ELEC-E4130

Lecture 2: mathematical review 2: Stokes and Divergence theorem

Aalto University
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ELEC-E4130 / Taylor

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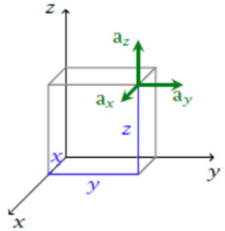
Nabla operations

Cartesian coordinates (x, y, z)



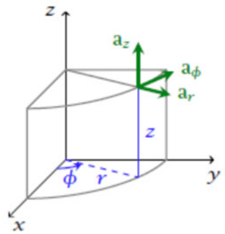
$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical coordinates (r, ϕ, z)



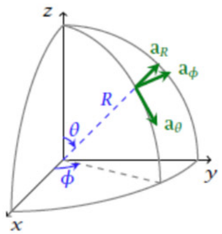
$$\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\phi \frac{1}{r} \frac{\partial V}{\partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi r & \mathbf{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & r A_\phi & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical coordinates (R, θ, ϕ)



$$\nabla V = \mathbf{a}_R \frac{\partial V}{\partial R} + \mathbf{a}_\theta \frac{1}{R} \frac{\partial V}{\partial \theta} + \mathbf{a}_\phi \frac{1}{R \sin\theta} \frac{\partial V}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R \sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & R A_\theta & (R \sin\theta) A_\phi \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 A_R) + \frac{1}{R \sin\theta} \frac{\partial}{\partial \theta} (A_\theta \sin\theta) + \frac{1}{R \sin\theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2}$$

Coordinate transformations

Cartesian ↔ Cylindrical

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z$$

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}, \qquad z = z$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Cartesian ↔ Spherical

$$x = R\sin\theta\cos\phi, \qquad y = R\sin\theta\sin\phi, \qquad z = R\cos\theta$$

$$R = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Cylindrical ↔ Spherical

$$r = R\sin\theta, \qquad \phi = \phi, \qquad z = R\cos\theta$$

$$R = \sqrt{r^2 + z^2}, \qquad \theta = \tan^{-1}\frac{r}{z}, \qquad \phi = \phi$$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

Other useful formulas

Cartesian coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_x\,dx + \mathbf{a}_y\,dy + \mathbf{a}_z\,dz$$
$$ds_x = dy\,dz$$
$$ds_y = dx\,dz$$
$$ds_z = dx\,dy$$
$$dv = dx\,dy\,dz$$

Cylindrical coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_r\,dr + \mathbf{a}_\phi r\,d\phi + \mathbf{a}_z\,dz$$
$$ds_r = r\,d\phi\,dz$$
$$ds_\phi = dr\,dz$$
$$ds_z = r\,dr\,d\phi$$
$$dv = r\,dr\,d\phi\,dz$$

Spherical coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_R\,dR + \mathbf{a}_\theta R\,d\theta + \mathbf{a}_\phi R\sin\theta\,d\phi$$
$$ds_R = R^2\sin\theta\,d\theta\,d\phi$$
$$ds_\theta = R\sin\theta\,dR\,d\phi$$
$$ds_\phi = R\,dR\,d\theta$$
$$dv = R^2\sin\theta\,dR\,d\theta\,d\phi$$

Divergence theorem $\int_V \nabla \cdot \mathbf{A}\,dv = \oint_S \mathbf{A} \cdot d\mathbf{s}$

Stokes' theorem $\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell}$

Constants

$$c = 299\,792\,458\ \frac{\text{m}}{\text{s}}$$
$$\mu_0 = 4\pi \times 10^{-7}\ \frac{\text{Vs}}{\text{Am}} \approx 1.257 \times 10^{-6}\ \frac{\text{H}}{\text{m}}$$
$$\varepsilon_0 = \frac{1}{\mu_0 c^2} \approx 8.854 \times 10^{-12}\ \frac{\text{As}}{\text{Vm}} \quad \left(= \frac{\text{F}}{\text{m}}\right)$$
$$e \approx 1.602 \times 10^{-19}\ \text{C}$$

