ELEC-E4130

Lecture 12: Transmission lines

Ch. 9.4

Aalto University
School of Electrical
Engineering

ELEC-E4130 / Taylor

Oct. 21, 2021

Impedance recap

Intrinsic Impedance

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

- Dependent only on the material properties of the medium
- Equal to wave impedance for unidirectional, uniform plane waves

Wave Impedance

$$Z_t = \frac{E_t}{H_t}$$

- Ratio of total electric field and total magnetic field
- May be space and angle dependent (e.g. dielectric interfaces)

Characteristic Impedance

$$Z_0 = \frac{V_0^+}{I_0^+}$$

- Ratio of voltage to current
- Voltage and current have a well defined relationship in a TEM guide (transmission line). Less well defined in TE or TM

Loaded, Finite Transmission line

- Chapter 9.4 is packed full of algebra. Please go through the algebra again, on your own
- The equations are tedious and the concepts are best understood by going back and forth between algebra and theory

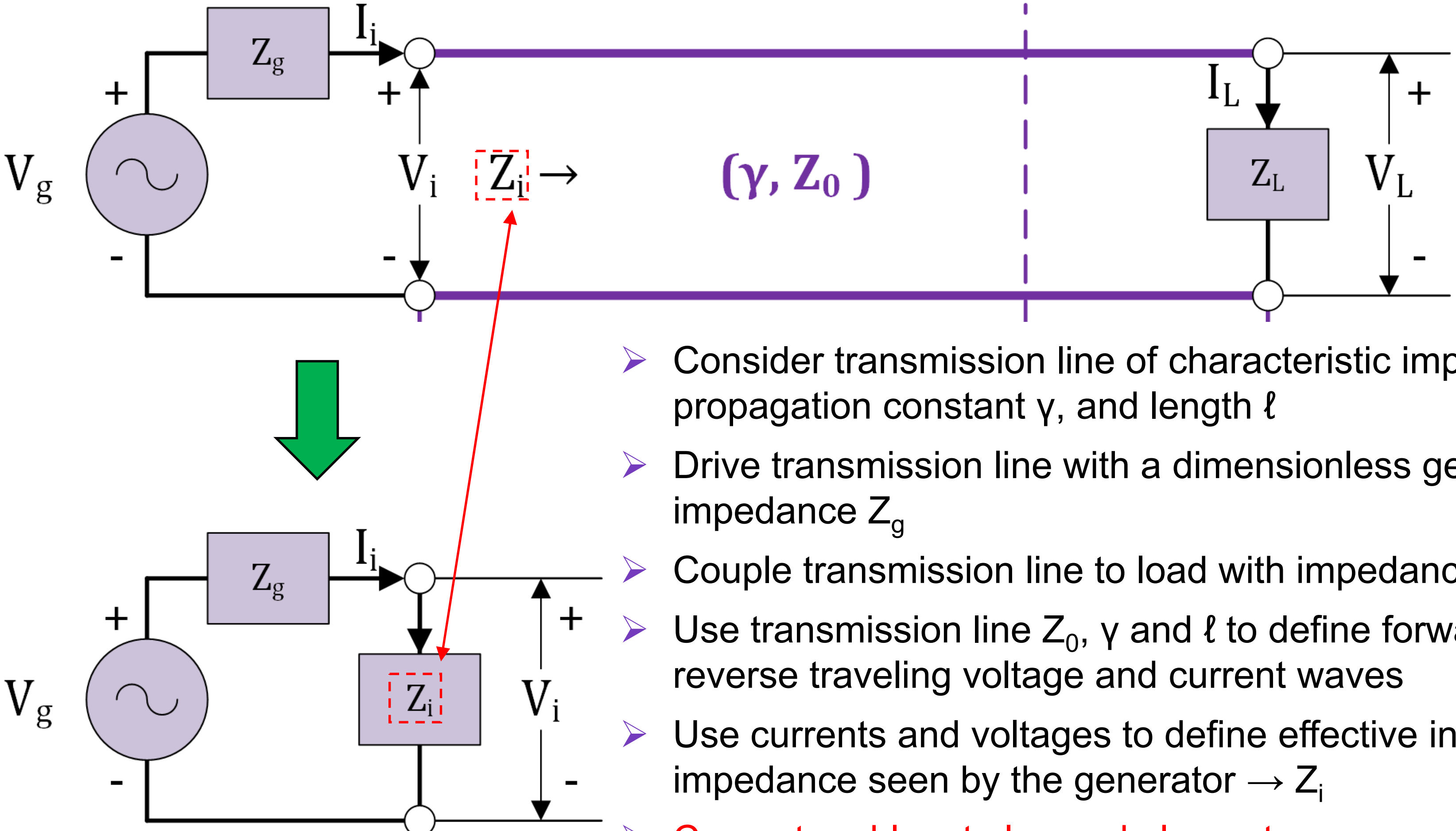
Last class

- In the last class we saw that electrostatics/magnetostatics could be used to compute the capacitance (**C**) and/or inductance (**L**) per unit length of a two conductor, TEM transmission line
- Boundary conditions can be used to define voltages and currents on the transmission line walls and to quantify the conductor loss per unit length (**R**) and dielectric fill loss per unit length (**G**)
- The characteristic impedance $\mathbf{Z_0}$ and propagation constant $\boldsymbol{\gamma}$ of an infinitely long line compose of L,C,R,G was defined

This class

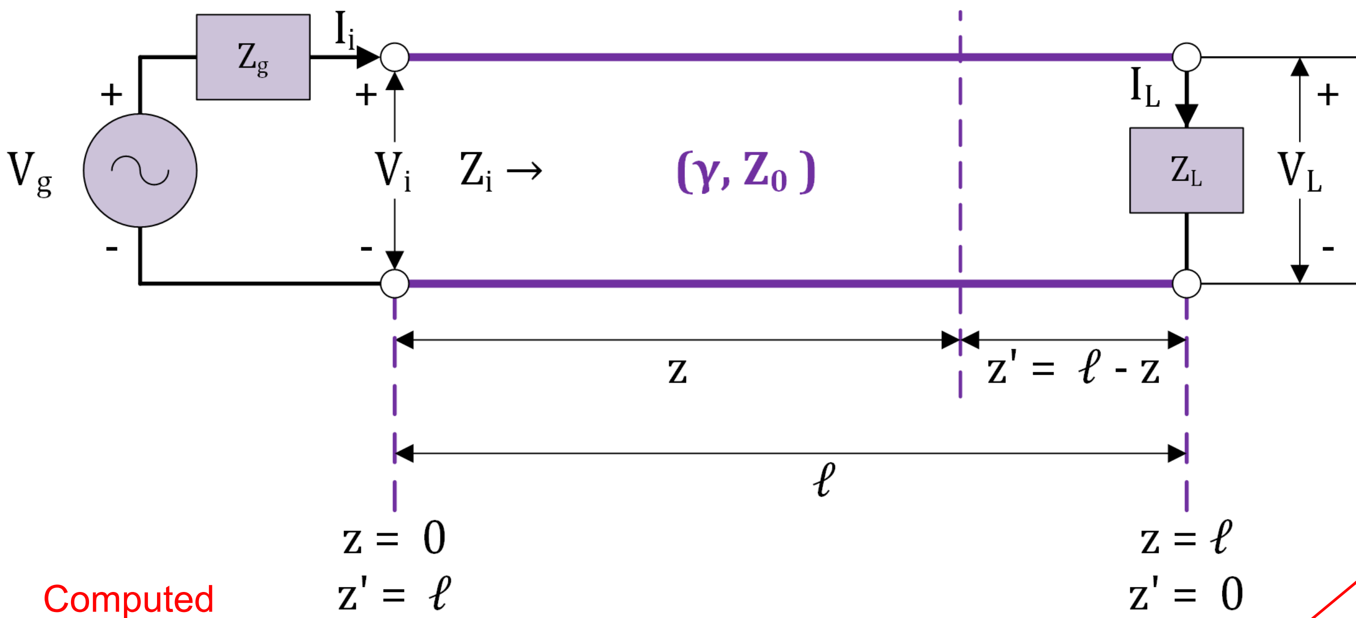
- What happens when we make the transmission line finite, load it with impedance $\mathbf{Z_L}$, and drive it with a generator signal $\mathbf{V_g}$ and generator impedance $\mathbf{Z_g}$

