Problem Set 4

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Exercise 1 - PS4

Consider the following matrix, where a and b are real numbers:

$$A = \begin{pmatrix} a & 1 & a \\ 1 & 1 & 0 \\ b & 0 & 2 \end{pmatrix}.$$

Find conditions on a and b under which A is:

(a) positive definite

(b) positive semidefinite

(c) negative definite

(d) negative semidefinite

(e) indefinite.

Exercise 1 - Solution

If a is negative, a-1 is also negative. Therefore, A cannot be negative semidefinite and, consequently, cannot be negative definite either. In order for A to be positive definite, it must be the case that the leading principal minors are as follows
i) $a > 0$, ii) $|A_2| > 0$, which is true iff $a > 1$, and iii) $|A_3| = |A| > 0$, which is true iff $2a - 2 - ab > 0$. In sum, A is positive definite iff $a > 1$ and $b < 2 - \frac{2}{a}$.
In order for A to be positive semidefinite, we must have $a \geq 0$ (first order principal minor), $a \geq 1$ and $2a - ab \geq 0$ (second order principal minors), and $2a - 2 - ab \geq 0$ (third order principal minor). In sum, A is positive semidefinite iff $a \geq 1$ and $b \leq 2 - \frac{2}{a}$.
Finally, if $a < 1$ or $b > 2 - \frac{2}{a}$, the matrix is indefinite.

Exercise 2

Consider the following two functions defined on $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Determine if they are concave or convex.

(a) $f(x, y) = -\sqrt{2}x + 17y + x^2 - 2xy + ay^2$, with $a \geq 1$

(b) $f(x, y, z) = x - 2x^2 + 3y - y^2 - 3z - xy - e^{x+y+z}$

Exercise 2 - Solution

(a) The Hessian matrix is

$$D^2 f(x,y) = \begin{pmatrix} 2 & -2 \\ -2 & 2a \end{pmatrix}$$

The two first order principal minors are positive, the second order principal minor is non-negative. Hence the Hessian is positive semidefinite and the function is convex.

(b) Define $u := x + y + z$. The Hessian is

$$\begin{pmatrix} -4 - e^u & -1 - e^u & -e^u \\ -1 - e^u & -2 - e^u & -e^u \\ -e^u & -e^u & -e^u \end{pmatrix}.$$

The leading principal minors are $|A_1| = -4 - e^u < 0$, $|A_2| = 7 + 4e^u > 0$ and $|A_3| = -7e^u < 0$. Hence the Hessian is negative definite and f is concave.

Exercise 3

Consider the following function defined over $\mathbb{R}^2$:

$$f(x, y) = e^x + e^y.$$

(a) Determine if f is concave or convex.

(b) Find all the local and global extrema of f.

(c) How can you reconcile what you found in question (b) with the proposition at p. 22 in the slides from Lecture 9?

Exercise 3 - Solution

(a) f is convex because the Hessian is always positive semidefinite.
(b) It does not have any critical point. Consequently, by the proposition at p. 22 in the slides from Lecture 9, it does not have any global extrema.
(c) Notice that the proposition under consideration gives us a *characterization* of extrema for convex and concave functions. That is, it says that, for these functions, critical points are the same as global extrema. The proposition does *not* say that a convex or concave function always has an extreme point.

Exercise 4

Find all the local and global extrema of the following functions defined over $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

(a) $f(x, y) = (1 + y)^3 x^2 + y^2$

(b) $f(x, y, z) = \exp\left(x^2 + y^2 + 3z^2 - xy + 2xz + yz\right)$, where $\exp(\alpha)$ stands for e^α

Exercise 4 - Solution

(a) The unique stationary point is the origin $(0,0)$. The Hessian is

$$D^2 f(x,y) = \begin{pmatrix} 2(1+y)^3 & 6(1+y)^2 x \\ 6(1+y)^2 x & 2+6(1+y)x^2 \end{pmatrix}$$

At $(0,0)$, the Hessian is positive definite, so implying that $(0,0)$ is a local minimizer. However, it is not a global minimizer. Notice that $f(x,-2) = -x^2 + 4$ tends to $-\infty$ as x goes to $+\infty$.

Exercise 4 - Solution

(b) We can simplify things and take a positive monotone transformation. Specifically, let $g(w) = \ln w$. Then

$$g \circ f = x^2 + y^2 + 3z^2 - xy + 2xz + yz. \tag{1}$$

Notice that (1) is a quadratic form, and its only stationary point is the origin $(0, 0, 0)$. The Hessian is

$$D^2(g \circ f)(x, y, z) = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{pmatrix}$$

The leading principal minors are $|D_1| = 2$, $|D_2| = 3$ and $|D_3| = 4$. Hence the Hessian is positive definite at every point of the domain. This means that $g \circ f$ is convex, and $(0, 0, 0)$ is the unique global (and local) minimizer of $g \circ f$ and f.

Exercise 5

Determine the definiteness of the following quadratic forms:

(a) $Q(x, y) = -(x - y)^2$

(b) $Q(x, y, z) = 5x^2 + 2xz + 2y^2 + 2yz + 4z^2$

(c) $Q(x, y, z) = -3x^2 + 2xy - y^2 + 4yz - 8z^2$.

Exercise 5 - Solution

(a) is negative semidefinite;
(b) is positive definite;
(c) is negative definite.