Stokes Theorem

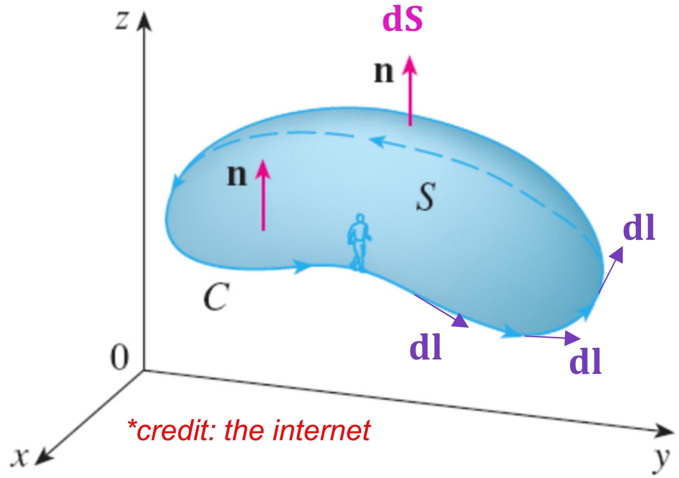
Stokes theorem

$\mathbf{A}(x, y, z) \rightarrow$ vector field

$$\mathbf{\nabla} = \frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y} + \frac{\partial}{\partial z}\mathbf{a_z}$$

$$\mathbf{A} = A_x\mathbf{a_x} + A_y\mathbf{a_y} + A_z\mathbf{a_z}$$

$$\mathbf{\nabla} \times \mathbf{A} = \begin{vmatrix} \mathbf{a_x} & \mathbf{a_x} & \mathbf{a_x} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$



$$\int_S \mathbf{\nabla} \times \mathbf{A} \cdot \mathbf{ds} = \oint_C \mathbf{A} \cdot \mathbf{dl}$$

C is a closed contour that bounds S

- The total flux of the curl vector field through surface S is equal to the sum of dot products between the tangent vector field of contour C and a vector field A at the contour

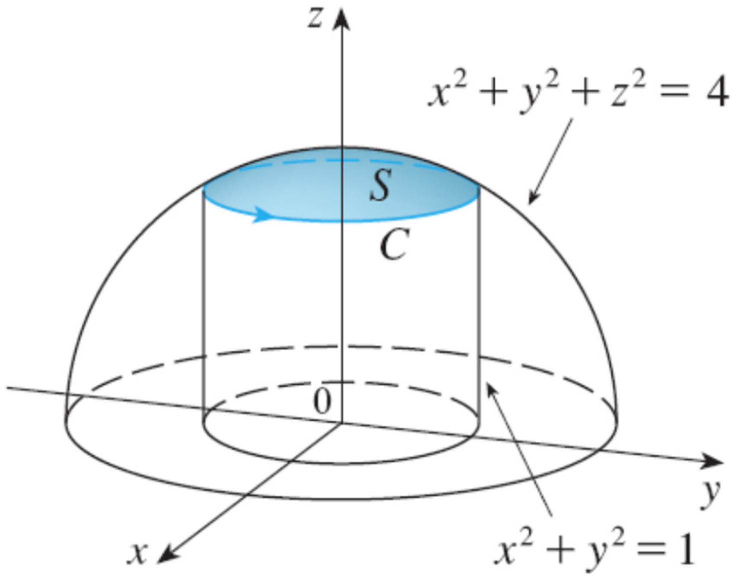
Example problem

$$\mathbf{F} = xz\mathbf{a_x} + yz\mathbf{a_y} + xy\mathbf{a_z}$$

$$\int_S \nabla \times \mathbf{F} \cdot \mathbf{ds} = \oint_C \mathbf{F} \cdot \mathbf{dl}$$

