ELEC-E4130

Lecture 11: Transmission lines

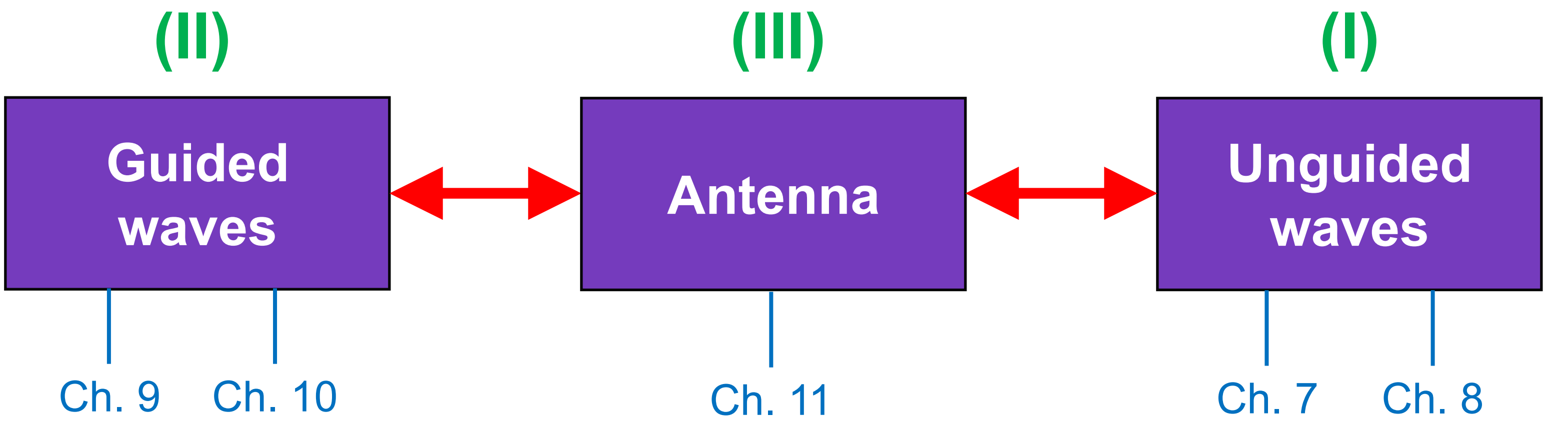
Ch. 9.2 – 9.3

Aalto University
School of Electrical
Engineering

ELEC-E4130 / Taylor

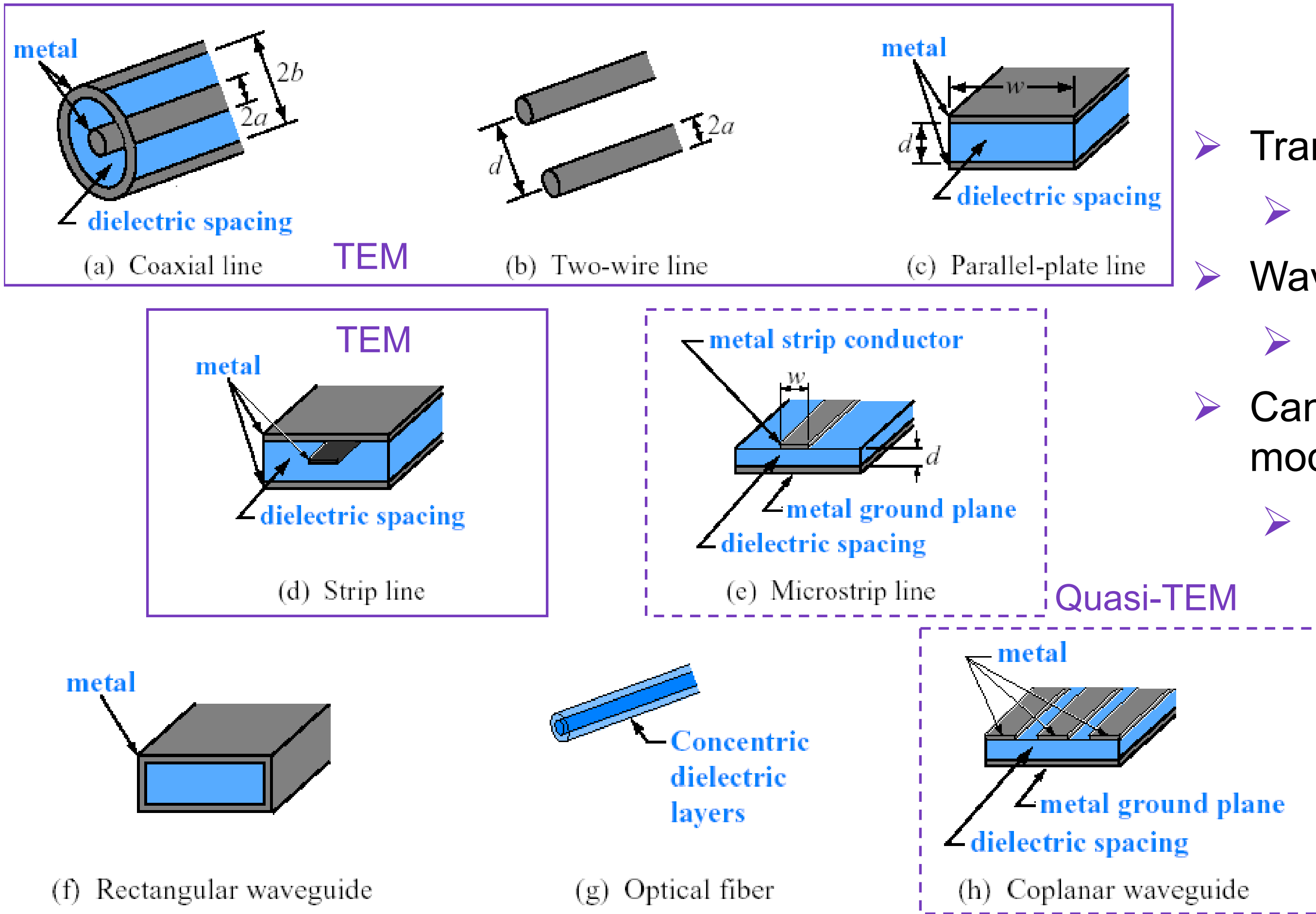
Oct. 18, 2021

Transmission lines and Waveguides



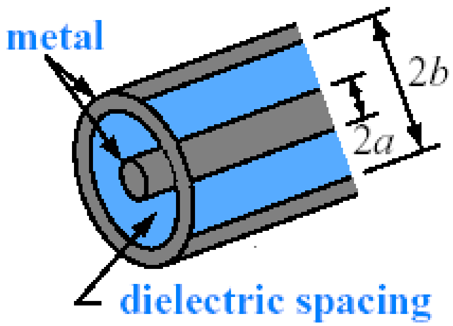
- Consider any RF communication system or bistatic RF sensing system
- The transmitter couples a signal to a transmission line and must get this energy to free space
  - Guided wave → Antenna → Unguided wave
- The receiver captures some energy from free space a must send it down a transmission line to some circuit
  - Unguided wave → Antenna → Guided wave

Transmission lines and Waveguides

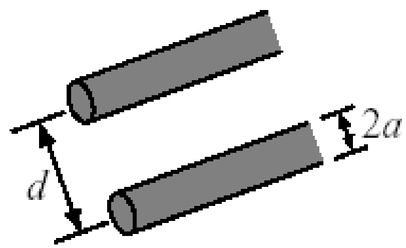


- Transmission lines
  - TEM or Quasi-TEM modes
- Waveguide
  - TE or TM modes
- Can a structure support at TEM mode?
  - Two conductors – voltages need to be defined

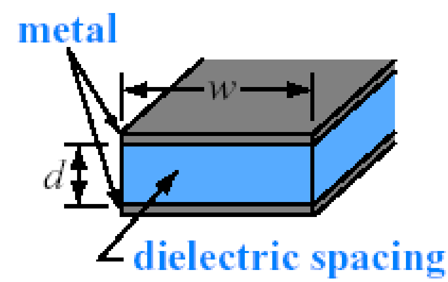
Chapter 9: TEM transmission lines



(a) Coaxial line



(b) Two-wire line



(c) Parallel-plate line

- Consider geometry and operational frequency where propagation occurs in a TEM mode
  - Constrain frequency upper limit to avoid TE or TM modes
- Use fields and boundary conditions to obtain voltages and currents
- Use voltages and currents to define lumped element equivalents
- Use lumped element equivalents to operate on phasors

Oct. 18
- 9.2
- 9.3

Oct. 21
- 9.4
- Recap

Nov. 1
- Recap
- + Examples

