Problem Set 3

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Exercise 1 - PS3

Consider the following two subsets of $\mathbb{R}^2$:

$$S_1 := \left\{(x, y) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } 0 < y < 1\right\},$$
$$S_2 := \left\{(x, y) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } 0 \leq y \leq 1\right\}.$$

(a) Consider the point $\left(\frac{1}{2}, \frac{1}{2}\right)$. Find an open ϵ-ball around $\left(\frac{1}{2}, \frac{1}{2}\right)$ that is entirely contained in S_1. [*Hint*: Sketch a diagram of S_1.]

(b) Let (x_0, y_0) be any point in S_1. Argue that you can always find an open ϵ-ball around (x_0, y_0) that is entirely contained in S_1.

(c) Is S_2 an open set? Explain why or why not.

(d) Is S_2 a closed set? Explain why or why not.

Exercise 1 - Solution

(a) One can choose the open ball having radius $\epsilon = \frac{1}{4}$

(b) Given (x_0, y_0), define

$$a := \min \{x_0, y_0, 1 - x_0, 1 - y_0\}.$$

By the definition of S_1, we have that $a > 0$. The open ball of radius $\epsilon = \frac{a}{2}$ around (x_0, y_0) is entirely contained in S_1.

(c) No. For instance, any open ball around $\left(\frac{1}{2}, 0\right)$ is not contained in S_2.

(d) No. For instance, take the sequence $\left\{\left(\frac{1}{2n}, 0\right)\right\}_{n=1}^{\infty}$. This sequence is contained in S_2. However, it converges to $(0, 0)$, which is not an element of S_2.

Exercise 2

(a) Let $z = \ln(x^2 + y^2)$, $x(t) = e^{-t}$, and $y(t) = e^t$. Calculate the derivative of z with respect to t.

(b) Calculate the Hessian matrix of the following function defined over $\mathbb{R}^3$:

$$f(x, y, z) = -2x^2 - z^2 - 2xz - 2yz + x + 7y.$$

Exercise 2 - Solution

(a) By the chain rule, the derivative is

$$\frac{dz}{dt} = 2\frac{e^{2t} - e^{-2t}}{e^{2t} + e^{-2t}}$$

(b) The Hessian matrix is

$$D^2 f(x, y, z) = \begin{pmatrix} -4 & 0 & -2 \\ 0 & 0 & -2 \\ -2 & -2 & -2 \end{pmatrix}$$

Exercise 3

Consider the following function defined over $\mathbb{R}^3$:

$$F(x_1, x_2, y) = x_1^2 - x_2^2 + y^3,$$

and consider the point $(x_1, x_2, y) = (6, 3, -3)$, where $F(x_1, x_2, y) = 0$.

(a) Suppose we want to express y as a function of x_1 and x_2 around the point $(6, 3, -3)$. Explain why we are able to apply the implicit function theorem in this case.

(b) Calculate the partial derivatives $\frac{\partial y}{\partial x_1}(6, 3)$ and $\frac{\partial y}{\partial x_2}(6, 3)$.

(c) Suppose that, given the point $(6, 3, -3)$, x_1 increases to 6.2 and x_2 decreases to 2.9. Use the total differential to estimate the corresponding change in y. *Note*: you're asked to estimate the change in y, not in F.

Exercise 3 - Solution

(a) We can apply the implicit function theorem because

- F is a continuously differentiable function,
- $F(6, 3, -3) = 0$, and
- $\frac{\partial F}{\partial y}(6, 3, -3) = 27 \neq 0$.

(b) By the implicit function theorem:

$$\frac{\partial y}{\partial x_1}(6, 3) = -\frac{2x_1}{3y^2} = -\frac{4}{9}, \quad \frac{\partial y}{\partial x_2}(6, 3) = \frac{2x_2}{3y^2} = \frac{2}{9}.$$

(c)

$$dy = \frac{\partial y}{\partial x_1}(6, 3)dx_1 + \frac{\partial y}{\partial x_2}(6, 3)dx_2 = -\frac{4}{9} \times 0.2 - \frac{2}{9} \times 0.1 = -\frac{1}{9}.$$

Exercise 4

Let $u : \mathbb{R}^2_+ \to \mathbb{R}$ be a C^1 utility function, and let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $f'(x) > 0$ for every $x \in \mathbb{R}$ (i.e. f is a strictly increasing function). Define the composite function $v := f \circ u$. Recall that the Marginal Rate of Substitution of u at a point (x_0, y_0) is

$$MRS^u(x_0, y_0) = -\frac{\frac{\partial u}{\partial x}(x_0, y_0)}{\frac{\partial u}{\partial y}(x_0, y_0)}.$$

(a) Write the expression of the MRS at (x_0, y_0) for the composite function v.

