Exercise Session 1

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Information

- You can give answers in English or Finnish.
- 20% of your final grades.
- 8 problem sets in total
- The first 3 sessions are held online. The last 3 sessions in November and December are scheduled to be in-person.

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Exercise 1

Consider the following 3×4 matrix, where $a \in \mathbb{R}$ is a parameter:

$$A = \begin{pmatrix} 1 & -2 & 4 & 2 \\ 2 & -3 & 5 & 3 \\ 3 & -4 & 6 & a \end{pmatrix}$$

1. Transform the matrix in row echelon form.
2. Find all the values of a such that $A = 2$.
3. Find all the values of a such that $A = 3$.

Exercise 1 - Solution

Terminology: A matrix is in row echelon form if each row begins with more zeros than the row above it.

$$A = \begin{pmatrix} 1 & -2 & 4 & 2 \\ 2 & -3 & 5 & 3 \\ 3 & -4 & 6 & a \end{pmatrix}$$

row2 = row2 - 2 * row1

row3 = row3 - 3 * row1

$$A = \begin{pmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 2 & -6 & a-6 \end{pmatrix}$$

row3 = row3 - 2 * row2

$$A = \begin{pmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & a-4 \end{pmatrix}$$

Exercise 1 - Solution

1. A possible row echelon form of A is

$$A = \begin{pmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & a-4 \end{pmatrix}$$

   Terminology: The rank of a matrix is the number of nonzero rows in its row echelon form (or, equivalently, in its reduced row echelon form).

2. rank $A = 2$ if and only if $a = 4$
3. rank $A = 3$ if and only if $a \neq 4$

Exercise 2

Consider the following 3×5 matrix:

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 3 & -3 & 10 & 1 & 7 \end{pmatrix}$$

1. Transform A in *reduced* row echelon form.
2. Calculate the rank of A.

Exercise 2 - Solution

Terminology: A matrix is in reduced row echelon form if it is in row echelon form with each pivot equal to one and each column that contains a pivot has no other nonzero entries.

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 3 & -3 & 10 & 1 & 7 \end{pmatrix}$$

row2 = row2 - 1 * row1

row3 = row3 - 3 * row2

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 3 & 1 & -2 & 1 \end{pmatrix}$$

row3 = row3 - 1 * row2

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 2 - Solution

Terminology: A matrix is in reduced row echelon form if it is in row echelon form with each pivot equal to one and each column that contains a pivot has no other nonzero entries.

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

row2 = 1/3 * row2

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

row1 = row1 + 2 * row2

$$A = \begin{pmatrix} 1 & 0 & \frac{11}{3} & -\frac{1}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 2 - Solution

1. The reduced row echelon form is

$$A = \begin{pmatrix} 1 & 0 & \frac{11}{3} & -\frac{1}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   Terminology: The rank of a matrix is the number of nonzero rows in its row echelon form (or, equivalently, in its reduced row echelon form).

2. rank $A = 2$

Exercise 3

Consider the following two $n \times 1$ matrices (i.e. vectors):

$$Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}, \qquad P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$$

Suppose that Q collects the quantities of n distinct grocery items you purchased last month, and P indicates the corresponding unit prices. For example, q_1 may be the quantity of milk you purchased, and p_1 is the corresponding unit price. This implies that last month you spent $p_1 q_1$ Euros on milk. Similarly, q_2 and p_2 may indicate the quantity and unit price of cereals, and so on.

1. Use matrix multiplication in such a way to determine how much money you spent on *all* the grocery items you purchased. [*HINT*: Recall how to calculate the transpose of a matrix.]
2. Calculate the product QP^T. What do the entries in QP^T represent? Do they all make economic sense?