Approach for this lecture

- Demonstrate theory with parallel plate transmission lines

- Assume lossless, ideal
  - E.g. fringing fields are ignored

- Perturb setup to make line lossy while keeping situation ideal
  - E.g. current in a finite conductor is confined to an infinitely thin surface sheet.

This Lecture

1. Compute inductance L using magnetostatics or capacitance C using electrostatics assuming a good conductor
2. Once one reactance is known, compute the other using the following good conductor relation

$$LC = \mu\epsilon$$

Lossless

3. Use the capacitance from steps 1 or 2 to compute the conductance via the following relation

$$\frac{G}{C} = \frac{\sigma}{\epsilon}$$

4. Compute the surface ohmic power dissipation per unit area via the surface current density (A/m) and the surface resistance (Ω/m) for both conductors

$$p_\sigma = \frac{1}{2}|J_{su}|^2 R_s$$

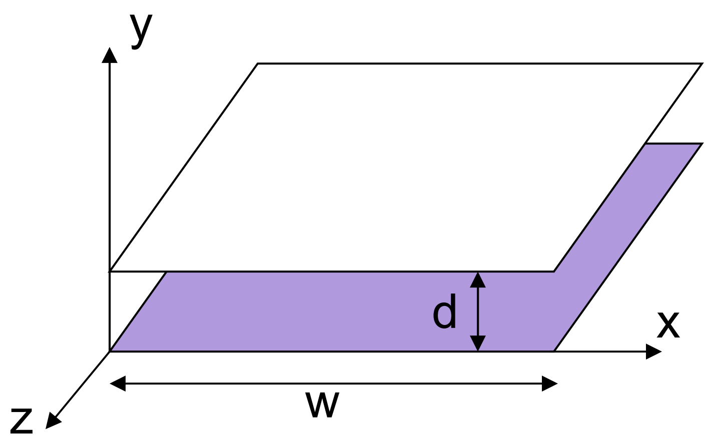
Pages 445 and 446 in Cheng

5. Convert this into V=IR circuit theory by multiplying the length width of the contour cross section and extract the resistance

$$P_\sigma = p_\sigma w = \frac{1}{2} I^2 R$$

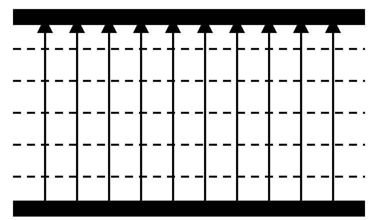
Lossy

Parallel Plate Transmission line

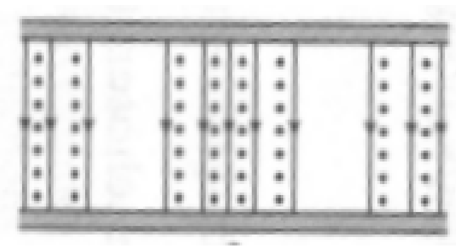


TEM wave

X-Y plane



Y-Z plane



$$\mathbf{E} = \mathbf{a_y} E_y(z) = \mathbf{a_y} E_0 e^{-\gamma z}$$

$$\mathbf{H} = \mathbf{a_x} H_x(z) = \mathbf{a_x} \frac{E_0}{\eta} e^{-\gamma z}$$

