Aalto University
School of Electrical Engineering

ELEC-E4130 Electromagnetic fields, Autumn 2021

Time-varying fields and Maxwell's equations

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Time-varying fields and Maxwell's equations

Outline for lecture weeks 2–3

Week 2 (September 20 & 23)

- The electromagnetic model [1, 7–1]
- Time-varying EM fields [7–2, 7–3]

Week 3 (September 27 & 30)

- Boundary conditions [7–5]
- Potentials [7–4]
- Wave equations [7–6]
- Time-harmonic fields [7–7]

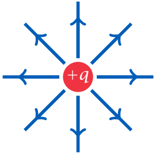
The contents cover Chapter 7 in the textbook D.K. Cheng, *Field and Wave Electromagnetics*, 2nd Ed., with some introductory parts from earlier chapters.

The electromagnetic model

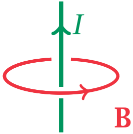
Cheng, Ch. 1 & 7—1 (+ some bits from Ch. 2 to 6)

The electromagnetic model

Electric charges are the fundamental sources of electromagnetic fields and forces.



$$\mathbf{E} = \mathbf{a}_R \frac{q}{4\pi\varepsilon_0 R^2}$$



$$\mathbf{B} = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r}$$

A charge q at rest or in motion produces an **electric field**.

Moving charges produce a current I that gives rise to a **magnetic field**.

The fields are defined using the (physically measurable) **electromagnetic force**

$$\mathbf{F} = q_0\,\mathbf{E} + q_0\,\mathbf{u} \times \mathbf{B} \qquad \text{(Lorentz's force equation)}$$

on a small test charge q_0 moving with velocity $\mathbf{u}$.

A time-varying electric field give rise to a magnetic field and vice versa

$\Rightarrow$ **electromagnetic radiation**

Fundamental equations

Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{(Ampère's law)}$$

$$\nabla \cdot \mathbf{D} = \rho \qquad \text{(Gauss' law)}$$

$$\nabla \cdot \mathbf{B} = 0$$

Lorentz's force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}$$

...depend on the media and can be more complicated

Following the textbook's **deductive approach**, we take these fundamental relations as postulates. All results of classical electromagnetics can be derived starting from these equations.

Example problem 2.1

Derive the electric field of a point charge q in free space

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R$$

starting from Gauss' law $\nabla \cdot \mathbf{D} = \rho$ and the constitutive relation $\mathbf{D} = \varepsilon_0 \mathbf{E}$.

Hint: Deduce the general form of $\mathbf{E}$ due to spherical symmetry and apply the divergence theorem.

Solution: Due to spherical symmetry, $\mathbf{E} = E_R(R)\,\mathbf{a}_R$. For the spherical volume V with radius R and surface S and the point charge in the center, we get

$$\begin{aligned}\int_V \nabla \cdot \mathbf{D}\, dv &= \oint_S \mathbf{D} \cdot d\mathbf{s} = \varepsilon_0 E_R \oint_S ds = \varepsilon_0 E_R\, 4\pi R^2 \\ &= \int_V \rho\, dv = q \qquad \Rightarrow \qquad E_R = \frac{q}{4\pi\varepsilon_0 R^2}\end{aligned}$$

Electromagnetic field quantities

$$\mathbf{E} = \text{Electric field intensity} \qquad [E] = \mathrm{V/m}$$

$$\mathbf{D} = \text{Electric flux density (displacement)} \qquad [D] = \mathrm{C/m^2} = \mathrm{As/m^2}$$

$$\mathbf{B} = \text{Magnetic flux density} \qquad [B] = \mathrm{T} = \mathrm{Wb/m^2} = \mathrm{Vs/m^2}$$

$$\mathbf{H} = \text{Magnetic field intensity} \qquad [H] = \mathrm{A/m}$$

In simple media $\mathbf{D} = \varepsilon\,\mathbf{E}$ and $\mathbf{B} = \mu\,\mathbf{H}$. Why bother using four field quantities?

Flux density D and B

- Surface integral gives the total flux through a surface
- Divergence of a flux density reveals the sources of the flux lines
- Divergence theorem

Field intensity E or H

- Line integral gives a meaningful quantity such as voltage or emf
- Curl of a field intensity is well defined
- Stokes' theorem

Electromagnetic source quantities

Charges

$$\rho = \text{volume charge density} \qquad [\rho] = \mathrm{C/m^3}$$
$$\rho_s = \text{surface charge density} \qquad [\rho_s] = \mathrm{C/m^2}$$
$$q = \text{point charge} \qquad [q] = \mathrm{C}$$
$$Q = \text{total charge (in a volume)} \qquad [Q] = \mathrm{C}$$

Currents

$$\mathbf{J} = \text{current density} \qquad [J] = \mathrm{A/m^2}$$
$$\mathbf{J}_s = \text{surface current density} \qquad [J_s] = \mathrm{A/m}$$
$$I = \text{current} \qquad [I] = \mathrm{A}$$

Note that $\mathbf{J}$ is a vector ($\sim$flux density) and I is a scalar.

In Maxwell's equations, these are *free charges* and *free currents*.

(Bound charges and bound currents are included in ε and μ.)

$\mathbf{J}$ can include both *conduction current*

$$\mathbf{J} = \sigma\,\mathbf{E}$$

and *convection current* due to other moving charges.