Exercise 3 - Solution

Matrix multiplication

$$\begin{matrix} \vec{a_1}\rightarrow \\ \vec{a_2}\rightarrow \end{matrix} \underset{A}{\begin{bmatrix} 1 & 7 \\ 2 & 4 \end{bmatrix}} \cdot \underset{B}{\overset{\begin{matrix} \vec{b_1} & \vec{b_2} \\ \downarrow & \downarrow \end{matrix}}{\begin{bmatrix} 3 & 3 \\ 5 & 2 \end{bmatrix}}} = \underset{C}{\begin{bmatrix} \vec{a_1} \cdot \vec{b_1} & \vec{a_1} \cdot \vec{b_2} \\ \vec{a_2} \cdot \vec{b_1} & \vec{a_2} \cdot \vec{b_2} \end{bmatrix}}$$

$$Q^T = \begin{pmatrix} q_1 & q_2 & \dots & q_n \end{pmatrix}, \qquad P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$$

$$Q^T P = \begin{pmatrix} q_1 * p_1 + q_2 * p_2 + \dots + q_n * p_n \end{pmatrix}$$

Q^T is a $1 \times n$ matrix, P is a $n \times 1$ matrix, $Q^T P$ is a 1×1 matrix.

Exercise 3 - Solution

Similarly,

$$P^T = \begin{pmatrix} p_1 & p_2 & \dots & p_n \end{pmatrix}, \qquad Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

$$\mathrm{P}^T Q = \begin{pmatrix} q_1 * p_1 + q_2 * p_2 + \ldots + q_n * p_n \end{pmatrix}$$

P^T is a $1 \times n$ matrix, Q is a $n \times 1$ matrix, $P^T Q$ is a 1×1 matrix.

Exercise 3 - Solution

$$Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}, \qquad P^T = \begin{pmatrix} p_1 & p_2 & \dots & p_n \end{pmatrix}$$

$$\mathrm{QP}^T = \begin{pmatrix} q_1 * p_1 & q_1 * p_2 & \dots & q_1 * p_n \\ q_2 * p_1 & q_2 * p_2 & \dots & q_2 * p_n \\ \vdots & \vdots & \vdots & \vdots \\ q_n * p_1 & q_n * p_2 & \dots & q_n * p_n \end{pmatrix}$$

Q is a $n \times 1$ matrix, P^T is a $1 \times n$ matrix, QP^T is a $n \times n$ matrix

Exercise 3 - Solution

1. The total money spent is determined by the product $Q^T P$ or, equivalently, $P^T Q$.
2. The product QP^T is an $n \times n$ matrix, whose entries are of the form $p_i q_j$, with $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Only the entries with $i = j$ make economic sense.

Exercise 4

Consider the following system of three linear equations in three unknowns:

$$\begin{aligned} 5x_1 + 3x_2 + x_3 &= -9 \\ 3x_1 + 2x_2 + 2x_3 &= 0 \\ x_1 + 4x_2 + 6x_3 &= -36. \end{aligned}$$

1. The system's coefficient matrix is invertible (and you can take this for granted). Calculate its inverse.
2. Use the inverse matrix you obtained in the previous question to solve the system.

Exercise 4 - Solution

The coefficient matrix

$$M = \begin{pmatrix} 5 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 4 & 6 \end{pmatrix}$$

Method 1: The augmented matrix

$$\left(\begin{array}{ccc|ccc} 5 & 3 & 1 & 1 & 0 & 0 \\ 3 & 2 & 2 & 0 & 1 & 0 \\ 1 & 4 & 6 & 0 & 0 & 1 \end{array}\right)$$

Switch row1 and row3

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 6 & 0 & 0 & 1 \\ 3 & 2 & 2 & 0 & 1 & 0 \\ 5 & 3 & 1 & 1 & 0 & 0 \end{array}\right)$$

Exercise 4 - Solution

Method 1: The augmented matrix

$$\begin{pmatrix} 1 & 4 & 6 & | & 0 & 0 & 1 \\ 3 & 2 & 2 & | & 0 & 1 & 0 \\ 5 & 3 & 1 & | & 1 & 0 & 0 \end{pmatrix}$$

row2 = row2 - 3 * row1

row3 = row3 - 5 * row1

$$\begin{pmatrix} 1 & 4 & 6 & | & 0 & 0 & 1 \\ 0 & -10 & -16 & | & 0 & 1 & -3 \\ 0 & -17 & -29 & | & 1 & 0 & -5 \end{pmatrix}$$

row2 = -1/10 $*$ row2

$$\begin{pmatrix} 1 & 4 & 6 & | & 0 & 0 & 1 \\ 0 & 1 & \frac{8}{5} & | & 0 & \frac{-1}{10} & \frac{3}{10} \\ 0 & -17 & -29 & | & 1 & 0 & -5 \end{pmatrix}$$