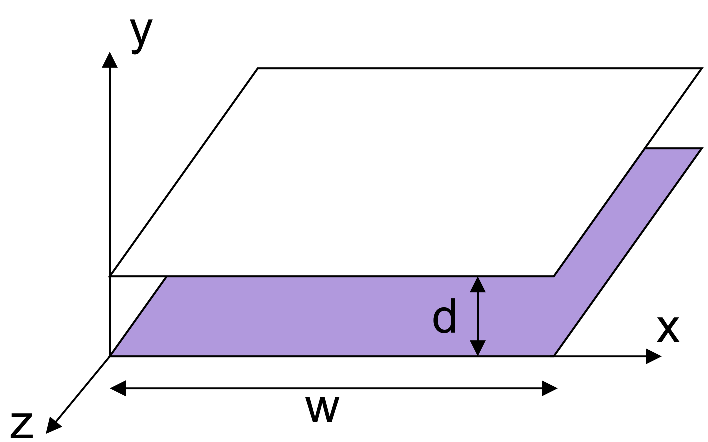
TEM Wave

$$\gamma = j\beta = j\omega\sqrt{\mu\epsilon}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

Assumes
- PEC plates
- Lossless dielectric fill

Parallel Plate Transmission line



Boundary conditions

$$\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

$$\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

Dielectric/PEC interface

$$\mathbf{a}_n \times \mathbf{H} = \mathbf{J}_s \qquad H_n = 0$$

$$\mathbf{a}_n \cdot \mathbf{D} = \rho_s \qquad E_t = 0$$

Lower plate (y = 0)

$$\mathbf{a}_n = \mathbf{a}_y$$

$$\mathbf{a}_y \times \mathbf{H} = \mathbf{J}_{sl} = -\mathbf{a}_z H_x(z) = \mathbf{a}_z \frac{E_0}{\eta} e^{-j\beta z}$$

$$\mathbf{a}_y \cdot \mathbf{D} = \rho_{sl} = \epsilon E_y(z) = \epsilon E_0 e^{-j\beta z}$$

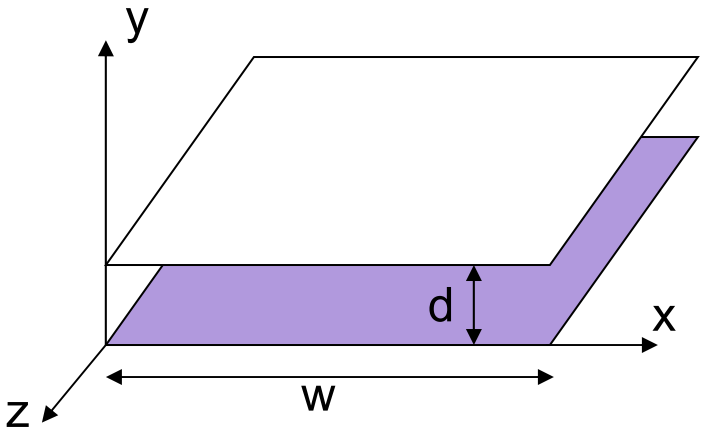
Upper plate (y = d)

$$\mathbf{a}_n = -\mathbf{a}_y$$

$$-\mathbf{a}_y \times \mathbf{H} = \mathbf{J}_{su} = \mathbf{a}_z J_{su}(z) = \mathbf{a}_z H_x(z) = -\mathbf{a}_z \frac{E_0}{\eta} e^{-j\beta z}$$

$$-\mathbf{a}_y \cdot \mathbf{D} = \rho_{su} = -\epsilon E_y(z) = -\epsilon E_0 e^{-j\beta z}$$

Parallel Plate Transmission line



Time Harmonic M.E.

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$$

Coupled fields (scalar)

$$\frac{d}{dz}E_y(z) = j\omega\mu H_x(z)$$

$$\frac{d}{dz}H_x(z) = j\omega\epsilon E_y(z)$$

$$\frac{d}{dz}E_y(z) = j\omega\mu H_x(z)$$

$$\frac{d}{dz}\int_0^d E_y(z)dy = j\omega\mu\int_0^d H_x(z)dy$$

$$-\frac{d}{dz}V(z) = j\omega\mu d H_x(z)$$

$$-\frac{d}{dz}V(z) = j\omega\left(\mu\frac{d}{w}\right)H_x(z)w$$

boundary conditions

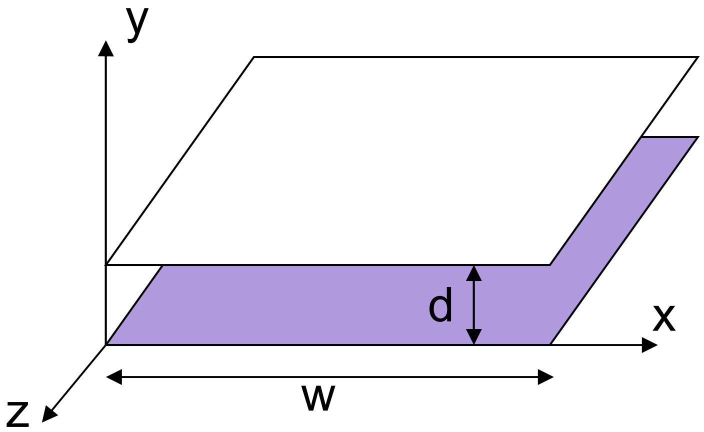
$$-\frac{d}{dz}V(z) = j\omega\left(\mu\frac{d}{w}\right)J_{su}(z)w$$

surface current density × width

$$-\frac{d}{dz}V(z) = j\omega L I_{su}(z)$$

$$L = \mu\frac{d}{w}$$

Parallel Plate Transmission line



Time Harmonic M.E.