Constitutive relations

In **simple conducting media**

$$\mathbf{D} = \varepsilon\,\mathbf{E} = \varepsilon_0\varepsilon_r\mathbf{E}, \qquad \varepsilon = \text{permittivity}, \qquad \varepsilon_r = \text{relative permittivity} \geq 1$$

$$\mathbf{B} = \mu\,\mathbf{H} = \mu_0\mu_\mathrm{r}\mathbf{H}, \qquad \mu = \text{permeability}, \qquad \mu_r = \text{relative permeability} \geq 1$$

$$\mathbf{J} = \sigma\mathbf{E}, \qquad \sigma = \text{conductivity}, \qquad [\sigma] = \mathrm{S/m} = \frac{\mathrm{A}}{\mathrm{Vm}}$$

This is a simple macroscopic model for the microscopic EM response of matter. Beware that tabulated material parameters typically are valid for limited frequency ranges and temperatures...

Speed of light in vacuum $$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 299\,792\,458\,\frac{\mathrm{m}}{\mathrm{s}}$$

Permeability of free space $$\mu_0 = 4\pi\times 10^{-7}\,\frac{\mathrm{Vs}}{\mathrm{Am}} \approx 1.257\times 10^{-6}\,\frac{\mathrm{H}}{\mathrm{m}}$$

Permittivity of free space $$\varepsilon_0 = \frac{1}{\mu_0\,c^2} \approx 8.854\times 10^{-12}\,\frac{\mathrm{As}}{\mathrm{Vm}} \quad \left(= \frac{\mathrm{F}}{\mathrm{m}}\right)$$

SI units in electromagnetics

Charge and electric flux	$\mathrm{C} = \text{coulomb} = \mathrm{As}$
Magnetic flux	$\mathrm{Wb} = \text{weber} = \mathrm{Vs}$
Magnetic flux density	$\mathrm{T} = \text{tesla} = \mathrm{Wb/m^2} = \mathrm{Vs/m^2}$
Capacitance	$\mathrm{F} = \text{farad} = \mathrm{C/V} = \mathrm{As/V}$
Inductance	$\mathrm{H} = \text{henry} = \mathrm{Wb/A} = \mathrm{Vs/A}$
Resistance	$\Omega = \text{ohm} = \mathrm{V/A}$
Conductance	$\mathrm{S} = \text{siemens} = 1/\Omega = \mathrm{A/V}$
Power	$\mathrm{W} = \text{watt} = \mathrm{VA}$
Work	$\mathrm{J} = \text{joule} = \mathrm{Ws} = \mathrm{VAs}$
Force	$\mathrm{N} = \text{newton} = \mathrm{J/m} = \mathrm{VAs/m}$

In EM it is convenient to expand SI units in $(\mathrm{V}, \mathrm{A}, \mathrm{m}, \mathrm{s})$ although volt is not one of the fundamental SI units:

$$\mathrm{V} = \frac{\mathrm{kg\,m^2}}{\mathrm{A\,s^3}}$$

Conservation of charge

The principle of conservation of charge is one very fundamental physical postulate. For an arbitrary volume V and its surface S, we have

$$I = \oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{dQ}{dt} = -\frac{d}{dt} \int_V \rho \, dv,$$

where the direction of $d\mathbf{s}$ (and the net current I) is outward.

Example problem 2.2: Show that electromagnetic fields always satisfy the above condition if the fields satisfy Maxwell's equations.

Hint: As an intermediate step, show that Maxwell's equations imply the **equation of continuity**

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}}$$

Solution

Take the divergence of Ampère's law and use Gauss' law:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) \quad \Rightarrow \quad \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Integrate over an arbitrary volume V with surface S and apply the divergence theorem:

$$\int_V \nabla \cdot \mathbf{J}\, dv = \oint_S \mathbf{J} \cdot d\mathbf{s} = I = -\int_V \frac{\partial \rho}{\partial t} dv \quad \Rightarrow \quad I = -\frac{dQ}{dt}$$

Static fields

If the sources are time invariant, the time derivatives in Maxwell's equations are zero and the electromagnetic coupling disappears:

Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\frac{\partial}{\partial t} = 0$$

Electrostatics

$$\nabla \times \mathbf{E} = \mathbf{0}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

→ Electric potential, Coulomb's law, . . .

Steady currents

$$\nabla \times \mathbf{E} = \mathbf{0}$$

$$\nabla \cdot \mathbf{J} = 0$$

$$\mathbf{J} = \sigma \mathbf{E}$$

Magnetostatics

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{B} = \mu \mathbf{H}$$

Lorentz's force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Time-varying electromagnetic fields

Cheng, Ch. 7-2 & 7-3

Dynamic fields

Time-varying fields create electromagnetic coupling

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\frac{\partial \mathbf{B}_1}{\partial t} \rightarrow \text{time-varying (and nonconservative) } \mathbf{E}_1 \rightarrow \frac{\partial \mathbf{D}_1}{\partial t}$$

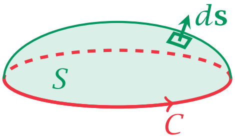
$$\frac{\partial \mathbf{D}_1}{\partial t} \rightarrow \text{time-varying } \mathbf{H}_2 \rightarrow \frac{\partial \mathbf{B}_2}{\partial t}$$

$$\rightarrow E_2 \rightarrow H_3 \ldots$$

Let us start with a time-varying magnetic field ($\mathbf{B}_1$ in Faraday's law) and assume that the secondary magnetic field ($\mathbf{B}_2$) is small enough so that it can be ignored.

