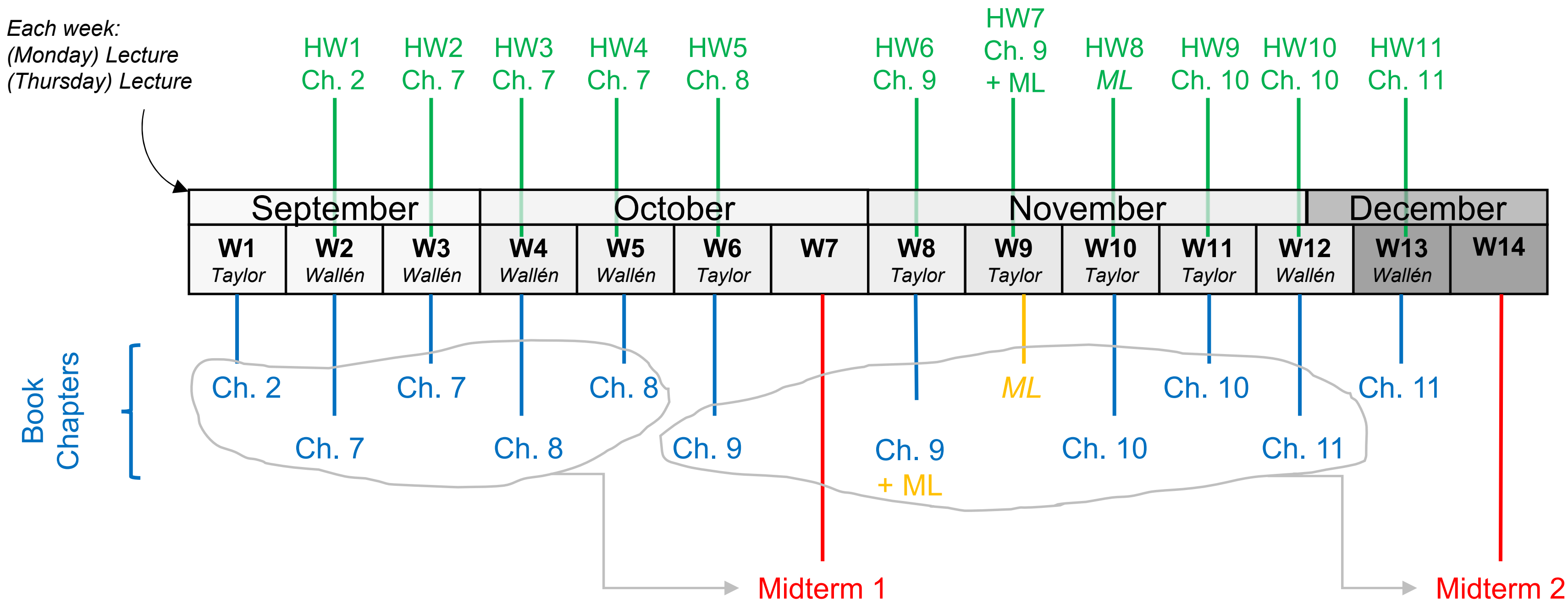
ELEC-E4130 Class structure cont'd

- Lectures will be recorded and posted to MyCourses later in the week

- Camera streams will NOT be included in the recording
  - Lecture is recorded by discussion is not

