Problem Set 2

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Exercise 1 - PS2

For each of the following sets of vectors, determine whether they are linearly dependent or independent:

$$(a)\quad \begin{pmatrix}1\\3\\4\end{pmatrix}, \begin{pmatrix}-2\\-7\\3\end{pmatrix}, \begin{pmatrix}3\\-2\\5\end{pmatrix}, \begin{pmatrix}-2\\4\\2\end{pmatrix};$$

$$(b)\quad \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\5\\-1\end{pmatrix}, \begin{pmatrix}-3\\2\\-4\end{pmatrix};$$

$$(c)\quad \begin{pmatrix}1\\2\\1\\1\end{pmatrix}, \begin{pmatrix}2\\5\\4\\3\end{pmatrix}, \begin{pmatrix}-1\\2\\7\\3\end{pmatrix}.$$

Exercise 1 - Solution

Proposition

If $k > n$, any set of k vectors in $\mathbb{R}^n$ is linearly **dependent**.

(a) Linearly dependent

$$A = \begin{pmatrix} 1 & -2 & 3 & -2 \\ 3 & -7 & -2 & 4 \\ 4 & 3 & 5 & 2 \end{pmatrix}$$

Exercise 1 - Solution

Proposition

A set of n vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ in $\mathbb{R}^n$ is linearly **independent** *if and only if*

$$\det\begin{pmatrix}\boldsymbol{u}_1 & \ldots & \boldsymbol{u}_n\end{pmatrix} \neq 0$$

(b) Linearly independent

$$B = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & 2 \\ 3 & -1 & -4 \end{pmatrix}$$

det(B)=61

Exercise 1 - Solution

Proposition

Vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$ in $\mathbb{R}^n$ are linearly dependent if and only if the linear system

$$A \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \mathbf{0}$$

has a **nonzero** *solution $(a_1, \ldots, a_m)$, where A is the $n \times m$ matrix whose columns are the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$.*

(c) Linearly dependent (form the system $Cx = 0$, which turns out to have infinitely many nonzero solutions).

Transform $C = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & 2 \\ 1 & 4 & 7 \\ 1 & 3 & 3 \end{pmatrix}$ into $\begin{pmatrix} 2 & 5 & 2 \\ 0 & 1.5 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Cx=0

Exercise 1 - Solution

$$\begin{pmatrix} 2 & 5 & 2 \\ 0 & 1.5 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 * x_1 & 5 * x_2 & 2 * x_3 \\ 0 & 1.5 * x_2 & 6 * x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, there are infinitely many non-zero solutions.

Alternative explanation: because rank=2<3, there are infinitely many non-zero solutions.

Exercise 2

Consider the following subset of $\mathbb{R}^3$

$$U := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = x_3\right\}$$

and the following three vectors in $\mathbb{R}^3$

$$\boldsymbol{u}_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

(a) Show that $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are linearly independent.

(b) Show that $U \subseteq \mathcal{L}[\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3]$, i.e. every element of U is a linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$.

(c) Show that $\mathcal{L}[\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3] \not\subseteq U$, i.e. *not* every linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ is an element of U.

(d) In light of the previous questions, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are not a basis of U. Find a basis of U and determine the dimension of U.

Exercise 2 - Solution

(a) The determinant of $A = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{pmatrix}$ is different from zero.

$\det \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 30$

(b) Any element in U has the form

$$\begin{pmatrix} a \\ a \\ a \end{pmatrix}$$

for some $a \in \mathbb{R}$. We need to find c1, c2, c3 such that

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a \\ a \\ a \end{pmatrix}$$

Exercise 2 - Solution

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} a \\ a \\ a \end{pmatrix}$$

$$= \begin{pmatrix} 4/15 & -1/15 & 0 \\ -1/15 & 4/15 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} a \\ a \\ a \end{pmatrix}$$

After finding c1, c2, c3, we have

$$\begin{pmatrix} a \\ a \\ a \end{pmatrix} = \frac{a}{5}\boldsymbol{u}_1 + \frac{a}{5}\boldsymbol{u}_2 + \frac{a}{2}\boldsymbol{u}_3$$

Exercise 2 - Solution

(c) It is enough to observe that $\boldsymbol{u}_1 \in \mathcal{L}[\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3]$ but $\boldsymbol{u}_1 \notin U$. There are infinitely many correct answers for this question.