sphere $\rightarrow x^2 + y^2 + z^2 = 4$

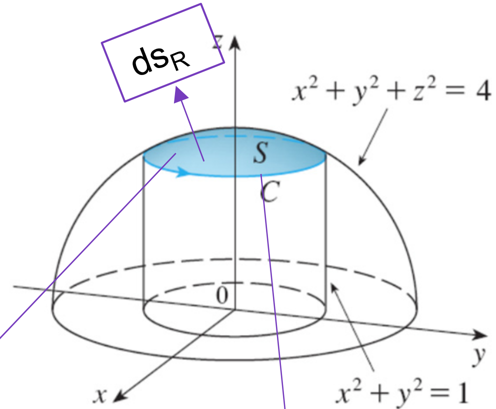
cylinder $\rightarrow x^2 + y^2 = 1$



- Show that stokes theorem is valid for the contour and surface shown in the figure.
- The closed contour C is defined by the intersection of the sphere and cylinder
- The surface S is the part of the sphere that lies inside of the cylinder

Example problem: Solution 1/7

$$\mathbf{F} = xz\mathbf{a_x} + yz\mathbf{a_y} + xy\mathbf{a_z}$$



Vector **F** field is in cartesian coordinates

Surface S is easily described in spherical coordinates

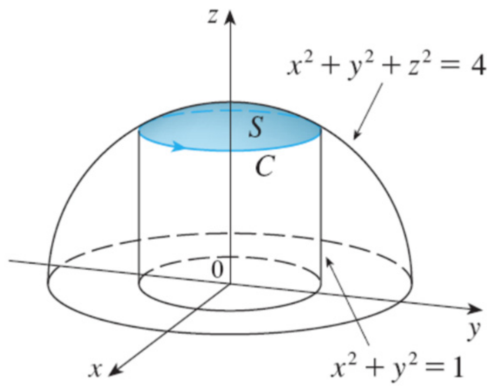
Contour C is easily described in cylindrical coordinates

Transform curl of F to **spherical** coordinates

Transform F to **cylindrical** coordinates

Parameterize S,C to **cartesian** coordinates

Example problem: Solution 2/7



$$\mathbf{F} = xz\mathbf{a_x} + yz\mathbf{a_y} + xy\mathbf{a_z}$$

$$\int_S \nabla \times \mathbf{F} \cdot \mathbf{ds}$$

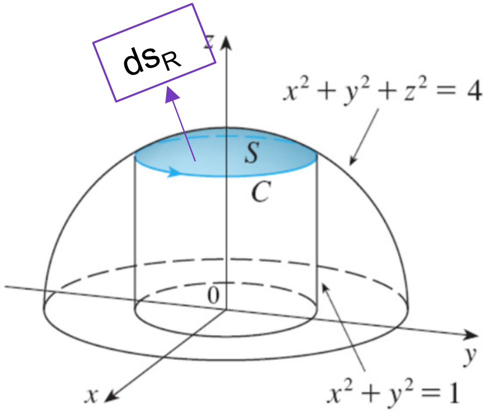
$$\mathbf{A} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{a_x} & \mathbf{a_x} & \mathbf{a_x} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = (x-y)\mathbf{a_x} + (x-y)\mathbf{a_y} + (0)\mathbf{a_z} = A_x\mathbf{a_x} + A_y\mathbf{a_y} + A_z\mathbf{a_z}$$

$$\mathbf{A} = A_x\mathbf{a_x} + A_y\mathbf{a_y} + A_z\mathbf{a_z}$$ } Convert to spherical coordinates and spherical base vectors

Example problem: Solution 3/7

$$\mathbf{F} = xz\mathbf{a_x} + yz\mathbf{a_y} + xy\mathbf{a_z}$$

$$\int_S \nabla \times \mathbf{F} \cdot \mathbf{ds}$$



$$\mathbf{A} = (x - y)\mathbf{a_x} + (x - y)\mathbf{a_y} + (0)\mathbf{a_z} = A_x\mathbf{a_x} + A_y\mathbf{a_y} + A_z\mathbf{a_z}$$

Formula Sheet

Cartesian ↔ Spherical

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\theta$$

$$R = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \quad \phi = \tan^{-1}\frac{y}{x}$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