Exercise 4 - Solution

Method 1: The augmented matrix

$$\begin{pmatrix} 1 & 4 & 6 & | & 0 & 0 & 1 \\ 0 & 1 & \frac{8}{5} & | & 0 & \frac{-1}{10} & \frac{3}{10} \\ 0 & -17 & -29 & | & 1 & 0 & -5 \end{pmatrix}$$

row1 = row1 - 4 * row2

row3 = row3 + 17 * row2

$$\begin{pmatrix} 1 & 0 & \frac{-2}{5} & | & 0 & \frac{2}{5} & \frac{-1}{5} \\ 0 & 1 & \frac{8}{5} & | & 0 & \frac{-1}{10} & \frac{3}{10} \\ 0 & 0 & \frac{-9}{5} & | & 1 & \frac{-17}{10} & \frac{1}{10} \end{pmatrix}$$

row3 = -5/9 $*$ row3

$$\begin{pmatrix} 1 & 0 & \frac{-2}{5} & | & 0 & \frac{2}{5} & \frac{-1}{5} \\ 0 & 1 & \frac{8}{5} & | & 0 & \frac{-1}{10} & \frac{3}{10} \\ 0 & 0 & 1 & | & \frac{-5}{9} & \frac{17}{18} & \frac{-1}{18} \end{pmatrix}$$

Exercise 4 - Solution

Method 1: The augmented matrix

$$\begin{pmatrix} 1 & 0 & \frac{-2}{5} & | & 0 & \frac{2}{5} & \frac{-1}{5} \\ 0 & 1 & \frac{8}{5} & | & 0 & \frac{-1}{10} & \frac{3}{10} \\ 0 & 0 & 1 & | & \frac{-5}{9} & \frac{17}{18} & \frac{-1}{18} \end{pmatrix}$$

row1 = row1 + 2/5 * row3
row2 = row2 - 8/5 * row3

$$\begin{pmatrix} 1 & 0 & 0 & | & \frac{-2}{9} & \frac{7}{9} & \frac{-2}{9} \\ 0 & 1 & 0 & | & \frac{-8}{9} & \frac{-29}{18} & \frac{7}{18} \\ 0 & 0 & 1 & | & \frac{-5}{9} & \frac{17}{18} & \frac{-1}{18} \end{pmatrix}$$

$$\frac{1}{9}\begin{pmatrix} 1 & 0 & 0 & | & -2 & 7 & -2 \\ 0 & 1 & 0 & | & 8 & \frac{-29}{2} & \frac{7}{2} \\ 0 & 0 & 1 & | & -5 & \frac{17}{2} & \frac{-1}{2} \end{pmatrix}$$

Exercise 4 - Solution

The inverse is

$$M^{-1} = \frac{1}{9}\begin{pmatrix} -2 & 7 & -2 \\ 8 & -\frac{29}{2} & \frac{7}{2} \\ -5 & \frac{17}{2} & -\frac{1}{2} \end{pmatrix}$$

Exercise 4 - Solution

Method 2: The matrix of cofactors

$$M^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{det(M)} \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}^{T} = \frac{1}{det(M)} \begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$$

A = (e*i-f*h)	B = -(d*i-f*g)	C = (d*h-e*g)
D = -(b*i-c*h)	E = (a*i-c*g)	F = -(a*h-b*g)
G = (b*f-c*e)	H = -(a*f-c*d)	I = (a*e-b*d)

Exercise 4 - Solution

$$M = \begin{pmatrix} 5 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 4 & 6 \end{pmatrix}$$