Faraday's law of electromagnetic induction

Integrate Faraday's law



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

over a stationary surface S with bounding contour C apply Stokes' theorem:

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = \int_S -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad \Rightarrow \quad \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \quad \Rightarrow \quad \boxed{\mathscr{V} = -\frac{d\Phi}{dt}}$$

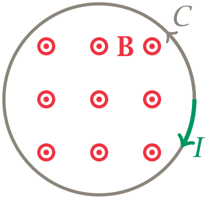
$\mathscr{V} = \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} =$ induced **electromotive force (emf)** in the contour C

$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} =$ magnetic flux through the surface S

The right-handed orientation of C and $d\mathbf{s}$ in the figure is also important!

Induced emf in a stationary loop (= transformer emf)

The magnetic flux density $\mathbf{B}$ (toward the viewer) is increasing.



The increasing magnetic flux induces an emf in the contour:

$$\mathscr{V} = -\frac{d\Phi}{dt} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} < 0$$

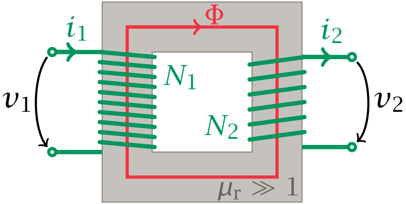
If the contour C is a conducting loop, the emf causes a current I to flow.

The current gives rise to a secondary magnetic field and magnetic flux through the loop away from the viewer, opposing the change in the primary magnetic field.

Lenz's law:

> The induced emf and current opposes the change in magnetic flux.

Example: An ideal transformer



An alternating current $i_1(t)$ in the primary winding with N_1 turns creates a time-varying magnetic flux Φ that induces an emf in the secondary winding $=$ secondary voltage v_2.

An ideal transformer (1) has no leakage flux and (2) is lossless.

(1) Faraday's law gives

$$v_1 = N_1 \frac{d\Phi}{dt}, \quad v_2 = N_2 \frac{d\Phi}{dt}$$

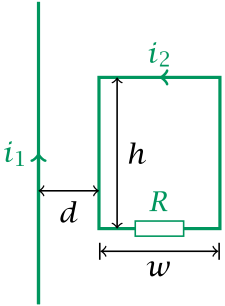
$$\Rightarrow \quad \boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}}$$

(2) The output power is the same as the input power

$$\boxed{v_2 i_2 = v_1 i_1}$$

The direction of the windings are chosen to eliminate minus signs from the formulas.

Example problem 2.3



A time-varying current i_1 in a long straight conductor induce a current i_2 in a nearby rectangular loop. The resistance of the loop is R and the dimensions are given in the figure. Assume that $di_1/dt > 0$ is given and derive i_2.

Check the direction (sign) of i_2 using Lenz' law.

Hint: You can use the expression for the magnetic field from slide 4

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{a}_\phi$$

and the self-inductance of the loop can be ignored.

Solution

The magnetic flux through the loop is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_{z=0}^{h} \int_{r=d}^{d+w} \frac{\mu_0 i_1}{2\pi r} \mathbf{a}_\phi \cdot (-\mathbf{a}_\phi)\, dr\, dz = -\frac{\mu_0 i_1 h}{2\pi} \ln \frac{d+w}{d}.$$

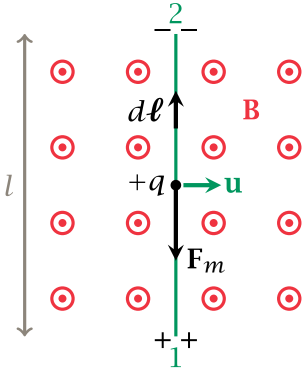
This gives the induced emf and current

$$\mathcal{V} = -\frac{d\Phi}{dt}, \qquad i_2 = \frac{\mathcal{V}}{R} = \frac{\mu_0 h}{2\pi R} \frac{di_1}{dt} \ln \frac{d+w}{d}.$$

The current i_2 is positive and this direction agrees with Lenz' law.

Motional emf / flux cutting emf

A conductor moves with velocity $\mathbf{u}$ in a static magnetic field $\mathbf{B}$:



The magnetic force $\mathbf{F}_m = q\,\mathbf{u} \times \mathbf{B}$ cause free electrons to drift until an eletric force balance the magnetic force.

$\mathbf{F}_m/q = \mathbf{u} \times \mathbf{B}$ can be interpreted as an induced electic field producing the voltage

$$V_{12} = \int_1^2 (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} = -uBl$$

The induced **motional emf** or **flux cutting emf** around a closed circuit C is

$$\mathcal{V}' = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}$$

where the parts of C moving across the magnetic field lines (cutting the magnetic flux) contribute to the emf.

Different kind of emfs?