- Nice if you can have your camera on during the discussion

ELEC-E4130 at a glance

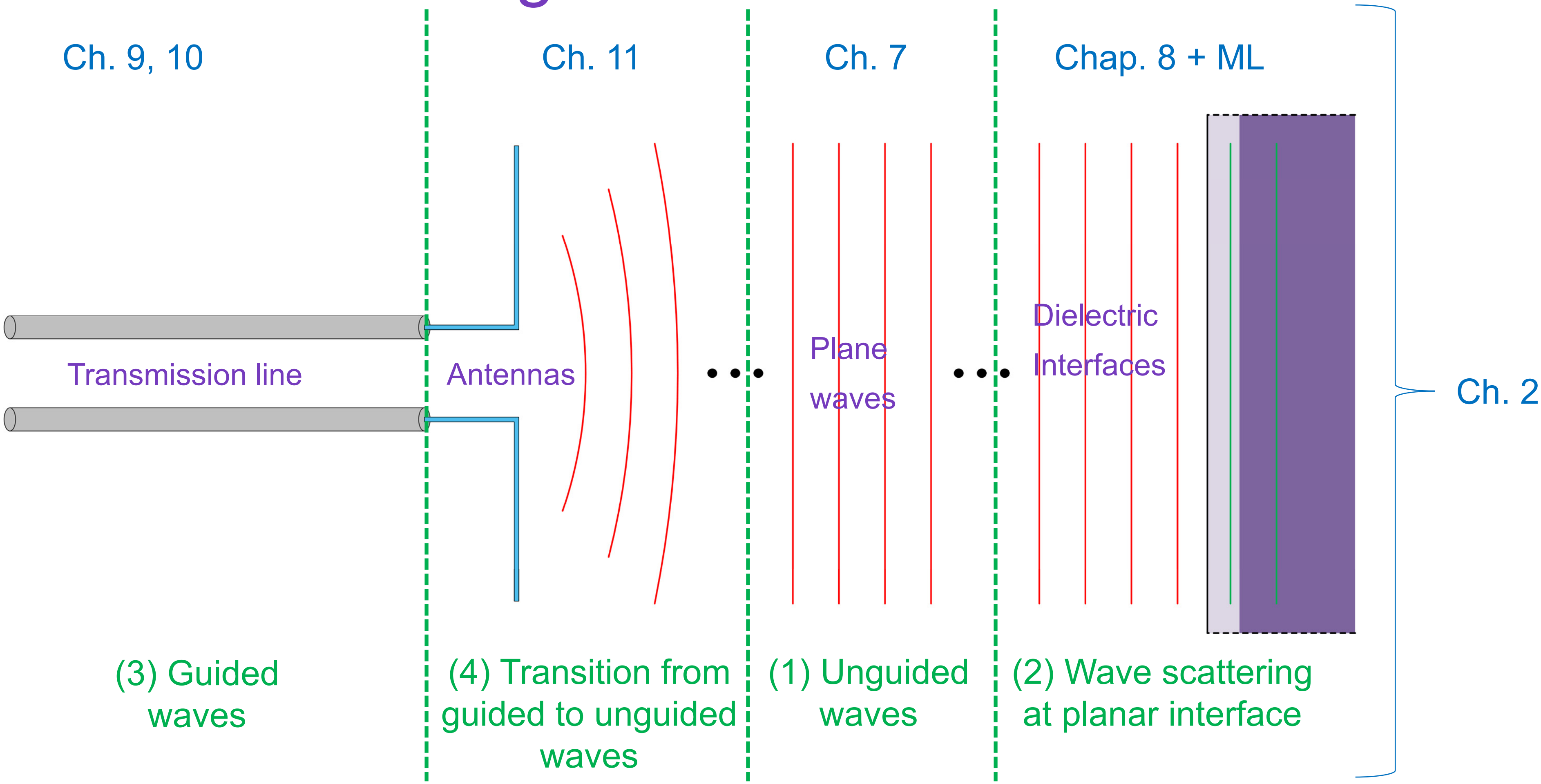


Grading
- Exercises 60%
- Midterm 1 (Oct 22): 20%
- Midterm 2 (Dec. 07): 20%

Lectures
- Monday: 12:15 – 14:00
- Thursday: 14:15 – 16:00

Exercises
- posted by the end of Mon.
- due the following Thurs. (+10 days) at **14:00**

ELEC-E4130: Big Picture



Mathematical review 1

Motivation via Maxwell's equations

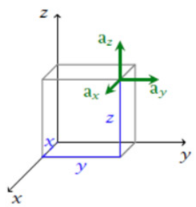
	Differential form		Integral form
Ampere's Law	$\nabla \times \mathbf{H} = \sigma\mathbf{E} + \varepsilon\frac{\partial \mathbf{E}}{\partial t}$	Stokes' Theorem	$\oint_C \mathbf{H} \cdot d\ell = \int_S \left(\sigma\mathbf{E} + \varepsilon\frac{\partial \mathbf{E}}{\partial t}\right) \cdot \boldsymbol{ds}$
Faraday's Law	$\nabla \times \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}$	Stokes' Theorem	$\oint_C \mathbf{E} \cdot d\ell = \int_S \left(-\mu\frac{\partial \mathbf{H}}{\partial t}\right) \cdot d\boldsymbol{s}$
Gauss's Law	$\nabla \cdot \varepsilon\mathbf{E} = \rho$	Divergence Theorem	$\oint_S \varepsilon\mathbf{E} \cdot d\boldsymbol{s} = \int_V \rho \, dv$
"Gauss's Law"	$\nabla \cdot \mu\mathbf{H} = 0$	Divergence Theorem	$\oint_S \mu\mathbf{H} \cdot d\boldsymbol{s} = 0$

Differential elements and other R^3 coordinates

Download from MyCourses!

Nabla operations

Cartesian coordinates (x, y, z)



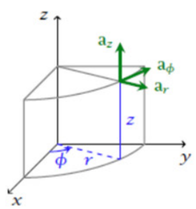
$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical coordinates (r, ϕ, z)



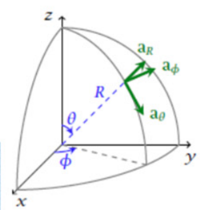
$$\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\phi \frac{1}{r}\frac{\partial V}{\partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi r & \mathbf{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical coordinates (R, θ, ϕ)



$$\nabla V = \mathbf{a}_R \frac{\partial V}{\partial R} + \mathbf{a}_\theta \frac{1}{R}\frac{\partial V}{\partial \theta} + \mathbf{a}_\phi \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 A_R\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Coordinate transformations

Cartesian ↔ Cylindrical

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z$$

$$r = \sqrt{x^2+y^2}, \quad \phi = \arctan\frac{y}{x}, \quad z = z$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Cartesian ↔ Spherical

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\theta$$

$$R = \sqrt{x^2+y^2+z^2}, \quad \theta = \arctan\frac{\sqrt{x^2+y^2}}{z}, \quad \phi = \arctan\frac{y}{x}$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Cylindrical ↔ Spherical

$$r = R\sin\theta, \quad \phi = \phi, \quad z = R\cos\theta$$

$$R = \sqrt{r^2+z^2}, \quad \theta = \arctan\frac{r}{z}, \quad \phi = \phi$$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

Other useful formulas

Cartesian coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_x\,dx + \mathbf{a}_y\,dy + \mathbf{a}_z\,dz$$
$$ds_x = dy\,dz$$
$$ds_y = dx\,dz$$
$$ds_z = dx\,dy$$
$$dv = dx\,dy\,dz$$