(d) Any vector of the form

$$\begin{pmatrix} a \\ a \\ a \end{pmatrix}$$

for some $a \in \mathbb{R}$ with $a \neq 0$ is a basis of U. Clearly, U has a dimension of 1

Exercise 3

Consider the following functions and determine if they are injective, surjective or bijective.

(a) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = e^x$

(b) $f : \mathbb{R} \to \mathbb{R}_{++}$ such that $f(x) = e^x$, where $\mathbb{R}_{++}$ is the set of strictly positive real numbers

(c) $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = x + y$

(d) $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = xy$

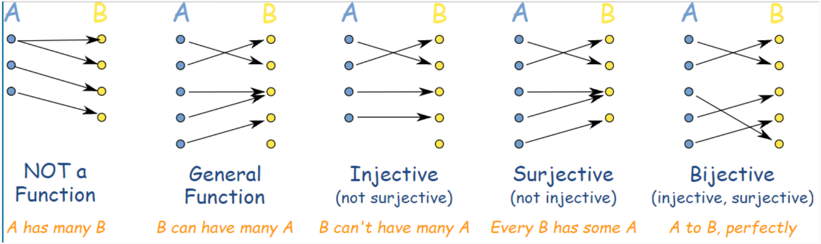
(e) $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = \min\{x, y\}$

(f) $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x_1, \ldots, x_n) = 17$

Exercise 3 - Solution

- A function $f : A \longrightarrow B$ is **one-to-one** or **injective** if, for every $x, y \in A$,
$$x \neq y \implies f(x) \neq f(y).$$
    - Example: $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that $f(x) = x^2$
- A function $f : A \longrightarrow B$ is **onto** or **surjective** if, for every $y \in B$, there exists an element $x \in A$ such that $f(x) = y$.
    - Example: $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ such that $f(x) = x^2$
- A function $f : A \longrightarrow B$ is **bijective** if it is both injective and surjective.
    - Example: $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $f(x) = x^2$

Exercise 3 - Solution



Exercise 3 - Solution

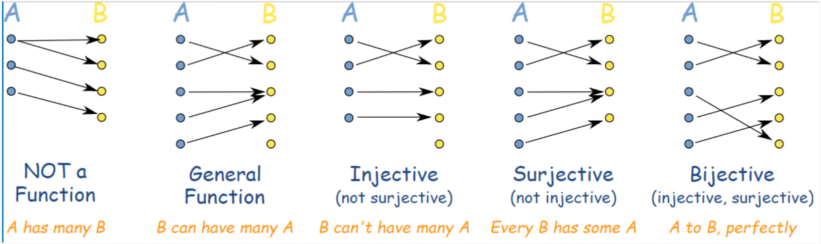
(a) $f(x) = e^x$ is injective but not surjective on R $\rightarrow \mathbb{R}$

Why injective? Because if a$\neq b$,e^a $\neq$e^b (Further explanation: function $f(x) = e^x$ is increasing in x)

Why not surjective? Because what if $f(x) = e^x$ $<$0, then we cannot find any x.

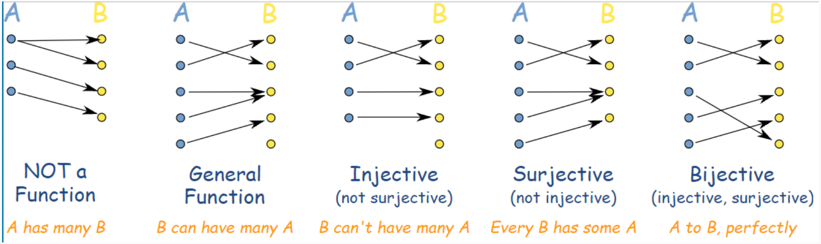
(b) $f(x) = e^x$ is bijective on R $\rightarrow \mathbb{R}$

Now that we have $f(x) > 0$, we can always find an x for every f(x). In particular, x=ln(f(x)).