1. **Transformer emf** for stationary loop

$$\mathscr{V} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

2. **Motional emf** for static magnetic field

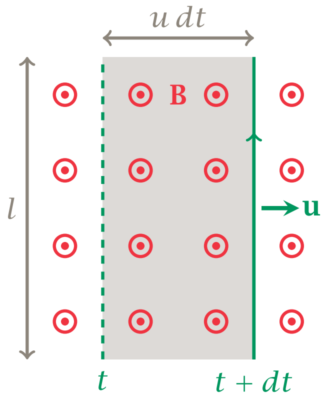
$$\mathscr{V}' = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}$$

3. **For a moving circuit in a time-varying magnetic field**, the division between transformer emf and motional emf depends on the frame of reference, but the sum is always unique and equal to

$$\boxed{\mathscr{V}' = -\frac{d\Phi}{dt}}$$

where the change in magnetic flux depends both on the moving circuit and the time-varying magnetic field.

Another interpretation of the moving conductor



In time dt, the conductor sweeps the area

$$dA = lu\,dt,$$

which can be interpreted as a change in magnetic flux:

$$\mathcal{V}' = -\frac{d\Phi}{dt} = -\frac{B\,dA}{dt} = -\frac{Blu\,dt}{dt} = -Blu$$

The emf is unique and we often have more than one way to calculate it.

Ampère's law

In Ampère's law

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

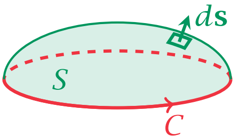
the **displacement current density** $\partial \mathbf{D}/\partial t$ is needed to

- explain how ac current can flow through a capacitor,
- fullfill the conservation of charge, and
- admit electromagnetic waves as a solution of Maxwell's equations.

Although the displacement current density $\partial \mathbf{D}/\partial t$ doesn't consist of moving charges, it is a similar source of magnetic fields as the current density $\mathbf{J}$.

Ampère's circuital law

Integrate Ampère's law



$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

over a surface S with bounding contour C apply Stokes' theorem:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s} \quad \Rightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = \int_S \mathbf{J} \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$$

$$\Rightarrow \quad \boxed{\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}}$$

The contour integral of the magnetic field around a contour or circuit C is equal to the total current (including displacement current) flowing through the circuit.

Example problem 2.4

An ideal voltage source $v(t) = V_0 \cos(\omega t)$ is connected to an ideal parallel-plate capacitor with capacitance C.

Show that the displacement current in the capacitor is the same as the conduction current in the wires connected to the capacitor.

Solution: For an ideal parallel-plate capacitor, we have

$$i = C\frac{dv}{dt}, \qquad C = \varepsilon\frac{A}{d}, \qquad E = \frac{v}{d}.$$

Thus, the conduction current is

$$i = C\frac{dv}{dt} = -\varepsilon\frac{A}{d}V_0\omega\sin(\omega t)$$

and the displacement current is

$$i_D = \int_A \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{s} = A\varepsilon\frac{dE}{dt} = \frac{A\varepsilon}{d}\frac{dv}{dt} = -\frac{A\varepsilon}{d}V_0\omega\sin(\omega t) = i.$$

Next week

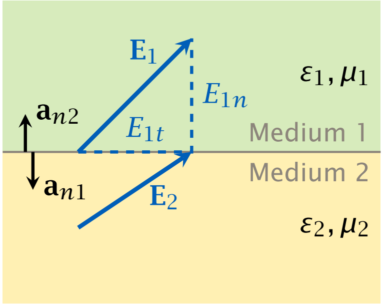
This is the end of Lecture week 2. We'll continue with the electromagnetic boundary conditions next week.

Electromagnetic boundary conditions

Cheng, Ch. 7-5

Electromagnetic boundary conditions

- In a homogeneous medium, the electromagnetic fields must be differentiable to satisfy Maxwell's equations ⇒ all components of $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{H}$ must be continuous.
- At the interface between two homogeneous media, we get certain boundary conditions from Maxwell's equations in integral form.

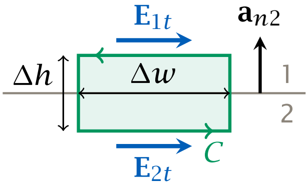


Notation:

Subscript $1, 2$ refer to medium $1, 2$
Subscript t = tangential component
Subscript n = normal component

$\mathbf{a}_{n2}$ = outward unit normal for medium 2
(we take this to be the positive normal direction)

Faraday's law



Apply Faraday's law of induction on the contour C:

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$$

If Δw is small and $\Delta h \to 0$, the magnetic flux through the contour vanishes and we get

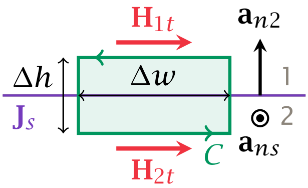
$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = (E_{2t} - E_{1t})\,\Delta w = 0 \quad \Rightarrow \quad \boxed{E_{1t} = E_{2t}}$$

The tangential component of the electric field intensity $\mathbf{E}$ is continuous at the interface.