Cylindrical coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_r\,dr + \mathbf{a}_\phi r\,d\phi + \mathbf{a}_z\,dz$$
$$ds_r = r\,d\phi\,dz$$
$$ds_\phi = dr\,dz$$
$$ds_z = r\,dr\,d\phi$$
$$dv = r\,dr\,d\phi\,dz$$

Spherical coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_R\,dR + \mathbf{a}_\theta R\,d\theta + \mathbf{a}_\phi R\sin\theta\,d\phi$$
$$ds_R = R^2\sin\theta\,d\theta\,d\phi$$
$$ds_\theta = R\sin\theta\,dR\,d\phi$$
$$ds_\phi = R\,dR\,d\theta$$
$$dv = R^2\sin\theta\,dR\,d\theta\,d\phi$$

Divergence theorem $\int_V \nabla\cdot\mathbf{A}\,dV = \oint_S \mathbf{A}\cdot d\mathbf{s}$

Stokes' theorem $\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{s} = \oint_C \mathbf{A}\cdot d\mathbf{l}$

Constants

$$c = 299\,792\,458\,\frac{\text{m}}{\text{s}}$$
$$\mu_0 = 4\pi\times10^{-7}\,\frac{\text{Vs}}{\text{Am}} \approx 1.257\times10^{-6}\,\frac{\text{H}}{\text{m}}$$
$$\varepsilon_0 = \frac{1}{\mu_0 c^2} \approx 8.854\times10^{-12}\,\frac{\text{As}}{\text{Vm}} \quad \left(=\frac{\text{F}}{\text{m}}\right)$$
$$e \approx 1.602\times10^{-19}\,\text{C}$$

- Cartesian, cylindrical, and spherical coordinates
- Matrix vector multiplication mapping between coordinate systems
- Divergence, gradient, and curl
- Differential elements
- Don't forget the Jacobian matrix determinants to preserve measure between coordinate systems

Review of Vectors (cartesian R^3) 1/(3)

Vector Field

$\mathbf{A} \rightarrow vector\ field$

$$\mathbf{A}(x,y,z) = A_x(x,y,z)\mathbf{a_x} + A_y(x,y,z)\mathbf{a_y} + A_z(x,y,z)\mathbf{a_z}$$

$A_x(x,y,z) \rightarrow$ scalar field

$A_y(x,y,z) \rightarrow$ scalar field

$A_z(x,y,z) \rightarrow$ scalar field

$\mathbf{a_x} \rightarrow$ unit vector in the **x** direction

$\mathbf{a_y} \rightarrow$ unit vector in the **y** direction

$\mathbf{a_z} \rightarrow$ unit vector in the **z** direction

Define two vectors

$$\mathbf{A} = A_x\mathbf{a_x} + A_y\mathbf{a_y} + A_z\mathbf{a_z}$$

$$\mathbf{B} = B_x\mathbf{a_x} + B_y\mathbf{a_y} + B_z\mathbf{a_z}$$

Review of Vectors (cartesian R^3) 2/(3)

Vector Sum

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

$$\mathbf{C} = (A_x + B_x)\mathbf{a_x} + (A_x + B_x)\mathbf{a_y} + (A_x + B_x)\mathbf{a_z}$$

Vector Dot Product

$$C = \mathbf{A} \cdot \mathbf{B}$$

$$C = A_x B_x + A_y B_y + A_z B_z$$

Vector Dot Product alternative

$$C = |\mathbf{A}||\mathbf{B}| \cos(\theta_{AB})$$

$$\theta_{AB} = \cos^{-1}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right)$$

$$|A| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$$

Review of Vectors (cartesian R³) 3/(3)

Vector Cross Product

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a_x} & \mathbf{a_y} & \mathbf{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Determinate of this 3x3 matrix can be solved

- recursively as the sum of the determinates of sub matricies: Laplace's formula for minors
- "Repeating the first two columns and multiplying"