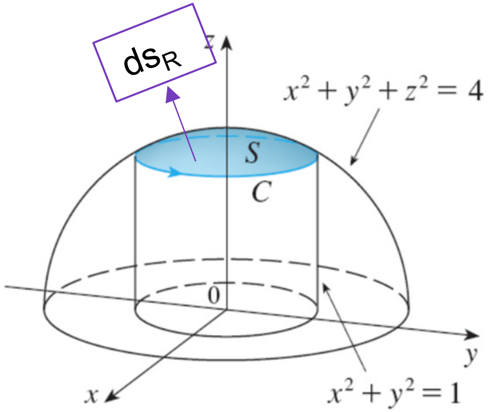
$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Note that its only necessary to compute A_R since ds always points in the $\mathbf{a_R}$ direction
ds → $ds_R\mathbf{a_R}$

Example problem: Solution 4/7

$\mathbf{F} = xz\mathbf{a_x} + yz\mathbf{a_y} + xy\mathbf{a_z}$

$\int_S \nabla \times \mathbf{F} \cdot \mathbf{ds}$



$$\mathbf{A} = (x-y)\mathbf{a_x} + (x-y)\mathbf{a_y} + (0)\mathbf{a_z} = A_x\mathbf{a_x} + A_y\mathbf{a_y} + A_z\mathbf{a_z}$$

$$A_R = \sin(\theta)\cos(\phi)A_x + \sin(\theta)\sin(\phi)A_y$$

$$x = R\sin(\theta)\cos(\phi)$$

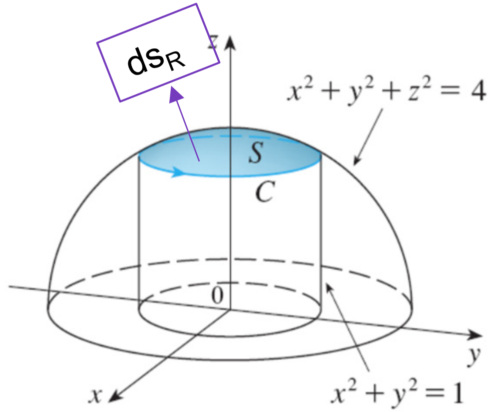
$$y = R\sin(\theta)\sin(\phi)$$

$$A_R = R[\sin(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)] \cdot [\sin(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)]$$

$$A_R = R[\sin^2(\theta)\cos^2(\phi) - \sin^2(\theta)\sin^2(\phi)]$$

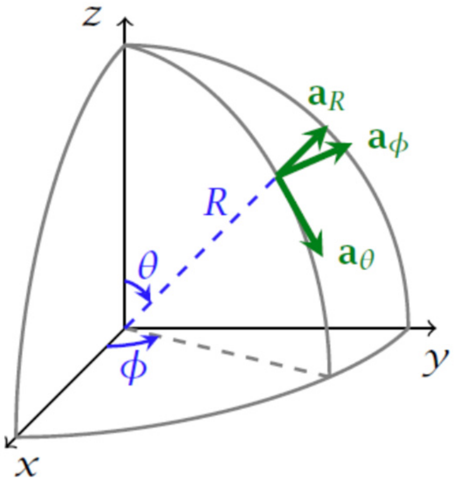
$$\mathbf{A} = A_r\mathbf{a_r} + \cdots$$

Example problem: Solution 5/7



$$\int_S \nabla \times \mathbf{F} \cdot \mathbf{ds} = \int_S A_R ds_R$$

$$= \int_S R[\sin^2(\theta)\cos^2(\phi) - \sin^2(\theta)\sin^2(\phi)]R^2\sin(\theta)d\theta d\phi$$