Equivalently, using rotated tangential components:

$$\mathbf{a}_{n2} \times \mathbf{E}_1 = \mathbf{a}_{n2} \times \mathbf{E}_2$$

Ampère's law



If we do the same procedure for Ampère's circuital law

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$$

and include the possible **surface current density $\mathbf{J}_s$** at the interface, we get

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = (H_{2t} - H_{1t})\,\Delta w = J_{sn}\Delta w$$

Using suitable unit vectors ($\mathbf{a}_{n2} \times \mathbf{a}_{ns}$ points to the right in the figure)

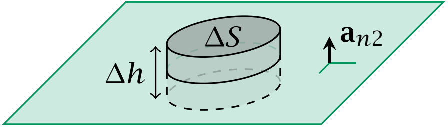
$$(\mathbf{H}_2 - \mathbf{H}_1) \cdot \mathbf{a}_{n2} \times \mathbf{a}_{ns}\,\Delta w = (\mathbf{H}_2 - \mathbf{H}_1) \times \mathbf{a}_{n2} \cdot \mathbf{a}_{ns}\,\Delta w = \mathbf{J}_s \cdot \mathbf{a}_{ns}\,\Delta w$$

gives the more general expression $\boxed{\mathbf{a}_{n2} \times \left(\mathbf{H}_1 - \mathbf{H}_2\right) = \mathbf{J}_s}$

No isolated magnetic charge

Integrate $\nabla \cdot \mathbf{B} = 0$ over a volume V and apply the divergence theorem:

$$\int_V \nabla \cdot \mathbf{B} \, dv = \oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$



Using a small cylinder with area ΔS and height $\Delta h \to 0$ placed on the boundary between medium 1 (above) and medium 2 (below), we get

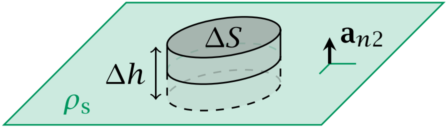
$$\oint_S \mathbf{B} \cdot d\mathbf{s} = \int_{\text{top}} \mathbf{B}_1 \cdot d\mathbf{s} + \int_{\text{bottom}} \mathbf{B}_2 \cdot d\mathbf{s} = \mathbf{B}_1 \cdot \mathbf{a}_{n2} \Delta S - \mathbf{B}_2 \cdot \mathbf{a}_{n2} \Delta S = 0$$

$$\Rightarrow \quad \boxed{\mathbf{a}_{n2} \cdot \mathbf{B}_1 = \mathbf{a}_{n2} \cdot \mathbf{B}_2}$$

The normal component of the magnetic flux density $\mathbf{B}$ is continuous at the interface.

Gauss' law

Similarly for Gauss' law $\nabla \cdot \mathbf{D} = \rho \quad \Rightarrow \quad \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho \, dv$



Same cylinder with ΔS small and $\Delta h \to 0$, but now we may have a surface charge density ρ_s on the boundary, and we get

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{\text{top}} \mathbf{D}_1 \cdot d\mathbf{s} + \int_{\text{bottom}} \mathbf{D}_2 \cdot d\mathbf{s} = \mathbf{D}_1 \cdot \mathbf{a}_{n2} \Delta S - \mathbf{D}_2 \cdot \mathbf{a}_{n2} \Delta S = \rho_s \, \Delta S$$

$$\Rightarrow \quad \boxed{\mathbf{a}_{n2} \cdot \left(\mathbf{D}_1 - \mathbf{D}_2\right) = \rho_s}$$

As in electrostatics, any charge distribution acts as a source for the electric flux density $\mathbf{D}$. Without surface charges, the normal component of $\mathbf{D}$ is continuous.

Interface between two lossless media

In lossless media, there are no free charges and the conductivity $\sigma = 0$

$$\Rightarrow \qquad \rho_s = 0, \quad \mathbf{J}_s = \mathbf{0}$$

and the boundary or **interface conditions** simplify to

$$\boxed{\begin{aligned} E_{1t} &= E_{2t} \\ H_{1t} &= H_{2t} \\ B_{1n} &= B_{2n} \\ D_{1n} &= D_{2n} \end{aligned}}$$

If there are no sources at the boundary, the tangential component of the field intensities and normal component of the flux densities are continuous.

This is typically also the case if we have a boundary between ordinary (low loss) materials.

Example problem 3.1: Steady electric currents

What are the boundary conditions for the normal and tangential components of the static current density $\mathbf{J}$ at the interface between two conducting media with conductivities σ_1 and σ_2?

Solution:

In statics, the equation of continuity is $\nabla \cdot \mathbf{J} = 0$. By analogy with $\nabla \cdot \mathbf{B} = 0$ we can conclude that the normal component of $\mathbf{J}$ is continuous:

$$J_{1n} = J_{2n}$$

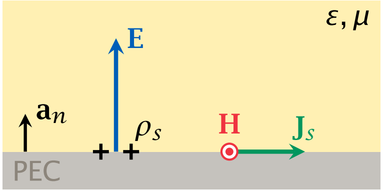
Since $\mathbf{J} = \sigma \mathbf{E}$ and $E_{1t} = E_{2t}$, we get

$$\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}.$$

PEC boundary conditions

In a **perfect electric conductor** (PEC) the conductivity $\sigma \to \infty$ and the electromagnetic fields must be zero. (Why?)

Setting $\mathbf{E}_2 = \mathbf{D}_2 = \mathbf{H}_2 = \mathbf{B}_2 = \mathbf{0}$ in the general conditions and omitting some unneccessary indices, we get the **PEC boundary conditions**



$$\begin{aligned} \mathbf{a}_n \times \mathbf{E} &= \mathbf{0} \\ \mathbf{a}_n \times \mathbf{H} &= \mathbf{J}_s \\ \mathbf{a}_n \cdot \mathbf{D} &= \rho_s \\ \mathbf{a}_n \cdot \mathbf{B} &= 0 \end{aligned}$$

at the boundary

These conditions are for the fields in the ordinary material, when the distance to the boundary tend to zero.