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

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Coupled fields (scalar)

$$\frac{d}{dz}E_y(z) = j\omega\mu H_x(z)$$

$$\frac{d}{dz}H_x(z) = j\omega\epsilon E_y(z)$$

$$\frac{d}{dz}H_x(z) = j\omega\epsilon E_y(z)$$

$$\frac{d}{dz}\int_0^w H_x(z)dx = j\omega\epsilon\int_0^w E_y(z)dy$$

$$-\frac{d}{dz}H_x(z)w = -j\omega\epsilon w E_y(z)$$

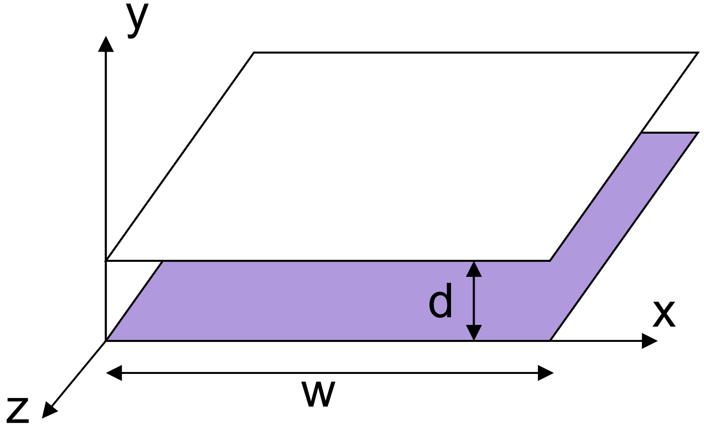
$$-\frac{d}{dz}J_{su}(z)w = j\omega\left(\epsilon\frac{w}{d}\right)\left(-E_y(z)d\right)$$

$$-\frac{d}{dz}I_{su}(z) = j\omega\left(\epsilon\frac{w}{d}\right)V(z)$$

$$-\frac{d}{dz}I_{su}(z) = j\omega CV(z)$$

$$C = \epsilon\frac{w}{d}$$

Parallel Plate Transmission line



Coupled

$$-\frac{d}{dz}I(z) = j\omega CV(z)$$

$$-\frac{d}{dz}V(z) = j\omega LI(z)$$

Uncoupled

$$\frac{d^2}{dz^2}I(z) = -\omega^2 LCI(z)$$

$$\frac{d^2}{dz^2}V(z) = -\omega^2 LCV(z)$$

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

Forward traveling

$$V(z) = V_0^+ e^{-j\beta z}$$

$$V(z) = V_0 e^{-j\beta z}$$

$$I(z) = I_0(V_0) e^{-j\beta z}$$

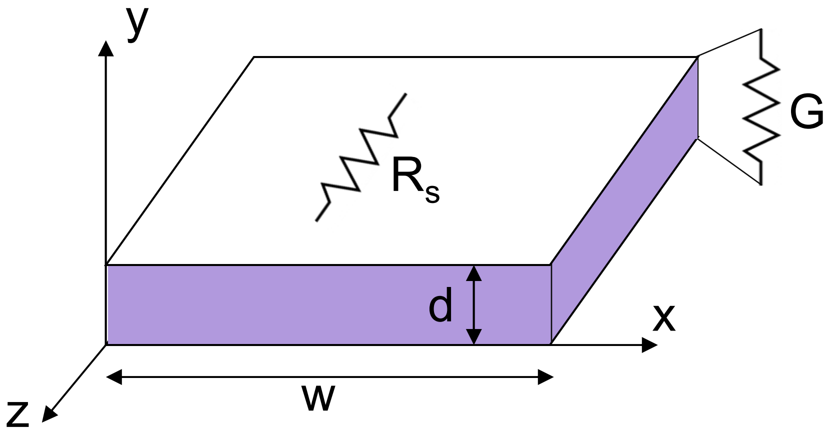
$$\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\epsilon}$$

$$Z_0 = \frac{V(z)}{I(z)} = \sqrt{\frac{L}{C}} = \frac{d}{w}\sqrt{\frac{\mu}{\varepsilon}} = \frac{d}{w}\eta$$

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\epsilon}}$$

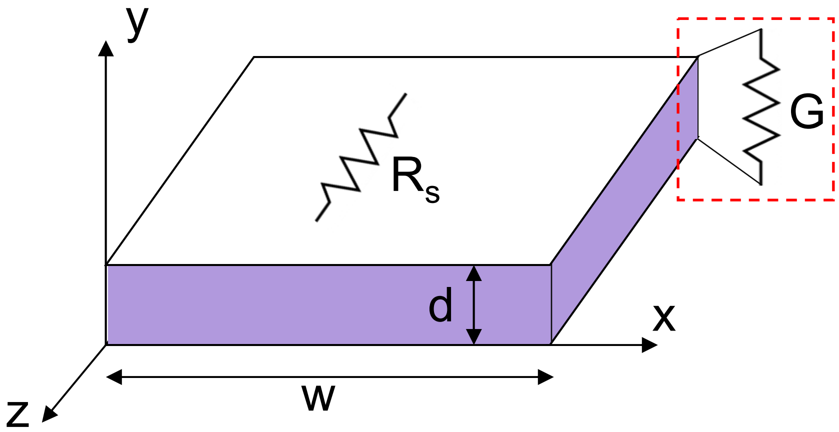
- TEM wave with propagation constant and phase velocity equal to free space planewave
- Wave impedance is a geometry modification of free space wave impedance

