31C01100 Taloustieteen matemaattiset menetelmät - Mathematics for Economists
Aalto University – Fall 2021
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Solutions to Problem Set 0

Exercise 1

Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at $x = 1$ and $x = 0$ but continuous at every other point of its domain.

Answer. An example is the following:

$$f(x) = \begin{cases} 2 & \text{if } x = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 2

Calculate the first derivative of each of the following functions:

1. $f(x) = x^a$, with $a > 0$;
2. $f(x) = e^{ax}$, with $a > 0$;
3. $f(x) = (3x + 2)^3$;
4. $f(x) = \frac{3x}{x^2+1}$;
5. $f(x) = 4e^{-3x}$;
6. $f(x) = x \ln x$.

Answer.

1. $f'(x) = ax^{a-1}$
2. $f'(x) = ae^{ax}$

3. $f'(x) = 3(3x+2)^2 \times 3 = 9(3x+2)^2$

4. $f'(x) = \frac{3(x^2+1)-3x(2x)}{(x^2+1)^2} = \frac{3(1-x^2)}{(x^2+1)^2}$

5. $f'(x) = -12e^{-3x}$

6. $f'(x) = \ln x + 1$

Exercise 3

Consider the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x - x^3$. Find all the points at which the function attains:

1. a local maximum;

2. a local minimum;

3. a global maximum;

4. a global minimum.

Answer.

1. Local maximum: $f(x) = \frac{2}{3\sqrt{3}}$ at $x = \frac{1}{\sqrt{3}}$;

2. Local minimum: $f(x) = -\frac{2}{3\sqrt{3}}$ at $x = -\frac{1}{\sqrt{3}}$

3. No global maximum;

4. No global minimum.

Exercise 4

Let $f : I \to \mathbb{R}$ be a function defined over an interval $I \subseteq \mathbb{R}$. We say that f is **convex** if, for all $x, y \in I$, and all $a \in [0,1]$, we have

$$f(ax + (1-a)y) \leq af(x) + (1-a)f(y).$$

Furthermore, we say that f is **concave** if, for all $x, y \in I$, and all $a \in [0,1]$, we have

$$f(ax + (1-a)y) \geq af(x) + (1-a)f(y).$$

For each of the following functions, determine whether it is convex or concave.

1. $f(x) = 3x^2$
2. $f(x) = e^x$
3. $f(x) = 2 + x$
4. $f(x) = -e^x$
5. $f(x) = \log x$
6. $f(x) = x^3 - 3x$

Answer.

1. Convex
2. Convex
3. Both concave and convex
4. Concave. Recall that if $f(x)$ is convex, then $-f(x)$ is concave
5. Concave
6. If we consider the entire domain $\mathbb{R}$, the function is neither concave nor convex. However, we can find regions of the domain over which the function is either concave or convex. Specifically, $f(x)$ is concave over the interval $(-\infty, 0]$, and convex over $[0, +\infty)$