Electromagnetic potentials

Cheng, Ch. 7-4

Electromagnetic potentials (1/3)

Since $\mathbf{B}$ is solenoidal, $\nabla \cdot \mathbf{B} = 0$, it can be expressed using the **vector magnetic potential $\mathbf{A}$**:

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

Substituting this into Faraday's law gives

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\left(\nabla \times \mathbf{A}\right) \quad \Rightarrow \quad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0.$$

Any conservative (= curl-free) vector field can be expressed as a gradient of a scalar field. To be consistent with electrostatics, choose

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \quad \Rightarrow \quad \boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}}$$

Electromagnetic potentials (2/3)

Combining the potentials with Ampére's law and the constitutive relations gives

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J} + \frac{\partial}{\partial t} \left[\varepsilon \left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right) \right]$$

If ε and μ are constant

$$\Leftrightarrow \quad \nabla \left(\nabla \cdot \mathbf{A} \right) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \mu\varepsilon \nabla \left(\frac{\partial V}{\partial t} \right) - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

$$\Leftrightarrow \quad \nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla \left(\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t} \right)$$

A vector field is unique (up to an additive constant) if both its curl and divergence are specified. At this point we choose the **Lorenz condition** or Lorenz gauge

$$\boxed{\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0}$$

Trivia: Ludvig Lorenz (DK) $\neq$ Hendrik Lorentz (NL), known for the Lorentz force

Electromagnetic potentials (3/3)

From Gauss' law we similarly get

$$\nabla \cdot \left[\varepsilon \left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right) \right] = \rho \quad \Leftrightarrow \quad \nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon}$$

and finally using the Lorenz condition

$$\Leftrightarrow \quad \nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

Thus, assuming constant ε, μ and using the Lorenz condition we have two separate (uncoupled) **inhomogeneous wave equations** for the potentials:

$$\boxed{\begin{aligned} \nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu \mathbf{J} \\ \nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} &= -\frac{\rho}{\varepsilon} \end{aligned}}$$

Example problem 3.2: Coulomb gauge

Another possible choice for the gauge condition is the so called **Coulomb gauge**

$$\nabla \cdot \mathbf{A} = 0.$$

Derive the equations that the scalar and vector potentials V and $\mathbf{A}$ must satisfy, if we choose this condition (and still assume that ε and μ are constants).

Solution:

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \mu\varepsilon \nabla \frac{\partial V}{\partial t}$$

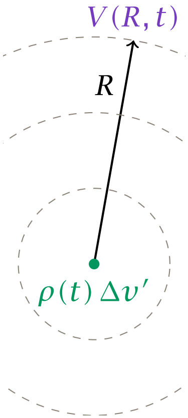
This gave an easier equation for V at the expense of a more difficult source term for the vector wave equation for $\mathbf{A}$. (We'll use the Lorenz condition for the rest of the course.)

Solution of the wave equations

Cheng, Ch. 7-6

Solution of the scalar wave equation (1/2)

Scalar potential $V(R,t)$ of a time-varying point charge $\rho(t)\,\Delta v'$ at the origin?



Except at the source, we have

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) - \mu\varepsilon\frac{\partial^2 V}{\partial t^2} = 0$$

Substituting $V = \frac{1}{R}U(R,t)$ gives

$$\frac{\partial^2 U}{\partial R^2} - \mu\varepsilon\frac{\partial^2 U}{\partial t^2} = 0$$

This is a 1D wave equation with solutions $f(t - R\sqrt{\mu\varepsilon})$ and $g(t + R\sqrt{\mu\varepsilon})$ for any twice-differentiable f and g. We choose $U(R,t) = f(t - R/u)$ since it is a solution traveling away from the source with velocity $u = 1/\sqrt{\mu\varepsilon}$.

Solution of the scalar wave equation (2/2)

Now we have a general solution for the homogeneous wave equation

$$V(R,t) = \frac{f(t - R/u)}{R}$$

and need to choose a specific function f based on the source.

For a static point charge $\rho(t)\,\Delta v' = q =$ constant, we know that

$$V = \frac{q}{4\pi\varepsilon R}$$

Comparing these two, we get

$$V = \frac{\rho(t - R/u)}{4\pi\varepsilon R}\Delta v' \qquad \text{for the point charge}$$

and so the potential due to a charge distribution is

$$V = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(t - R/u)}{R} dv'$$

where R is now the distance between dv' and the observation point.

Retarded potentials

Since the components of the vector potential $\mathbf{A}$ satisfy a similar wave equation as the scalar potential V, we thus have the solutions

$$V = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(t - R/u)}{R} dv'$$

$$\mathbf{A} = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J}(t - R/u)}{R} dv'$$

where the volume integral is over the whole source region and R is the distance between dv' and the observation point. (This distance is approximately the same as the spherical coordinate R only for a small source at the origin.)

These are called **retarded potentials** since the potentials at time t depends on the sources at an earlier time $(t - R/u)$.