LOSSY Parallel Plate Transmission line



- What happens when the parallel plate transmission line plates have finite conductivity
  - $1/\sigma_s > 0$ ⟶ Conductor loss
- What happens when medium bordered by the parallel plate transmission line has a non-0 conductivity?
  - $\sigma_d > 0$ ⟶ "Fill" loss

LOSSY Parallel Plate Transmission line



Lossy medium

ϵ_d = dielectric of medium

σ_d = cond. of medium

$$RC = \frac{C}{G} = \frac{\epsilon_d}{\sigma_d}$$ → Shape independent

$$G = \frac{\sigma_d}{\epsilon_d} C = \sigma_d \frac{w}{d}$$ → Parallel plate

Electrostatics

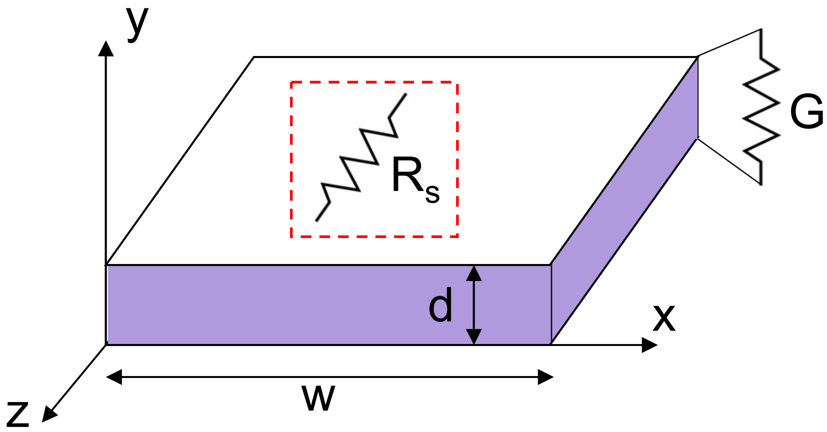
$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E} \cdot \mathbf{d}\ell}{\oint_S \mathbf{J} \cdot \mathbf{ds}} = \frac{-\int_L \mathbf{E} \cdot \mathbf{d}\ell}{\oint_S \sigma_d \mathbf{E} \cdot \mathbf{ds}}$$

Electrostatics

$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D} \cdot \mathbf{ds}}{-\int_L \mathbf{E} \cdot \mathbf{d}\ell} = \frac{\oint_S \epsilon_d \mathbf{E} \cdot \mathbf{ds}}{-\int_L \mathbf{E} \cdot \mathbf{d}\ell}$$

- Surface **S** encapsulates the higher potential terminal (positive)
- Line integral **L** is from the lower potential terminal to the higher potential terminal (negative to positive)

LOSSY Parallel Plate Transmission line



Lossy conductor

σ_s = finite conductivity of metallic plates

- Something is dissipated in the conductor\
- Non-zero tangential field in the axial direction
  - $E_z \neq 0$

Average power dissipated in top plate (y-direction)

$$\wp_{AVE} = \mathbf{a_y} p_{\sigma_s}$$

$$\wp_{AVE} = \frac{1}{2} \Re\{\mathbf{a_z} \mathrm{E_z} \times \mathbf{a_x} \mathrm{H_x^*}\}$$

Surface impedance

$$Z_s = \frac{E_t}{J_s} = \frac{E_z}{J_{su}} = \frac{E_z}{H_x} = R_s + jX_s = (1+j)\sqrt{\frac{\pi f \mu_c}{\sigma_c}}$$

Tangential E-field

Resistance and reactance

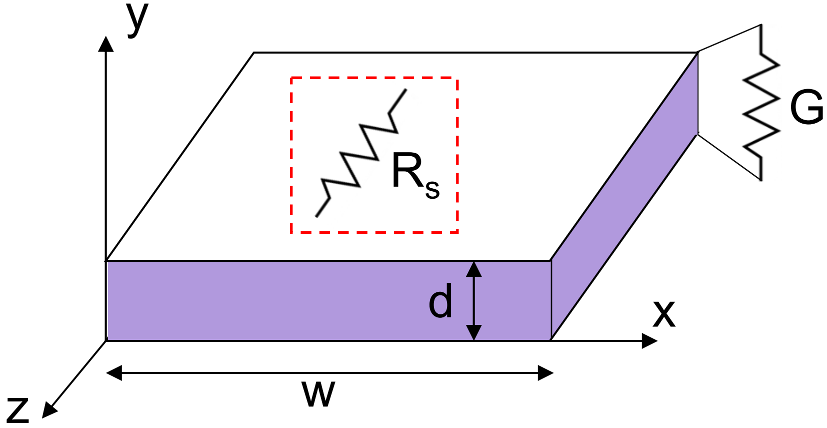
Surface current density

Surface current density UPPER PLATE

Good conductor

E-field must point in direction of current density (z-direction) for energy dissipation

LOSSY Parallel Plate Transmission line



Lossy conductor

σ_s = finite conductivity of metallic plates

- Something is dissipated in the conductor\
- Non-zero tangential field in the axial direction
  - $E_z \neq 0$