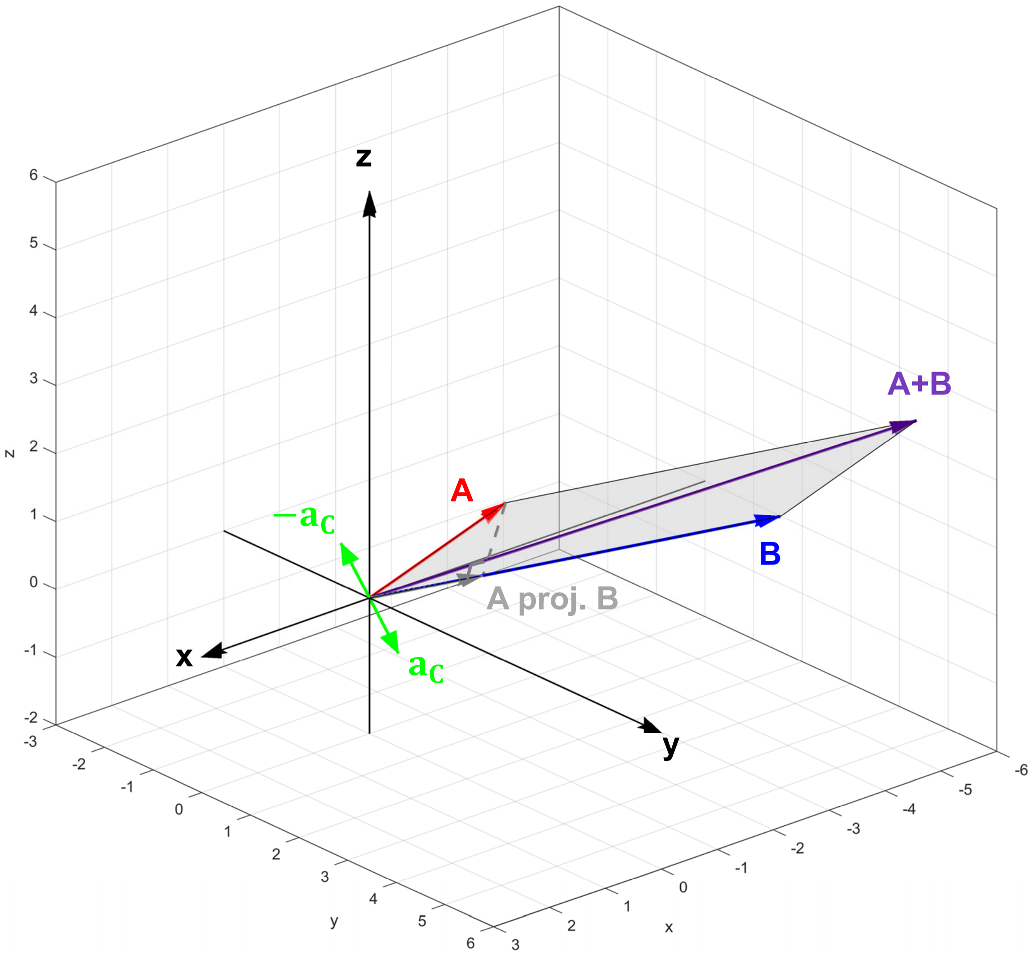
$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = + \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \mathbf{a_x} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \mathbf{a_y} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \mathbf{a_z}$$

Don't forget the negative

$$\mathbf{C} = +(A_y B_z - A_z B_y)\mathbf{a_x} - (A_x B_z - A_z B_x)\mathbf{a_y} + (A_x B_y - A_y B_x)\mathbf{a_z}$$

$\mathbf{C}$ is orthogonal to the plane defined by $\mathbf{A}, \mathbf{B}$

Graphical Review



$$\mathbf{A} = -2\mathbf{a_x} + 0.5\mathbf{a_y} + 1\mathbf{a_z}$$

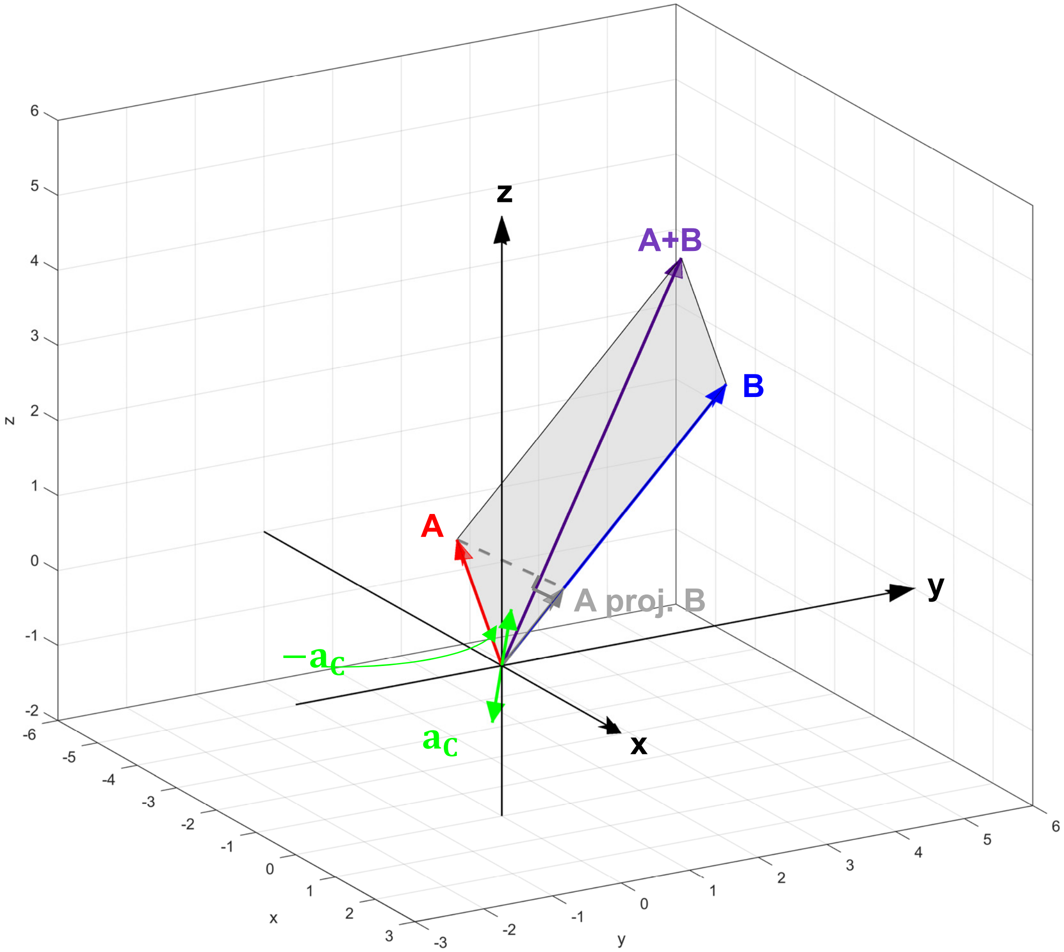
$$\mathbf{B} = -3\mathbf{a_x} + 5\mathbf{a_y} + 2\mathbf{a_z}$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{a_C} = \frac{\mathbf{C}}{|\mathbf{C}|} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$$

$$\mathbf{A\ proj.\ B} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B} \cdot \mathbf{B}|}\mathbf{B}$$