Source-free wave equations for the fields

For a non-conducting ($\sigma = 0$) source-free medium with constant ε and μ, Maxwell's equations reduce to

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \qquad \nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \qquad \nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{H} = 0$$

Taking the curl of Faraday's law, we get

$$\begin{aligned} \nabla \times \left(\nabla \times \mathbf{E}\right) &= \nabla\left(\nabla \cdot \mathbf{E}\right) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} \\ &= -\mu \frac{\partial}{\partial t}\left(\nabla \times \mathbf{H}\right) = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \end{aligned}$$

Starting from the curl of Ampère's law we get exactly the same **homogeneous vector wave equation** also for $\mathbf{H}$:

$$\boxed{\nabla^2 \mathbf{E} - \frac{1}{u^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0}} \qquad \boxed{\nabla^2 \mathbf{H} - \frac{1}{u^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = \mathbf{0}} \qquad u = \frac{1}{\sqrt{\mu\varepsilon}}$$

Time-harmonic fields

Cheng, Ch. 7-7

Phasors

A **phasor** is a complex time-independent quantity that has a one-to-one mapping to the corresponding **time-harmonic signal**.

Using a $\cos(\omega t)$-reference, we have the definition

$$v(t) = \mathcal{R}e\left[V\, e^{j\omega t}\right]$$

From this definition follows for real constants A, B, ϕ:

$$\begin{aligned} v(t) &= A\cos(\omega t) && \longleftrightarrow && V = A \\ v(t) &= A\cos(\omega t + \phi) && \longleftrightarrow && V = A\, e^{j\phi} = A\underline{/\phi} \\ v(t) &= A\cos(\omega t) - B\sin(\omega t) && \longleftrightarrow && V = A + j\,B \end{aligned}$$

Beware that other courses and books may use different definitions of the mapping above! Another popular choice is to use $e^{-i\omega t}$.

Example problem 3.3: Phasors

Convert the phasor

$$V_1 = jV_0\, e^{-\alpha z - j\beta z}$$

to the corresponding time-harmonic signal $v_1(t)$, and the signal

$$v_2(t) = A\sin(\omega t + \phi)$$

to the corresponding phasor V_2. All given constants can be assumed real.

Solution:

$$\begin{aligned} v_1(t) &= \mathcal{R}e\left[V_1\, e^{j\omega t}\right] = \mathcal{R}e\left[jV_0\, e^{-\alpha z}\, e^{j(\omega t - \beta z)}\right] \\ &= \mathcal{R}e\left[jV_0\, e^{-\alpha z}\Big(\cos(\omega t - \beta z) + j\sin(\omega t - \beta z)\Big)\right] \\ &= -V_0 e^{-\alpha z}\sin(\omega t - \beta z) \end{aligned}$$

$$v_2(t) = A\sin(\omega t + \phi) = A\cos\left(\omega t + \phi - \frac{\pi}{2}\right)$$

$$\Rightarrow \qquad V_2 = A\, e^{j(\phi - \pi/2)} = -jA\, e^{j\phi}$$

Derivatives and integrals

Time derivatives and integrals are algebraic operations for phasors:

$$\frac{d}{dt} v(t) = \frac{d}{dt} \mathcal{R}e\left[V\, e^{j\omega t}\right] = \mathcal{R}e\left[V \frac{d}{dt} e^{j\omega t}\right] = \mathcal{R}e\left[V\, j\omega\, e^{j\omega t}\right]$$

$$\Rightarrow \quad \boxed{\frac{d}{dt} \longleftrightarrow j\omega} \quad \text{and similarly} \quad \boxed{\int dt \longleftrightarrow \frac{1}{j\omega}}$$

The fundamental usefulness of phasors is that complex arithmetic is easier than calculus and trigonometric functions.

Mathematically, you could also define phasors
as the Fourier transform of time-harmonic signals.

Time-harmonic electromagnetic fields

A time-harmonic electric field $\mathbf{E}(x, y, z, t)$ can be represented with a **vector phasor** $\mathbf{E}(x, y, z)$:

$$\mathbf{E}(t) = \mathscr{Re}\left[\mathbf{E}\, e^{j\omega t}\right]$$

Note: We use the same symbol $\mathbf{E}$ for the time-harmonic electric field and its vector phasor and typically omit the space-coordinate arguments.

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D}$$
$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$

For the rest of the course we will mostly deal with phasors and Maxwell's equations for time-harmonic fields represented by phasors.

Time-harmonic potentials

The wave equations for the potentials are transformed into Helmholtz' equations

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}, \qquad \nabla^2 V + k^2 V = -\frac{\rho}{\sigma}, \qquad k = \omega\sqrt{\varepsilon\mu}$$

and the corresponding retarded potential solution becomes

$$V = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\rho\, e^{-jkR}}{R} dv'$$

$$\mathbf{A} = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}\, e^{-jkR}}{R} dv'$$

For a given time-harmonic source current:

$$\mathbf{J} \xrightarrow{\int} \mathbf{A} \xrightarrow{\nabla\times} \mathbf{H} \xrightarrow{\nabla\times} \mathbf{E}$$

This strategy will be used to calculate antenna radiation. . .