Average power dissipated differential section

$$p_{\sigma_s} = \frac{1}{2}\Re\{|J_{su}|^2 Z_s\}$$

$$p_{\sigma_s} = \frac{1}{2}|J_{su}|^2 R_s = \frac{1}{2}\left(\frac{I}{w}\right)^2 R_s$$

Average power dissipated in bottom plate per untit length

$$P_{\sigma_s} = p_{\sigma_s} w = \frac{1}{2} I^2 \frac{R_s}{w}$$

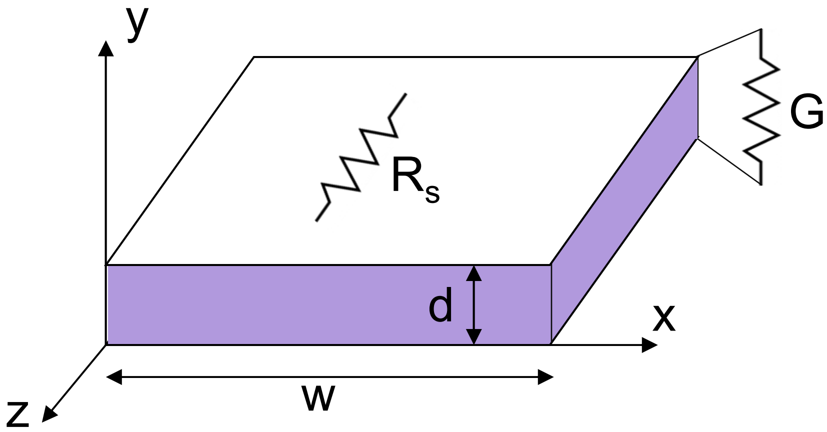
Resistance of the top plate

$$R_{\sigma_s} = \frac{R_s}{w}$$

Total resistance of PP
(contribution from both plates)

$$R = R_{\sigma_s} + R_{\sigma_s} = \frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}}$$

Parallel Plate Transmission line Summary



$$C = \epsilon_d \frac{w}{d} \qquad G = \sigma_d \frac{w}{d}$$

$$L = \mu \frac{d}{w} \qquad R = \frac{2R_s}{w}$$

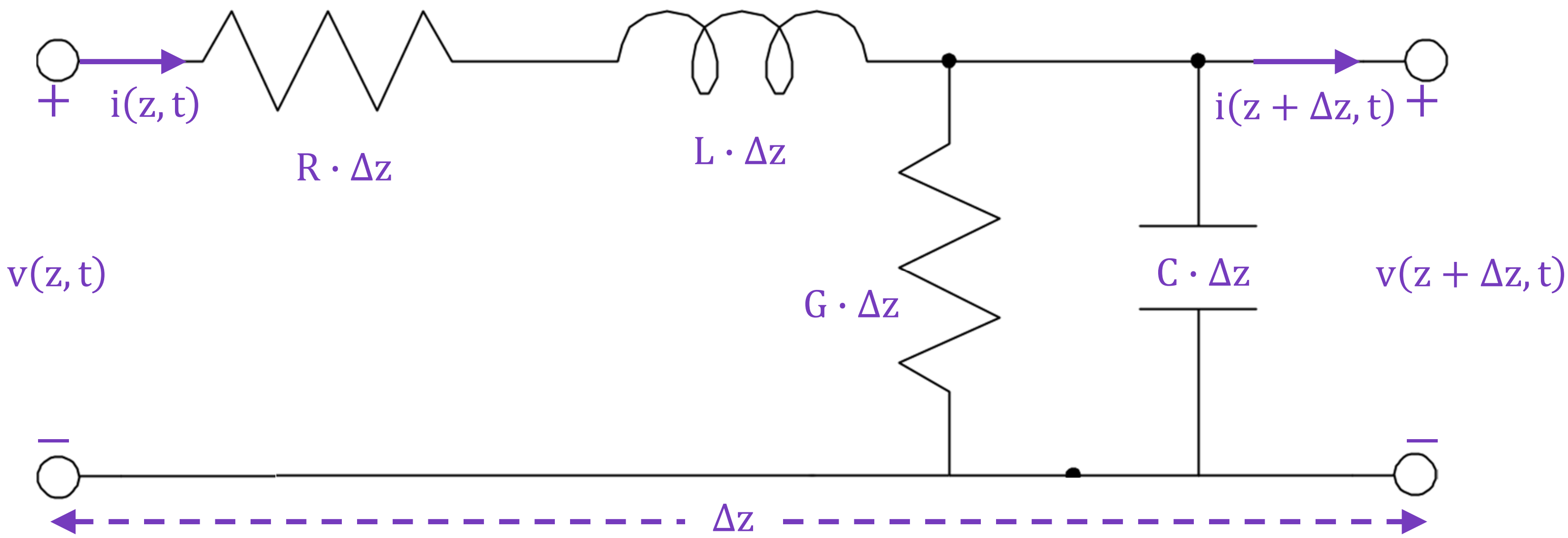
- Use boundary conditions and statics to compute L and C
- Use the capacitance to compute the conductance
- Compute the surface ohmic power dissipation per unit area via the surface current density (A/m) and the surface resistance (Ω/m) for both conductors
- Convert this into V=IR circuit theory by multiplying the length of the contour cross section and extract the resistance

- **C** and **L** → lossless energy storage per unit length
- **G** → energy loss in the dielectric per unit length
- **R** → energy loss in the conductor per unit length

$$Z_s = \frac{E_t}{J_s} = \frac{E_z}{J_{su}}$$

This field is no longer TEM!!!!!!