Graphical Review



$$\mathbf{A} = -2\mathbf{a_x} + 0.5\mathbf{a_y} + 1\mathbf{a_z}$$

$$\mathbf{B} = -3\mathbf{a_x} + 5\mathbf{a_y} + 2\mathbf{a_z}$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{a_C} = \frac{\mathbf{C}}{|\mathbf{C}|} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$$

$$\mathbf{A}\ \text{proj.}\ \mathbf{B} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B} \cdot \mathbf{B}|}\mathbf{B}$$

Example Problem 1

Find a unit vector $\mathbf{a_D}$ that lies in the plane defined by A and B and bisects the angle made by A and B. Define a unit vector $\mathbf{a_C}$ that is perpendicular to both A and B. Prove that is $\mathbf{a_C}$ normal to $\mathbf{a_D}$

$$\mathbf{A} = -2\mathbf{a_x} + 0.5\mathbf{a_y} + 1\mathbf{a_z}$$

$$\mathbf{B} = -3\mathbf{a_x} + 5\mathbf{a_y} + 2\mathbf{a_z}$$

Example Problem 1: Solution (1/2)

Find a unit vector $\mathbf{a_D}$ that lies in the plane defined by A and B and bisects the angle made by A and B. Define a unit vector $\mathbf{a_C}$ that is perpendicular to both A and B. Prove that is $\mathbf{a_C}$ normal to $\mathbf{a_D}$

$$\mathbf{A} = -2\mathbf{a_x} + 0.5\mathbf{a_y} + 1\mathbf{a_z}$$

$$\mathbf{B} = -3\mathbf{a_x} + 5\mathbf{a_y} + 2\mathbf{a_z}$$

$$\mathbf{a_A} = \frac{\mathbf{A}}{|\mathbf{A}|} = -0.8729\mathbf{a_x} + 0.2182\mathbf{a_x} + 0.4364\mathbf{a_x}$$

$$\mathbf{a_B} = \frac{\mathbf{B}}{|\mathbf{B}|} = -0.4867\mathbf{a_x} + 0.8111\mathbf{a_y} + 0.3244\mathbf{a_z}$$

$$\mathbf{D} = \mathbf{a_A} + \mathbf{a_B} = -1.3595\mathbf{a_x} + 1.0293\mathbf{a_y} + 0.7609\mathbf{a_z}$$

$$\boxed{\mathbf{a_D} = \frac{\mathbf{D}}{|\mathbf{D}|} = -0.7281\mathbf{a_x} + 0.5512\mathbf{a_y} + 0.4075\mathbf{a_z}}$$

Example Problem 1: Solution (2/2)

Find a unit vector $\mathbf{a_D}$ that lies in the plane defined by A and B and bisects the angle made by A and B. Define a unit vector $\mathbf{a_C}$ that is perpendicular to both A and B. Prove that is $\mathbf{a_C}$ normal to $\mathbf{a_D}$

$$\mathbf{A} = -2\mathbf{a_x} + 0.5\mathbf{a_y} + 1\mathbf{a_z}$$

$$\mathbf{B} = -3\mathbf{a_x} + 5\mathbf{a_y} + 2\mathbf{a_z}$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{D} = \frac{\mathbf{A}}{|\mathbf{A}|} + \frac{\mathbf{B}}{|\mathbf{B}|}$$

For normality test, sufficient to work with **C** and **D** instead of their normal vectors $\mathbf{a_C}$ and $\mathbf{a_D}$

$$\mathbf{D} \cdot \mathbf{C} = \left(\frac{\mathbf{A}}{|\mathbf{A}|} + \frac{\mathbf{B}}{|\mathbf{B}|}\right) \cdot (\mathbf{A} \times \mathbf{B}) = \frac{\mathbf{A}}{|\mathbf{A}|} \cdot (\mathbf{A} \times \mathbf{B}) + \frac{\mathbf{B}}{|\mathbf{B}|} \cdot (\mathbf{A} \times \mathbf{B})$$