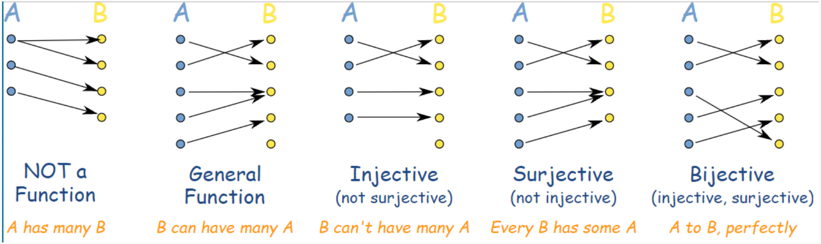
Exercise 3 - Solution

(c) $f(x, y) = x + y$ is surjective but not injective on $R^2 \to \mathbb{R}$
Why not injective? For example f(2,3)=f(4,1)=5 although (2,3) $\neq (4, 1)$.
Why surjective? Because for example, we have f(x, y) = 5. We can find infinitely many pairs of (x,y) that satisfies the condition such as (2,3), (2.5,2.5), etc
When we compare the definitions of injectivity and surjectivity to this question, x (in the definitions) is (x,y). y (in the definitions) is f(x, y).



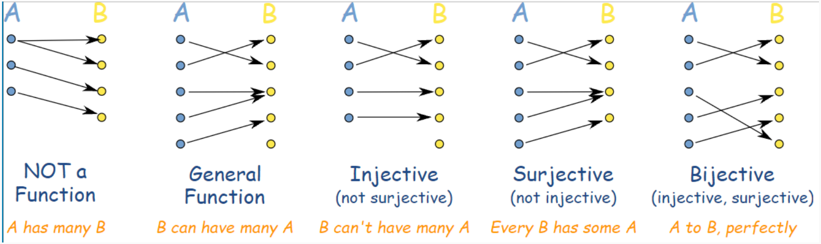
Exercise 3 - Solution

(d) $f(x, y) = xy$ is surjective but not injective on $R^2 \to \mathbb{R}$
Why not injective? For example f(2,3)=f(6,1)=6 although $(2{,}3) \neq (6, 1)$.
Why surjective? Because for example, we have f(x, y) = 6. We can find infinitely many pairs of (x,y) that satisfies the condition such as (2,3), (-2,-3), etc



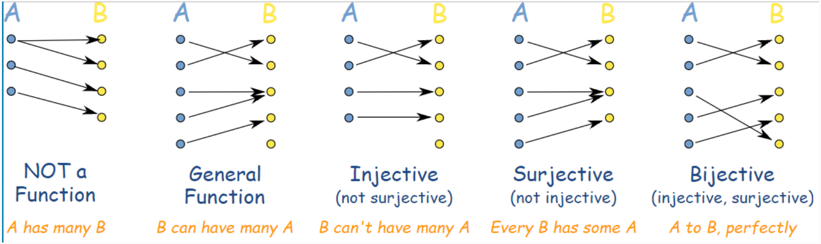
Exercise 3 - Solution

(e) $f(x, y) = \min\{x, y\}$ is surjective but not injective on $R^2 \to \mathbb{R}$
Why not injective? For example f(2,3)=f(2,4)=2 although (2,3) $\neq (2, 4)$.
Why surjective? Because for example, we have f(x, y) = 2. We can find infinitely many pairs of (x,y) that satisfies the condition such as (2,3), (2,4), etc



Exercise 3 - Solution

(f) $f(x_1, \ldots, x_n) = 17$ is neither injective nor surjective on $R^n \to \mathbb{R}$
Basically, no matter what point $(x_1, \ldots, x_n)$ we have,the function f gives us the result of 17.
Why not injective? For example f$(1, 1 \ldots, 1)$=f$(2, 2 \ldots, 2)$=17 although $(1, 1 \ldots, 1) \neq$(2, 2 . . . , 2).
Why not surjective? For example, when we are given a value of f(x) equal 18, we cannot find any point x. Remember, the range of f(x) in this question is R.