sphere → $x^2 + y^2 + z^2 = 4$

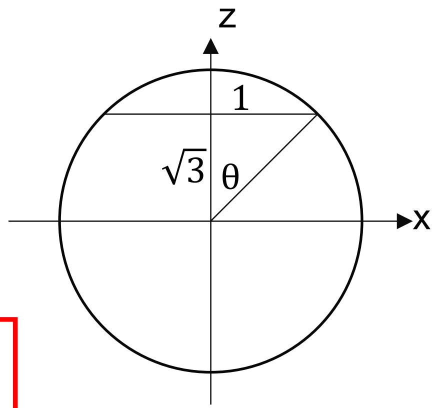
cylinder → $x^2 + y^2 = 1$

$$x^2 + y^2 + z^2 - 4 = x^2 + y^2 - 1$$

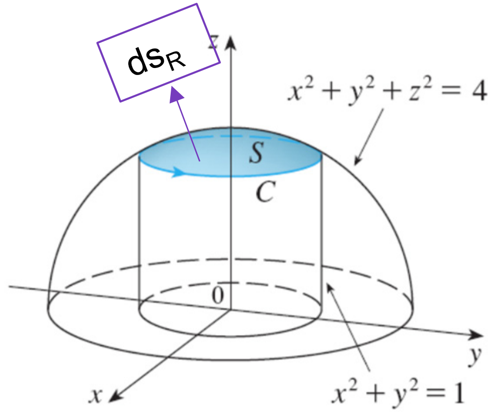
$$z^2 = 3 \rightarrow z = \sqrt{3}$$

$$\theta \in \left[0, \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$\phi \in [0, 2\pi]$$



Example problem: Solution 6/7



$$\int_S \nabla \times \mathbf{F} \cdot \mathbf{ds} = \int_S \mathrm{A_R ds_R}$$

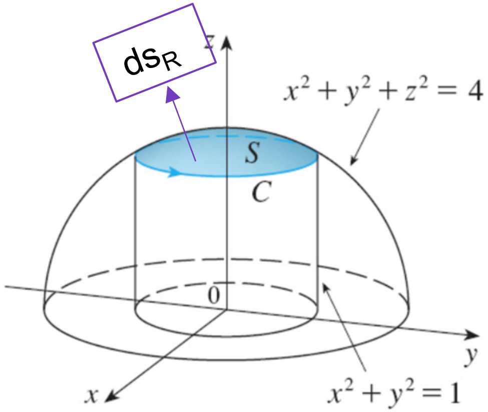
$$= \int_S R[\sin^2(\theta)\cos^2(\phi) - \sin^2(\theta)\sin^2(\phi)]R^2\sin(\theta)d\theta d\phi$$

$$= \int_S R^3\sin^3(\theta)[\cos^2(\phi) - \sin^2(\phi)]d\theta d\phi$$

$$= R^3 \int_0^{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)} \sin^3(\theta)d\theta \int_0^{2\pi} \cos^2(\phi) - \sin^2(\phi)d\phi$$

$$= R^3 \int_0^{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)} \sin^3(\theta)d\theta \left[\int_0^{2\pi} \cos^2(\phi)d\phi - \int_0^{2\pi} \sin^2(\phi)d\phi\right]$$

Example problem: Solution 7/7



$$\int_S \nabla \times \mathbf{F} \cdot \mathbf{ds} = \int_S A_R ds_R$$

$$= R^3 \int_{\mathbf{0}}^{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)} \sin^3(\theta)d\theta \left[\int_{\mathbf{0}}^{\mathbf{2\pi}} \cos^2(\phi)d\phi - \int_{\mathbf{0}}^{\mathbf{2\pi}} \sin^2(\phi)d\phi\right]$$

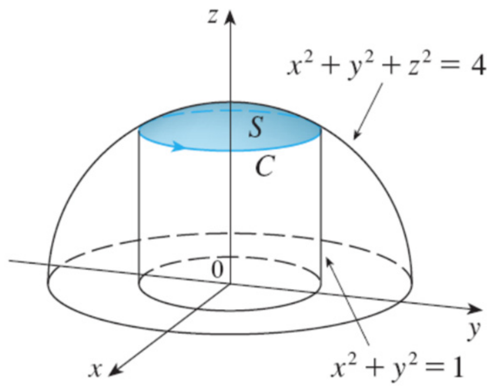
$$= R^3 \int_0^{\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)} \sin^3(\theta)d\theta \underbrace{\left[\left(\frac{1}{2}\phi + \frac{1}{2}\sin(2\phi)\right)\Big|_0^{2\pi} - \left(\frac{1}{2}\phi - \frac{1}{2}\sin(2\phi)\right)\Big|_0^{2\pi}\right]}_{0}$$