(b) Use the chain rule to show that the MRS of u and v at (x_0, y_0) is the same.

(c) Now assume that u is also homogeneous of degree k. Show that the MRS of u is a homogeneous function of degree zero.

Exercise 4 - Solution

(a)

$$MRS^v(x_0, y_0) = -\frac{\frac{\partial v}{\partial x}(x_0, y_0)}{\frac{\partial v}{\partial y}(x_0, y_0)}.$$

(b)

$$MRS^v(x_0, y_0) = -\frac{\frac{\partial v}{\partial x}(x_0, y_0)}{\frac{\partial v}{\partial y}(x_0, y_0)} = -\frac{\frac{\partial f}{\partial x}(u(x_0, y_0))}{\frac{\partial f}{\partial y}(u(x_0, y_0))} = -\frac{f'(u(x_0, y_0)) * \frac{\partial u}{\partial x}(x_0, y_0)}{f'(u(x_0, y_0)) * \frac{\partial u}{\partial y}(x_0, y_0)}$$

$= \mathrm{MRS}^u(x_0, y_0)$.

(c) For every $t > 0$

$$MRS^u(tx_0, ty_0) = -\frac{\frac{\partial u}{\partial x}(tx_0, ty_0)}{\frac{\partial u}{\partial y}(tx_0, ty_0)} = -\frac{t^{k-1}\frac{\partial u}{\partial x}(x_0, y_0)}{t^{k-1}\frac{\partial u}{\partial y}(x_0, y_0)} = MRS^u(x_0, y_0),$$

If $f : \mathbb{R}^n_+ \to \mathbb{R}$ is a C^1 function homogeneous of degree k, its first order partial derivatives are homogeneous of degree k-1.

Exercise 5

For each of the following production functions, determine whether the corresponding returns to scale are *decreasing*, *increasing*, or *constant*. Throughout the exercise, assume that the parameters a, b, and c are all strictly positive.

(a) $f(x_1, x_2) = ax_1 + bx_2$.

(b) $f(x_1, x_2) = ax_1^c + bx_2^c$.

(c) $f(x_1, x_2) = \min\{ax_1, bx_2\}$.

(d) $f(x_1, x_2) = \max\{ax_1, bx_2\}$.

(e) $f(x_1, x_2) = x_1^a x_2^b$.

(f) $f(x_1, x_2) = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2}}$.

Exercise 5 - Solution

(a) $f(x_1, x_2) = ax_1 + bx_2$

$$\begin{aligned} f(tx_1, tx_2) &= atx_1 + btx_2 \\ &= t(ax_1 + bx_2) \\ &= t\,f(x_1, x_2). \end{aligned}$$

Returns to scale are constant.

(b) $f(x_1, x_2) = ax_1^c + bx_2^c$

$$\begin{aligned} f(tx_1, tx_2) &= at^c x_1^c + bt^c x_2^c \\ &= t^c(ax_1^c + bx_2^c) \\ &= t^c f(x_1, x_2). \end{aligned}$$

Returns to scale are constant if $c = 1$, increasing if $c > 1$, and decreasing if $c < 1$.

Exercise 5 - Solution

(c) $f(x_1, x_2) = \min\{ax_1, bx_2\}$

$$\begin{aligned} f(tx_1, tx_2) &= \min\{atx_1, btx_2\} \\ &= t\ \min\{ax_1, bx_2\} \\ &= t\, f(x_1, x_2). \end{aligned}$$

Returns to scale are constant.

(d) $f(x_1, x_2) = \max\{ax_1, bx_2\}$

$$\begin{aligned} f(tx_1, tx_2) &= \max\{atx_1, btx_2\} \\ &= t\ \max\{ax_1, bx_2\} \\ &= t\, f(x_1, x_2). \end{aligned}$$

Returns to scale are constant.

Exercise 5 - Solution

(e) $f(x_1, x_2) = x_1^a x_2^b$

$$f(tx_1, tx_2) = t^a x_1^a t^b x_2^b = t^{a+b} x_1^a x_2^b = t^{a+b} f(x_1, x_2).$$

Returns to scale are constant if $(a+b) = 1$, increasing if $(a+b) > 1$, and decreasing if $(a+b) < 1$.

(f) $f(x_1, x_2) = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2}}$

$$\begin{aligned} f(tx_1, tx_2) &= \frac{1}{\frac{1}{tx_1} + \frac{1}{tx_2}} \\ &= \frac{1}{\frac{1}{t}\left(\frac{1}{x_1} + \frac{1}{x_2}\right)} \\ &= t\,\frac{1}{\frac{1}{x_1} + \frac{1}{x_2}} \\ &= t\,f(x_1, x_2). \end{aligned}$$

Returns to scale are constant.