General Transmission Line



Kirchhoff nodal analysis, $\Delta z \rightarrow 0$

$$-\frac{\partial v(z,t)}{\partial z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t}$$

$$-\frac{\partial i(z,t)}{\partial z} = Gv(z,t) + C\frac{\partial v(z,t)}{\partial t}$$

→

Time harmonic series

$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z)$$

$$-\frac{dI(z)}{dz} = (G + j\omega C)V(z)$$

→

Uncoupled, 2nd order, ODE

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z)$$

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z)$$

General Transmission Line

Uncoupled, 2nd order, ODE

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z)$$

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z)$$

General solution

$$V(z) = V_0^+ e^{-j\gamma z} + V_0^- e^{j\gamma z}$$

$$I(z) = I_0^+ e^{-j\gamma z} + I_0^- e^{j\gamma z}$$

Forward traveling waves

$$V(z) = V_0^+ e^{-j\gamma z}$$

$$I(z) = I_0^+ e^{-j\gamma z}$$

Complex propagation factor

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

Complex wave impedance

$$Z_0 = \frac{V(z)}{I(z)} = R_0 + jX_0 = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}}$$

True for any lossy medium with σ_d

$$\frac{G}{C} = \frac{\sigma_d}{\epsilon_d}$$

True for good conductors, σ_s large

$$LC = \mu_d \epsilon_d$$

Summary for lumped element calculation

1. Compute inductance L using magnetostatics or capacitance C using electrostatics assuming a good conductor
2. Once one reactance is known, compute the other using the following good conductor relation

$$LC = \mu\epsilon$$

Lossless

3. Use the capacitance from steps 1 or 2 to compute the conductance via the following relation

$$\frac{G}{C} = \frac{\sigma}{\epsilon}$$

4. Compute the surface ohmic power dissipation per unit area via the surface current density (A/m) and the surface resistance (Ω/m) for both conductors

$$p_\sigma = \frac{1}{2}|J_{su}|^2 R_s$$

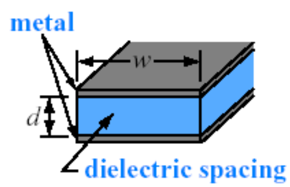
Pages 445 and 446 in Cheng

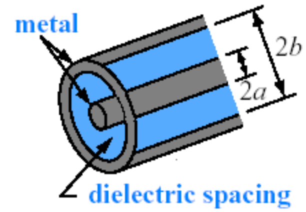
5. Convert this into V=IR circuit theory by multiplying the length width of the contour cross section and extract the resistance

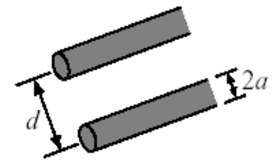
$$P_\sigma = p_\sigma w = \frac{1}{2} I^2 R$$

Lossy

Summary for lumped element calculation







Loss due to **Conductor**.

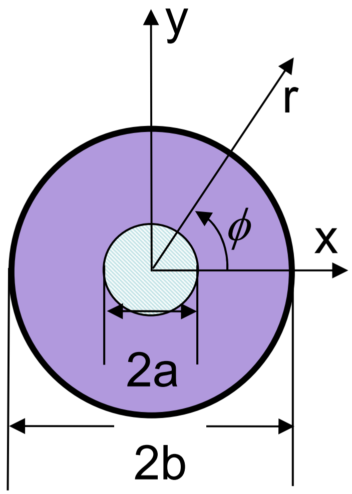
Assumes good conductivity i.e. surface current deenity and sheet resistance

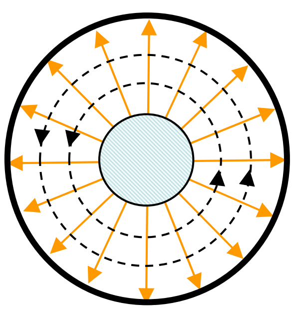
Loss due to **Dielectric**.

Assumes power is dissipated via displacement current ∝ ω^{-1}

$R = \frac{2R_s}{w}$	$R = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right)$	$R = \frac{R_s}{\pi a}$	$\boldsymbol{\frac{\Omega}{m}}$
$L = \mu\frac{d}{w}$	$L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$	$L = \frac{\mu}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right)$	$\boldsymbol{\frac{H}{m}}$
$G = \sigma_d\frac{w}{d}$	$G = \frac{2\pi\sigma_d}{\ln\left(\frac{b}{a}\right)}$	$G = \frac{\pi\sigma_d}{\cosh^{-1}\left(\frac{D}{2a}\right)}$	$\boldsymbol{\frac{S}{m}}$
$C = \epsilon_d\frac{w}{d}$	$C = \frac{2\pi\epsilon_d}{\ln\left(\frac{b}{a}\right)}$	$C = \frac{\pi\epsilon_d}{\cosh^{-1}\left(\frac{D}{2a}\right)}$	$\boldsymbol{\frac{F}{m}}$

EX. Coax transmission line analysis





$$R = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$$

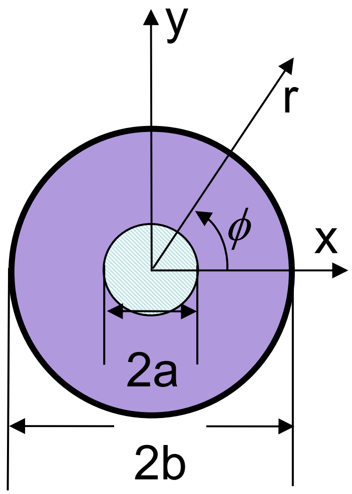
$$G = \frac{2\pi\sigma_d}{\ln\left(\frac{b}{a}\right)}$$

$$C = \frac{2\pi\epsilon_d}{\ln\left(\frac{b}{a}\right)}$$

$$R_0 = \sqrt{\frac{L}{C}} = \frac{\eta}{2\pi}\ln\left(\frac{b}{a}\right)$$

- Define the ratio of outer to inner radii as D = b/a
- Find a ratio D that minimizes loss and compute the resulting wave impedance of the coax line
- Assume good conductor and lossless dielectric
- Inspect different medium relative permittivity
  - ϵ_d = 2.25 (polyethylene)
  - ϵ_d = 1.00 (air)