$$\int_S \nabla \times \mathbf{F} \cdot \mathbf{ds} = 0$$

In class exercise

$$\mathbf{F} = xz\mathbf{a_x} + yz\mathbf{a_y} + xy\mathbf{a_z}$$

$$\oint_C \mathbf{F} \cdot \mathbf{dl}$$

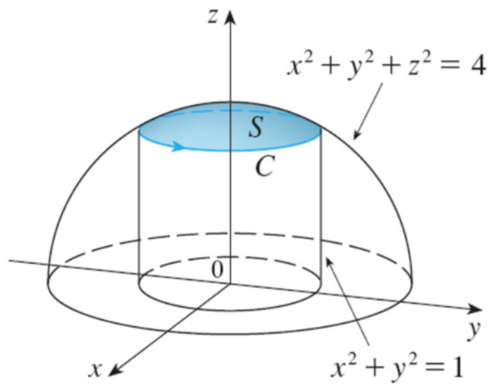


Let's solve the other side

In class exercise: Solution 1/2

$$\mathbf{F} = xz\mathbf{a_x} + yz\mathbf{a_y} + xy\mathbf{a_z}$$

$$\oint_C \mathbf{F} \cdot \mathbf{dl}$$



Formula Sheet

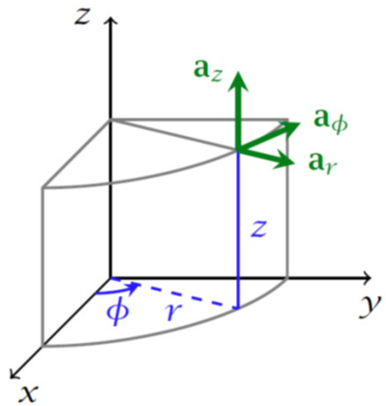
Cartesian ↔ Cylindrical

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z$$

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}, \qquad z = z$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$



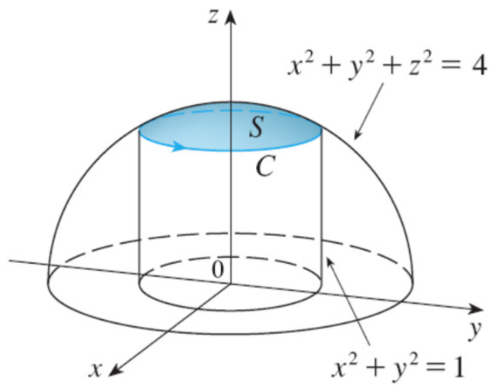
$\phi \in [0, 2\pi]$
@
$z = \sqrt{3}$
$r = 1$

Note that its only necessary to compute A_Φ since **dl** always points in the $\mathbf{a_\Phi}$ direction
dl → $r\,d\Phi\,\mathbf{a_\Phi}$

In class exercise: Solution 2/2

$$\mathbf{F} = xz\mathbf{a_x} + yz\mathbf{a_y} + xy\mathbf{a_z}$$

$$\oint_C \mathbf{F} \cdot \mathbf{dl}$$



$$A_\phi = -\sin(\phi)A_x + \cos(\phi)A_y$$

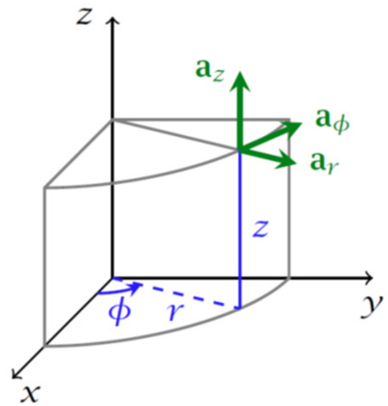
$$A_\phi = -xz\sin(\phi) + yz\cos(\phi)$$