A = (2*6-2*4)=4	B = -(3*6-2*1)=-16	C = (3*4-2*1)=10
D = -(3*6-1*4)=-14	E = (5*6-1*1)=29	F = -(5*4-3*1)=-17
G = (3*2-1*2)=4	H = -(5*2-1*3)=-7	I = (5*2-3*3)=1

det(M) = 5*(2*6-2*4) - 3*(3*6-2*1) + 1*(3*4-2*1) = -18

$$M^{-1} = \frac{1}{-18} \begin{pmatrix} 4 & -16 & 10 \\ -14 & 29 & -17 \\ 4 & -7 & 1 \end{pmatrix}^T = \frac{1}{-18} \begin{pmatrix} 4 & -14 & 4 \\ -16 & 29 & -7 \\ 10 & -17 & 1 \end{pmatrix}$$

$$M^{-1} = \frac{1}{9} \begin{pmatrix} -2 & 7 & -2 \\ 8 & \frac{-29}{2} & \frac{7}{2} \\ -5 & \frac{17}{2} & \frac{-1}{2} \end{pmatrix}$$

Exercise 4 - Solution

$$\mathrm{M} = \begin{pmatrix} 5 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 4 & 6 \end{pmatrix}, \mathrm{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{aligned} 5x_1 + 3x_2 + x_3 &= -9 \\ 3x_1 + 2x_2 + 2x_3 &= 0 \\ x_1 + 4x_2 + 6x_3 &= -36. \end{aligned}$$

$$\mathrm{Mx} = \begin{pmatrix} -9 \\ 0 \\ -36 \end{pmatrix}$$

$$x = M^{-1}Mx = M^{-1} * \begin{pmatrix} -9 \\ 0 \\ -36 \end{pmatrix}$$

Exercise 4 - Solution

1. The inverse is

$$M^{-1} = \frac{1}{9}\begin{pmatrix} -2 & 7 & -2 \\ 8 & -\frac{29}{2} & \frac{7}{2} \\ -5 & \frac{17}{2} & -\frac{1}{2} \end{pmatrix}$$

2. The system's solution is

$$\frac{1}{9}\begin{pmatrix} -2 & 7 & -2 \\ 8 & -\frac{29}{2} & \frac{7}{2} \\ -5 & \frac{17}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} -9 \\ 0 \\ -36 \end{pmatrix} = \begin{pmatrix} 10 \\ -22 \\ 7 \end{pmatrix}$$

Exercise 5

Consider the following system of three linear equations in three unknowns:

$$x_1 + 2x_2 + 3x_3 = 1$$
$$3x_1 + 2x_2 + x_3 = 2$$
$$x_1 + 3x_2 + 2x_3 = 3.$$

1. Check that the system coefficient's matrix is nonsingular by calculating its determinant.
2. Use Cramer's rule to solve the system.

Exercise 5 - Solution

The coefficient matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$

$\det(A) = 1*(2*2-1*3) - 2*(3*2-1*1) + 3*(3*3-2*1) = 12$

Proposition: For a square matrix A, the following are equivalent:

- A is nonsingular
- A is invertible
- $\det(A) \neq 0$

Exercise 5 - Solution

Cramer's rule: For a nonsingular matrix A, the unique solution to $Ax = b$ is obtained as $x_i = \frac{det(B_i)}{det(A)}$
where B-i is the matrix A with the ith column replaced by b.

$$B_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 3 & 2 \end{pmatrix}$$

$$\det(B_1) = 1*(2*2-1*3) - 2*(2*2-1*3) + 3*(2*3-2*3) = -1$$

$$B_2 = \begin{pmatrix} 1 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\det(B_2) = 1*(2*2-1*3) - 1*(3*2-1*1) + 3*(3*3-2*1) = 17$$

$$B_3 = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}$$

$$\det(B_3) = 1*(2*3-2*3) - 2*(3*3-2*1) + 1*(3*3-2*1) = -7$$

Exercise 5 - Solution

1. $\det A = 12 \neq 0$
2.

$$
\begin{aligned}
x_1 &= \frac{\det B_1}{\det A} = \frac{-1}{12} = -\frac{1}{12} \\
x_2 &= \frac{\det B_2}{\det A} = \frac{17}{12} \\
x_3 &= \frac{\det B_3}{\det A} = \frac{-7}{12} = -\frac{7}{12}
\end{aligned}
$$