Exercise 4

(a) Consider the example at p. 17 in the slides from Lecture 5. Use the same type of argument as in the example to show that

$$\lim_{n \longrightarrow \infty} \frac{n+2}{5n} = \frac{1}{5}.$$

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 4 \\ 1 & \text{if } x = 4. \end{cases}$$

Show that f is *not* continuous at $x = 4$. [*Hint*: Have a look at the example at p. 22 in the slides from Lecture 5.]

Exercise 4 - Solution

- **Example.** $\lim_{n\longrightarrow\infty} \frac{1}{n^2} = 0$
- How to check that 0 is actually the limit of this sequence?

  1. Fix a small number $\epsilon > 0$
  2. Choose any positive integer N such that $N > \frac{1}{\sqrt{\epsilon}}$
  3. For any $n \geq N$, we have

$$|x_n - L| = |\frac{1}{n^2} - 0| \leq |\frac{1}{N^2} - 0| < |\frac{1}{(1/\sqrt{\epsilon})^2} - 0| = \epsilon.$$

(a) $\lim_{n\longrightarrow\infty} \frac{n+2}{5n} = \frac{1}{5}$
For any given ϵ >0, it suffices to choose $N > \frac{2}{5\epsilon}$

$|x_n - L| = |\frac{n+2}{5n} - \frac{1}{5}| = |\frac{2}{5n}| \leq |\frac{2}{5 * \frac{2}{5\epsilon}}| = \epsilon$

Exercise 4 - Solution

- An example of a discontinuous function is $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

- To see why this function is discontinuous at $x = 0$, take the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ in $\mathbb{R}$. This sequence converges to zero, but the sequence $\{f(\frac{1}{n})\}_{n=1}^{\infty}$ converges to 1

(b) To prove that the function is discontinuous at x=4, it is sufficient to point out a sequence of x_n that converges to 4 but the sequence of $f(x_n)$ does not converge to f(4). Take the sequence $\{4 + \frac{1}{n}\}$. This sequence converges to 4 as n goes to infinity. Now, the sequence $\{(4 + \frac{1}{n})^2\} = \{16 + \frac{1}{n^2} + \frac{8}{n}\}$ converges to 16. But then we have $16 \neq f(4) = 1$. This shows the discontinuity at $x = 4$.

Exercise 5

Calculate all the partial derivatives of the following functions:

(a) $f(x, y) = ax^b y^c$

(b) $f(x, y) = a \ln(1 - x) + b \ln(y)$

(c) $f(x, y) = \dfrac{ay^d}{bx^c}$

(d) $f(x, y, z) = e^{ax - by} - z$

(e) $f(x, y, z) = \sqrt{x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5z^2}.$

Exercise 5 - Solution

(a)

$$\frac{\partial f}{\partial x} = abx^{b-1}y^c$$
$$\frac{\partial f}{\partial y} = acx^b y^{c-1}$$

(b)

$$\frac{\partial f}{\partial x} = -\frac{a}{1-x}$$
$$\frac{\partial f}{\partial y} = \frac{b}{y}$$

(c)

$$\frac{\partial f}{\partial x} = \frac{-ay^d bcx^{c-1}}{b^2x^{2c}} = \frac{-ay^d c}{bx^{c+1}}$$
$$\frac{\partial f}{\partial y} = \frac{ady^{d-1}}{bx^c}$$

Exercise 5 - Solution

(d)

$$\frac{\partial f}{\partial x} = ae^{ax-by}$$
$$\frac{\partial f}{\partial y} = -be^{ax-by}$$
$$\frac{\partial f}{\partial z} = -1$$

(e)

$$\frac{\partial f}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5z^2}}\frac{1}{2}x^{-\frac{1}{2}}$$
$$\frac{\partial f}{\partial y} = \frac{1}{2}\frac{1}{\sqrt{x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5z^2}}\frac{1}{3}y^{-\frac{2}{3}}$$
$$\frac{\partial f}{\partial z} = \frac{1}{2}\frac{1}{\sqrt{x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5z^2}}10z$$