$$x = r\cos(\phi)$$

$$y = r\sin(\phi)$$

$$A_\phi = -rz\cos(\phi)\sin(\phi) + rz\cos(\phi)\sin(\phi)$$

$$A_\phi = 0$$



$$\oint_C \mathbf{F} \cdot \mathbf{dl} = \mathbf{0}$$

Divergence Theorem

Divergence theorem

$\mathbf{A}(x, y, z) \rightarrow$ vector field

$$\mathbf{A} = A_x \mathbf{a_x} + A_y \mathbf{a_y} + A_z \mathbf{a_z}$$

$$\mathbf{\nabla} = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

$$\mathbf{\nabla} \cdot \mathbf{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

$$\int_V \mathbf{\nabla} \cdot \mathbf{A} dv = \oint_S \mathbf{A} \cdot d\mathbf{s}$$

S is a closed surface that bounds V

- The volume integral of the divergence of A is equal to the flux of A through S where S bounds A
- The sum of infinitesimal outward flows across vector field A within volume V is equal to the total flux of A through S where S bounds A
- **Wikipedia:** *sum of all sources of the field in a region (with sinks regarded as negative sources) gives the net flux out of the region*

Example problem

$$\mathbf{F} = z\mathbf{a_x} + y\mathbf{a_y} + x\mathbf{a_z}$$

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F} dv = \oint_S \mathbf{F} \cdot d\mathbf{s}$$

- Find the flux of the vector field F over the unit sphere using both sides of the divergence theorem equality

$$\text{unit sphere} \rightarrow x^2 + y^2 + z^2 = 1$$

Example problem: Solution 1/4

$$\mathbf{F} = z\mathbf{a_x} + y\mathbf{a_y} + x\mathbf{a_z}$$

$$\oint_S \mathbf{F} \cdot d\mathbf{s}$$

- Find the flux of the vector field F over the unit sphere using both sides of the divergence theorem equality

$$\text{unit sphere} \rightarrow x^2 + y^2 + z^2 = 1$$

Formula Sheet

Cartesian ↔ Spherical

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\theta$$

$$R = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \quad \phi = \tan^{-1}\frac{y}{x}$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\mathbf{F} = A_x\mathbf{a_x} + A_y\mathbf{a_y} + A_z\mathbf{a_z}$$

$$\mathbf{F} = A_r\mathbf{a_r} + A_\theta\mathbf{a_\theta} + A_\phi\mathbf{a_\phi}$$

Note that its only necessary to compute A_R since ds always points in the $\mathbf{a_R}$ direction
$\mathbf{ds} \rightarrow ds_R\mathbf{a_R}$

Example problem: Solution 2/4

$$\mathbf{F} = z\mathbf{a_x} + y\mathbf{a_y} + x\mathbf{a_z}$$

$$\int_V \nabla \cdot \mathbf{F} dv = \oint_S \mathbf{F} \cdot d\mathbf{s}$$

- Find the flux of the vector field F over the unit sphere using both sides of the divergence theorem equality

$$\text{unit sphere} \rightarrow x^2 + y^2 + z^2 = 1$$

$$F_R = \sin(\theta)\cos(\phi)F_x + \sin(\theta)\sin(\phi)F_y + \cos(\theta)F_z$$

$$x = R\sin(\theta)\cos(\phi)$$

$$y = R\sin(\theta)\sin(\phi)$$

$$z = R\cos(\theta)$$

$$F_R = \sin(\theta)\cos(\phi)z + \sin(\theta)\sin(\phi)y + \cos(\theta)x$$

$$F_R = R\sin(\theta)\cos(\phi)\cos(\theta) + R\sin^2(\theta)\sin^2(\phi) + R\sin(\theta)\cos(\phi)\cos(\theta)$$

$$F_R = 2R\sin(\theta)\cos(\phi)\cos(\theta) + R\sin^2(\theta)\sin^2(\phi)$$

Example problem: Solution 3/4

$$\oint_S \mathbf{F} \cdot d\mathbf{s} = \oint_S F_R R^2 \sin(\theta) d\theta d\phi$$