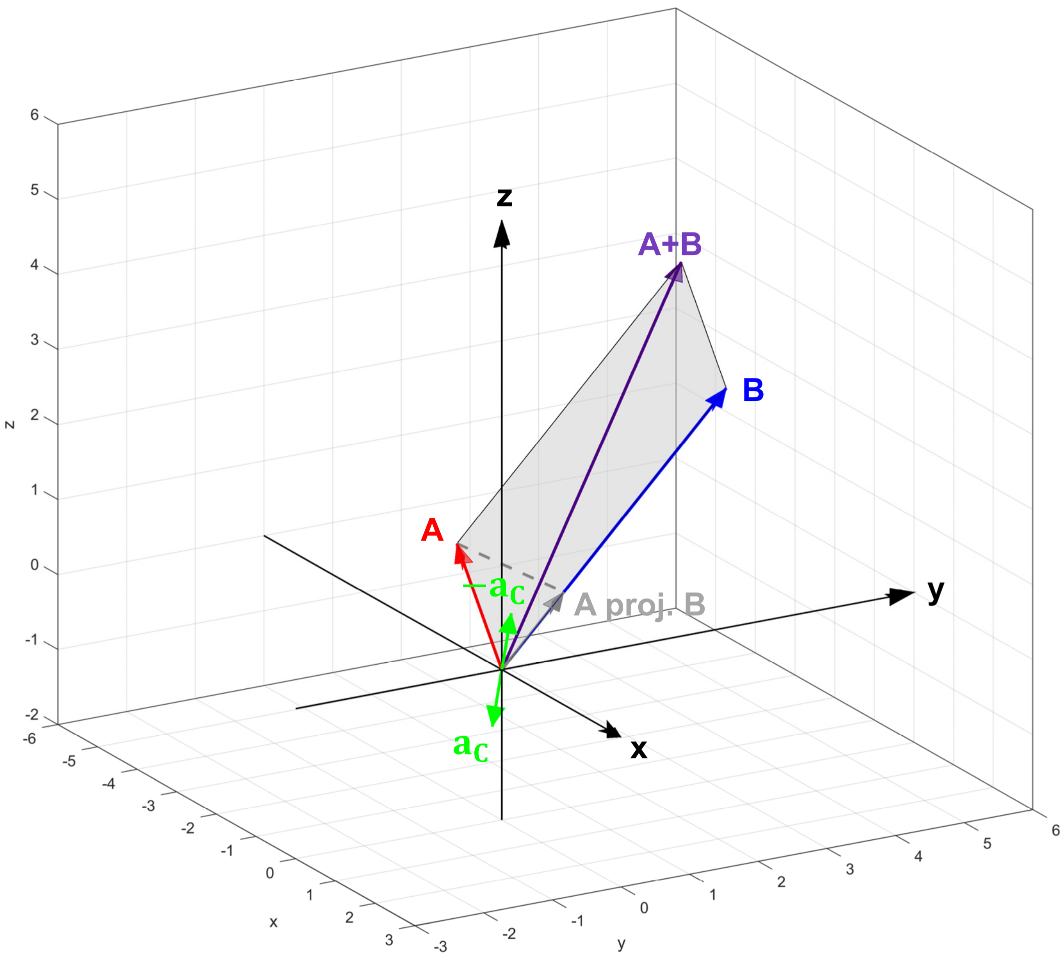
$$= \frac{\mathbf{1}}{|\mathbf{A}|} \mathbf{B} \cdot \underbrace{\mathbf{A} \times \mathbf{A}}_{0} + \frac{\mathbf{1}}{|\mathbf{B}|} \mathbf{A} \cdot \underbrace{\mathbf{B} \times \mathbf{B}}_{0}$$

$$= \mathbf{0}$$

Rewritten identity from inside cover of your book.

$$\mathbf{F} \cdot \mathbf{G} \times \mathbf{H} = \mathbf{G} \cdot \mathbf{H} \times \mathbf{F} = \mathbf{H} \cdot \mathbf{F} \times \mathbf{G}$$