Source-free fields in simple media

The previously derived homogeneous vector wave equation can easily be converted to **homogeneous vector Helmholtz's equations**

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = \mathbf{0}, \qquad \nabla^2\mathbf{H} + k^2\mathbf{H} = \mathbf{0}, \qquad k = \omega\sqrt{\mu\varepsilon}$$

by simply replacing $\partial/\partial t \Rightarrow j\omega$. The same results can, of course be obtained from the time-harmonic Maxwell's equations.

Example problem 3.4:

Derive the equation $\nabla^2\mathbf{H} + k^2\mathbf{H} = \mathbf{0}$ starting from Ampère's law for time-harmonic fields. Assume a simple medium with constant ε and μ, $\sigma = 0$, and no sources.

Solution

For a non-conducting ($\sigma = 0$) source-free medium with constant ε and μ, Ampère's law for time-harmonic fields becomes

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}.$$

Taking the curl of the left-hand side

$$\nabla \times \left(\nabla \times \mathbf{H}\right) = \nabla\left(\underbrace{\nabla \cdot \mathbf{H}}_{=0}\right) - \nabla^2\mathbf{H} = -\nabla^2\mathbf{H}$$

and the right-hand side

$$j\omega\varepsilon\nabla \times \mathbf{E} = j\omega\varepsilon\left(-j\omega\mu\mathbf{H}\right) = \omega^2\mu\varepsilon\mathbf{H} = k^2\mathbf{H}$$

gives the wanted result

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = \mathbf{0}.$$

Conducting media and complex permittivity

If the simple media (constant ε, μ) is conducting, we have a conduction current $\mathbf{J} = \sigma \mathbf{E}$, and Ampère's law can be put in the form

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D} = \sigma \mathbf{E} + j\omega\varepsilon \mathbf{E} = j\omega \left(\frac{\sigma}{j\omega} + \varepsilon \right) \mathbf{E} = j\omega \varepsilon_c \mathbf{E},$$

using the **complex permittivity**

$$\varepsilon_c = \varepsilon' - j\varepsilon'' = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon_0 \left(\varepsilon_\mathrm{r} - j\frac{\sigma}{\omega\varepsilon_0} \right)$$

Including the conductance in the complex permittivity eliminates the conduction current, if we replace ε with ε_c in all equations.

Loss tangent, good conductors and insulators

The complex permittivity makes the wave number k in Helmholtz equation complex:

$$k_c = \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu(\varepsilon' - j\varepsilon'')}$$

We will later discuss how $\mathcal{Im}[k_c]$ is related to losses. One measure for power loss in the medium is the **loss tangent**

$$\tan\delta_c = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon}.$$

A medium is a **good conductor** if $\sigma \gg \omega\varepsilon$ and a **good insulator** if $\omega\varepsilon \gg \sigma$.

The same material may be a good conductor at low frequencies and a good insulator at high frequencies.

The electromagnetic spectrum (1/2)

Maxwell's equations (in time-harmonic form) are exactly the same regardless of frequency.

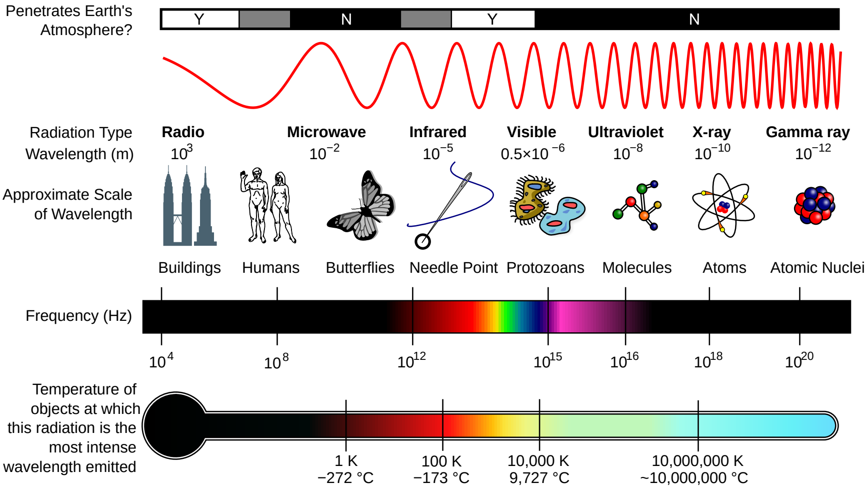
- All EM waves, including radio waves and visible light, are mathematically the same.
- All EM waves propagate with the same velocity $u = 1/\sqrt{\mu\varepsilon}$.
- The interaction between EM waves and matter do, however, depend on the frequency. For instance ultraviolet radiation and X-rays are dangerous, while infrared radiation (heat) is not.
- Material parameters depend on frequency: $\varepsilon = \varepsilon(\omega)$, etc. One prime example is water:

  For radio waves ($f < 1\,\mathrm{GHz}$) water has high permittivity $\varepsilon_r \approx 80$ and the conductivity depends heavily on the impurity: $\sigma \approx 1\,\mathrm{mS/m}$ for fresh water and $\sigma \approx 4\,\mathrm{S/m}$ for seawater.

  For visible light, water is transparent (nearly lossless) and $\varepsilon_r \approx 1.8$.

The electromagnetic spectrum (2/2)

Frequency and wavelength range of EM waves



Outlook

In the following weeks, we'll be using time-harmonic fields (vector phasors) extensively for plane waves, guided waves, and antennas.