EX. Coax transmission line analysis



$$\alpha = \frac{1}{2R_0}\left(R + G|Z_0|^2\right) = \frac{R}{2R_0}$$

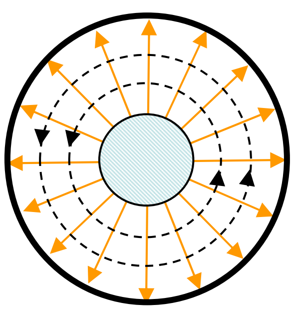
(G → 0) → Equation 9-90

$Z_0 = R_0 + jX_0$

$$\alpha = \frac{R_s}{2\eta}\left(\frac{1}{\ln\left(\frac{b}{a}\right)}\right)\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$D = \frac{b}{a}$$

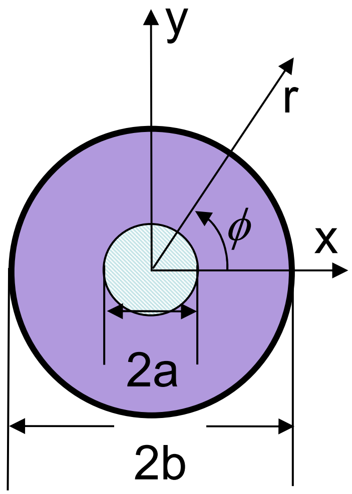
$$\alpha = \frac{R_s}{2\eta b}\left(\frac{1}{\ln(D)}\right)(D + 1)$$

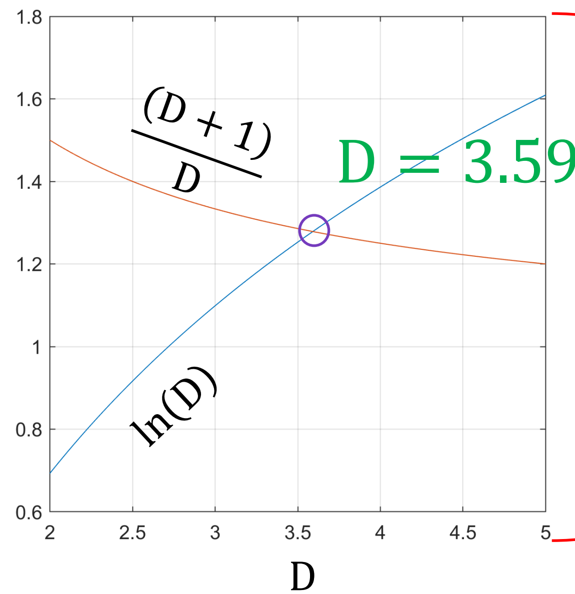


$$\alpha = \frac{R_s}{2\eta b}\left(\frac{D+1}{\ln(D)}\right)$$

$$\frac{d\alpha}{dD} = \frac{R_s}{2\eta b}\left(\frac{\ln(D) - \frac{(D+1)}{D}}{(\ln(D))^2}\right) = 0$$

EX. Coax transmission line analysis





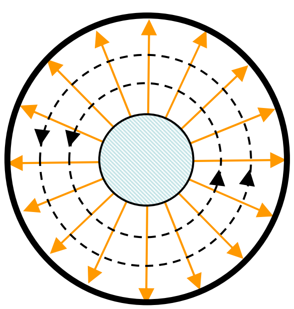
$$\ln(D) = \frac{(D+1)}{D}$$

$$R_0 = \frac{\eta}{2\pi}\ln\left(\frac{b}{a}\right)$$

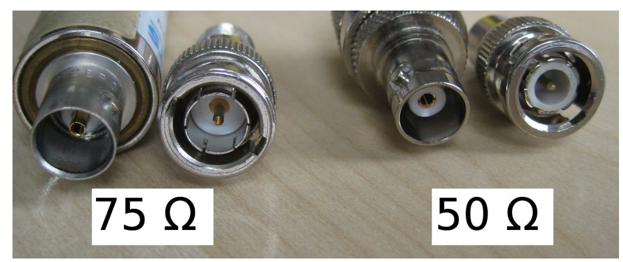
$$R_0 = \frac{\eta_0}{2\pi\sqrt{\epsilon_r}}\ln(D)$$

$$R_0 = \frac{60}{\sqrt{2.25}}\ln(3.59) = 51.1\ \Omega$$

$$R_0 = \frac{60}{\sqrt{1.00}}\ln(3.59) = 76.7\ \Omega$$



- This is one explanation of why 50 ohms vs 75 ohms
- We will investigate other reasons in the next lectures



Conclusions

Summary

- Transmission line analysis mapping fields to lumped elements leverages many assumption
  - Good conductor
  - Ignore internal inductance
  - Ignore reactance in the sheet impedance of good conductors
  - Ignore skin effect
  - Radiative losses
- Exact results need simulation: HFSS, CST, etc.
- Formulations use the following form of complex medium permittivity for lossy dielectrics

$$\epsilon_c = \epsilon_d - j\frac{\sigma_d}{\omega}$$

- We have added the subscript d to help keep track of what parameters are for what part of the transmission line

$$\times_d \rightarrow \text{dielectric}$$

$$\times_s \rightarrow \text{surface} \rightarrow \text{conductor}$$

- Current in a real conductor treated as a surface current which violates skin effect
  - Equivalent surface current