Example Problem 2

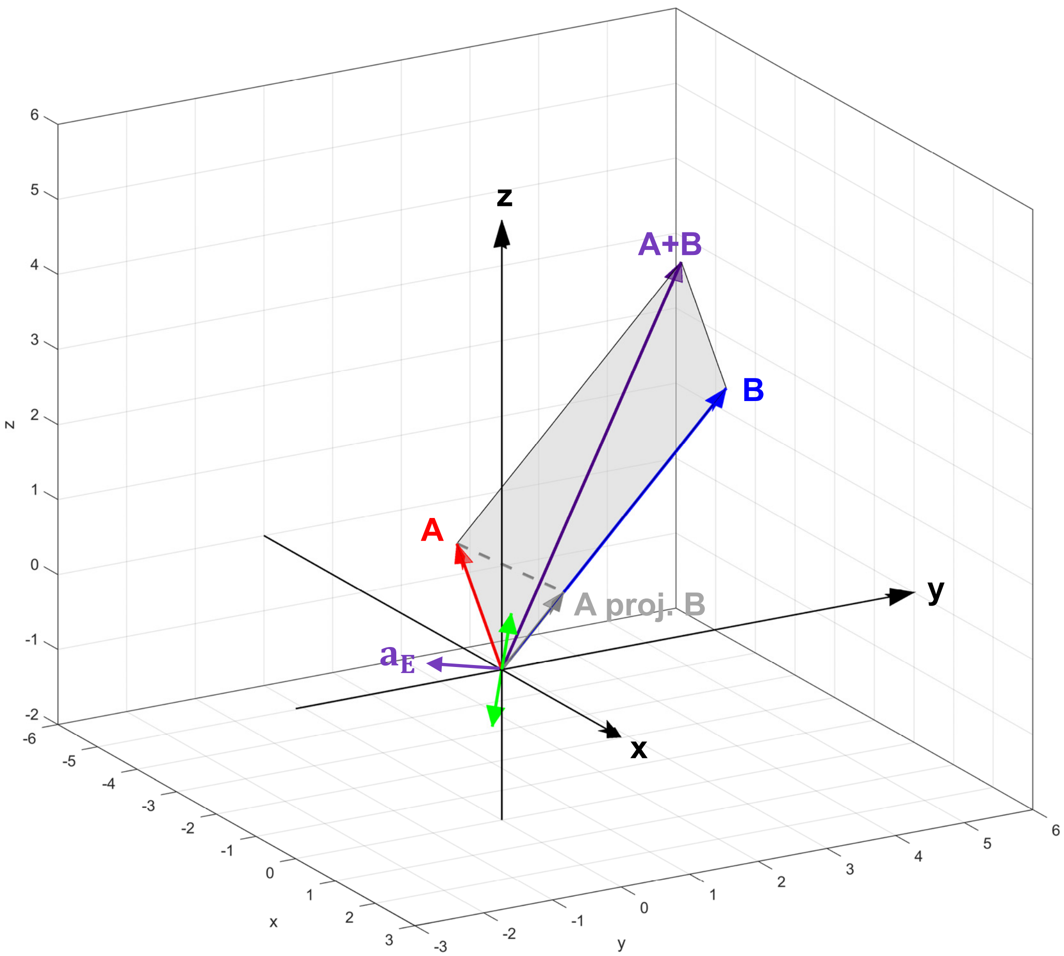


Find a unit vector $\mathbf{a}_E$ that lies in the plane defined by **A** and **B** and is normal to **A**

$$\mathbf{A} = -2\mathbf{a}_x + 0.5\mathbf{a}_y + 1\mathbf{a}_z$$

$$\mathbf{B} = -3\mathbf{a}_x + 5\mathbf{a}_y + 2\mathbf{a}_z$$

Example Problem 2: Solution



Find a unit vector $\mathbf{a_E}$ that lies in the plane defined by **A** and **B** and is normal to **A**

$$\mathbf{A} = -2\mathbf{a_x} + 0.5\mathbf{a_y} + 1\mathbf{a_z}$$

$$\mathbf{B} = -3\mathbf{a_x} + 5\mathbf{a_y} + 2\mathbf{a_z}$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{E} = \mathbf{A} \times \mathbf{C} = \mathbf{A} \times \mathbf{A} \times \mathbf{B}$$

$$\mathbf{E} = \mathbf{A}(\mathbf{A} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{A} \cdot \mathbf{A})$$

$$\mathbf{a_E} = \frac{\mathbf{E}}{|\mathbf{E}|} = -0.2425\mathbf{a_x} - 0.9701\mathbf{a_y} + 0\mathbf{a_z}$$

perpendicular ~ orthogonal ~ normal

Review of Vectors (cartesian R^3) (1/4)

Gradient

$$\nabla = \frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y} + \frac{\partial}{\partial z}\mathbf{a_z}$$

Gradient is an operator and also a vector

$$A(x, y, z) \rightarrow \text{scalar field}$$

$$\nabla A = \frac{\partial}{\partial x}A(x, y, z)\mathbf{a_x} + \frac{\partial}{\partial y}A(x, y, z)\mathbf{a_y} + \frac{\partial}{\partial z}A(x, y, z)\mathbf{a_z}$$

- Vector quantity
- Points in the direction of greatest **ascent**
- Magnitude is the rate of change

- Use the formula sheet for cylindrical and spherical coordinates!

Review of Vectors (cartesian R^3) (2/4)

Divergence

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y} + \frac{\partial}{\partial z}\mathbf{a_z}$$

← Gradient is an operator and also a vector

$$\mathbf{A}(x, y, z) \rightarrow \text{vector field}$$

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

- Scalar quantity
- Measure of the **outward** flow of an infinitesimally small volume
- Differential flux

- Use the formula sheet for cylindrical and spherical coordinates!

Review of Vectors (cartesian R^3) (3/4)

Curl

$$\nabla = \frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y} + \frac{\partial}{\partial z}\mathbf{a_z}$$ ← Gradient is an operator and also a vector

$\mathbf{A}(x, y, z) \rightarrow$ vector field

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a_x} & \mathbf{a_x} & \mathbf{a_x} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = +\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right)\mathbf{a_x} - \left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right)\mathbf{a_y} + \left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)\mathbf{a_z}$$

- Infinitesimal rotation
- Direction of the resulting vector is the axis of rotation w.r.t. counter clockwise rotation
- Magnitude of the curl is the magnitude of rotation

- Use the formula sheet for cylindrical and spherical coordinates!

Review of Vectors (cartesian R^3) (4/4)

"BAC-CAB" rule

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\mathbf{A} = \boldsymbol{\nabla}$$
$$\mathbf{B} = \boldsymbol{\nabla}$$
$$\mathbf{C} = \mathbf{E}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{E}$$

$$= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \boldsymbol{\nabla}^2\mathbf{E}$$

Div/grad/crul

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = \mathbf{0}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\mathrm{A}) = \mathbf{0}$$

Inside cover of Cheng Book

- Keep track of vector vs scalar quantities

Example Problem 3

Find the tangent plane and normal line to $x^2 + y^2 + z^2 = 25$ at the point $(x_0, y_0, z_0) = (3, 3, \sqrt{7})$

Example Problem 3: solution (1/2)

Find the tangent plane and normal line to $x^2+y^2+z^2=25$ at the point $(x_0,y_0,z_0)=(3,3,\sqrt{7})$

$$\boldsymbol{\nabla}=\frac{\partial}{\partial x}\mathbf{a_x}+\frac{\partial}{\partial y}\mathbf{a_y}+\frac{\partial}{\partial z}\mathbf{a_z}$$