$$= \oint_S [2R \sin(\theta)\cos(\phi)\cos(\theta) + R \sin^2(\theta)\sin^2(\phi)] R^2 \sin(\theta) d\theta d\phi$$

$$= R^3 \oint_S [2\sin^2(\theta)\cos(\theta)\cos(\phi) + \sin^3(\theta)\sin^2(\phi)] d\theta d\phi$$

$$= R^3 \left[\underbrace{\mathbf{2}\int_{\mathbf{0}}^{\boldsymbol{\pi}} \sin^2(\theta)\cos(\theta) d\theta}_{I_1} \underbrace{\int_{\mathbf{0}}^{\mathbf{2}\boldsymbol{\pi}} \cos(\phi) d\phi}_{I_2} + \underbrace{\int_{\mathbf{0}}^{\boldsymbol{\pi}} \sin^3(\theta) d\theta}_{I_3} \underbrace{\int_{\mathbf{0}}^{\mathbf{2}\boldsymbol{\pi}} \sin^2(\phi) d\phi}_{I_4} \right]$$

Example problem: Solution 4/4

I_1 $\int_0^{\pi} \sin^2(\theta)\cos(\theta)d\theta = \mathbf{0}$ → Integration by substitution

I_2 ~~$\int_0^{2\pi} \cos(\phi)d\phi$~~ → Not necessary since $I_1 = 0$

I_3 $\int_0^{\pi} \sin^3(\theta)d\theta = \left(-\frac{1}{3}\left(2+\sin^2(\theta)\right)\cos(\theta)\right)\Big|_0^{\pi} = \frac{4}{3}$ → Trigonometric identity

I_4 $\int_0^{2\pi} \sin^2(\phi)d\phi = \left(\frac{1}{2}\phi - \frac{1}{4}\sin(2\phi)\right)\Big|_0^{2\pi} = \pi$ → Trigonometric identity

$$\oint_S \mathbf{F}\cdot d\mathbf{s} = I_1 I_2 + I_3 I_4 = \frac{4\pi}{3}$$

In class exercise:

$$\mathbf{F} = \mathrm{z}\mathbf{a_x} + \mathrm{y}\mathbf{a_y} + \mathrm{x}\mathbf{a_z}$$

$$\boxed{\int_V \nabla \cdot \mathbf{F} dv}$$

- Find the flux of the vector field F over the unit sphere using both sides of the divergence theorem equality

$$\text{unit sphere} \rightarrow x^2 + y^2 + z^2 = 1$$

Let's solve the other side

In class exercise: Solution 1/1

$$\mathbf{F} = z\mathbf{a_x} + y\mathbf{a_y} + x\mathbf{a_z}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

Formula Sheet

Spherical coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_R\, dR + \mathbf{a}_\theta R\, d\theta + \mathbf{a}_\phi R \sin\theta\, d\phi$$

$$ds_R = R^2 \sin\theta\, d\theta\, d\phi$$

$$ds_\theta = R \sin\theta\, dR\, d\phi$$

$$ds_\phi = R\, dR\, d\theta$$

$$dv = R^2 \sin\theta\, dR\, d\theta\, d\phi$$

$$\int_V \nabla \cdot \mathbf{F} dv = \iiint 1\, R^2 \sin(\theta) dR d\theta d\phi$$

$$= \int_0^1 R^2 dR \int_0^\pi \sin(\theta) d\theta \int_0^{2\pi} d\phi$$

$$= \left(\frac{1}{3}\right)(2)(2\pi) = \frac{4\pi}{3}$$

Conclusions and Next Time

Summary

- Your choice to parameterize the contour/surface or map the vector field to something that is “naturally aligned” with the contour/surface
  - Many EM problems have some sort of cylindrical or radial symmetry giving rise to circular contours and spherical and cylindrical test surfaces. Its often not a bad idea to map the vector field to match
- Stoke’s and divergence theorems map between N dimensional and to an N-1 dimensional integrals.
  - One side of each theorem is typically easier to evaluate than the other although this depends on the vector field and surface

- Next week Dr. Wallen starts with Chapter 7
- Have a good weekend!