Gradient points orthogonal to the level set/contour

$$A(x,y,z)=x^2+y^2+z^2-25$$

$$\boldsymbol{\nabla}A=\frac{\partial}{\partial x}A(x,y,z)\mathbf{a_x}+\frac{\partial}{\partial y}A(x,y,z)\mathbf{a_y}+\frac{\partial}{\partial z}A(x,y,z)\mathbf{a_z}=2x\mathbf{a_x}+2y\mathbf{a_y}+2z\mathbf{a_z}$$

$$\boldsymbol{\nabla}A\left(3,3,\sqrt{7}\right)=6\mathbf{a_x}+6\mathbf{a_y}+2\sqrt{7}\mathbf{a_z}=\mathbf{F}=F_x\mathbf{a_x}+F_y\mathbf{a_y}+F_z\mathbf{a_z}$$

$$F_x(x-x_0)+F_y(y-y_0)+F_z(z-z_0)=0$$

Point Normal Form

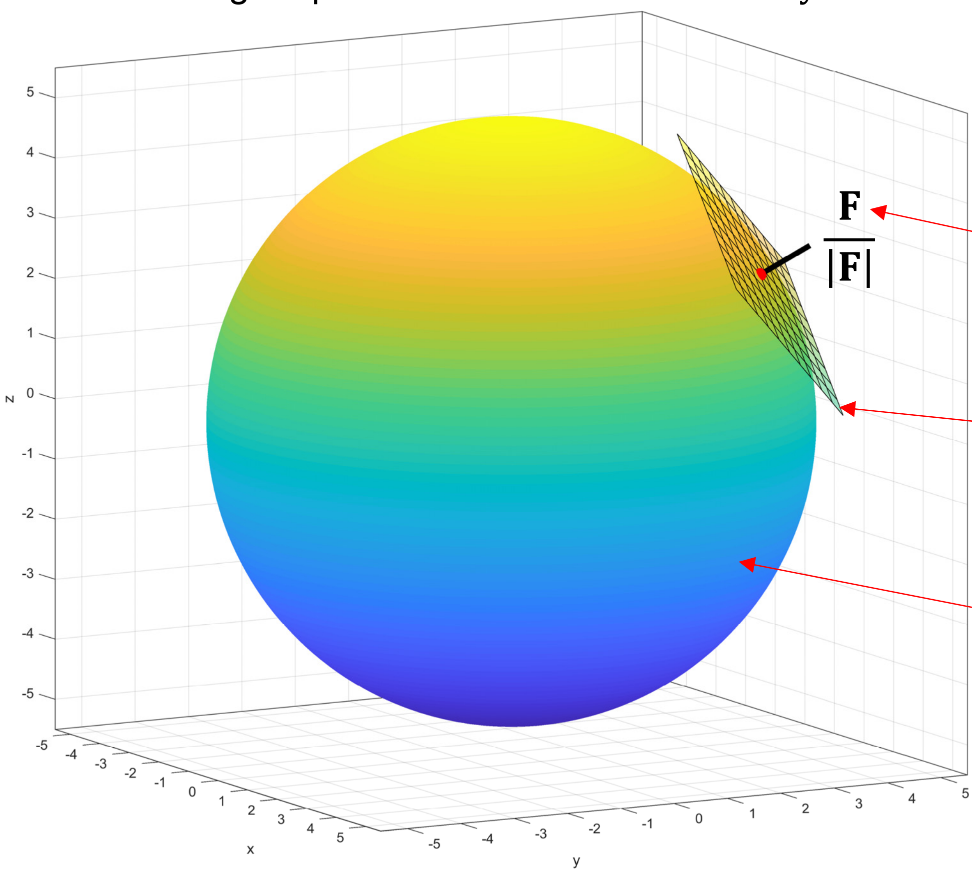
Equation of a plane normal to **F** and containing (x_0,y_0,z_0)

$$6(x-3)+6(y-3)+2\sqrt{7}\left(z-2\sqrt{7}\right)=0$$

$$\boxed{6x+6y+2\sqrt{7}z=64}$$

Example Problem 3: solution (2/2)

Find the tangent plane and normal line to $x^2 + y^2 + z^2 = 25$ at the point $(x_0, y_0, z_0) = (3, 3, \sqrt{7})$



$$\frac{\mathbf{F}}{|\mathbf{F}|}$$

$$\nabla A = \mathbf{F} = 6\mathbf{a_x} + 6\mathbf{a_y} + 2\sqrt{7}\mathbf{a_z}$$

$$6x + 6y + 2\sqrt{7}z = 64$$

$$x^2 + y^2 + z^2 = 25$$

Example Problem 4

Compute the curl and divergence of the following vector field F

$$\mathbf{F} = (x^2 y)\mathbf{a_x} + (z^3 - 3x)\mathbf{a_y} + (4y^2)\mathbf{a_z}$$

Example Problem 4: Solution (1/1)

Compute the curl and divergence of the following vector field F

$$\mathbf{F} = (x^2 y)\mathbf{a_x} + (z^3 - 3x)\mathbf{a_y} + (4y^2)\mathbf{a_z}$$

Divergence

$$\nabla \cdot \mathbf{F} = 2\text{xy}$$ ← Scalar field

Curl

$$\nabla \times \mathbf{F} = (8y - 3z^2)\mathbf{a_x} - (0)\mathbf{a_y} - (3 + x^2)\mathbf{a_z}$$ ← Vector field

Next time

- We'll review stokes theorem and divergence theorem on Thursday
- Homework up later today on MyCourses
- Submission links on MyCourses for HomeWorks are up

- See you Thursday at 14:15.