Loaded, Finite Transmission line



- Consider transmission line of characteristic impedance Z_0, propagation constant γ, and length ℓ
- Drive transmission line with a dimensionless generator with impedance Z_g
- Couple transmission line to load with impedance Z_L
- Use transmission line Z_0, γ and ℓ to define forward and reverse traveling voltage and current waves
- Use currents and voltages to define effective input impedance seen by the generator → Z_i
- Convert problem to lumped element

Loaded, Finite Transmission line



V_g $\rightarrow$ Generator voltage

V_L $\rightarrow$ Voltage across the load

V_i $\rightarrow$ Voltage across the combined load + transmission line

Z_g $\rightarrow$ Generator internal impedance

Z_L $\rightarrow$ Load impedance

$\boxed{Z_0}$ $\rightarrow$ transmission line characteristic impedance

Z_i $\rightarrow$ impedance seen by the generator

I_L $\rightarrow$ Current through the load

I_i $\rightarrow$ Current sourced by the generator

Computed last lecture

ℓ $\rightarrow$ transmission line length

$\boxed{\gamma}$ $\rightarrow$ transmission line propagation constant

z $\rightarrow$ distance measured from generator to the load

z' $\rightarrow$ distance measured from the load to the generator

Computed last lecture

From last time

Uncoupled, 2nd order, ODE

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z)$$

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z)$$

General solution

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

Forward traveling waves

$$V(z) = V_0^+ e^{-\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z}$$

Complex propagation factor

$$\boxed{\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}}$$

Complex wave impedance

$$\boxed{Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = R_0 + jX_0 = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}}}$$

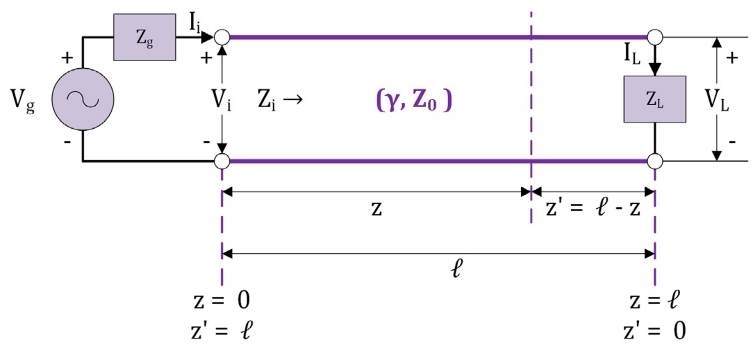
True for any lossy medium with σ_d

$$\boxed{\frac{G}{C} = \frac{\sigma_d}{\epsilon_d}}$$

True for good conductors, σ_s large (vanishingly small non TEM fields)

$$\boxed{LC = \mu_d \epsilon_d}$$

Loaded, Finite Transmission line



Assume forward and reverse traveling waves due to impedance mismatch

General solution

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

$$V(z=\ell) = V_L = V_0^+ e^{-\gamma\ell} + V_0^- e^{\gamma\ell}$$

$$I(z=\ell) = I_L = I_0^+ e^{-\gamma\ell} + I_0^- e^{\gamma\ell}$$

$$= \frac{V_0^+}{Z_0} e^{-\gamma\ell} + \frac{V_0^-}{Z_0} e^{\gamma\ell}$$

algebra

$$V_0^+ = \frac{1}{2}(V_L + I_L Z_0) e^{\gamma\ell}$$

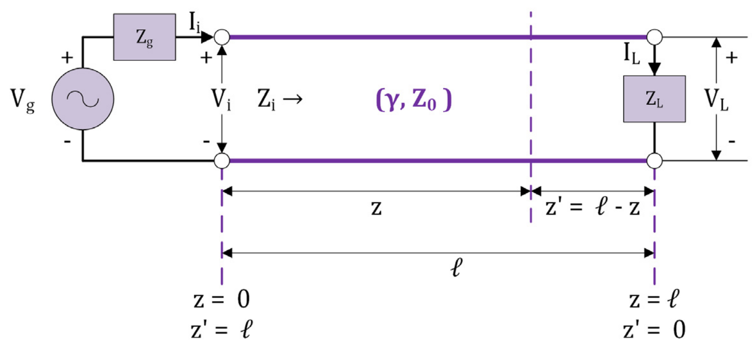
$$V_0^- = \frac{1}{2}(V_L - I_L Z_0) e^{-\gamma\ell}$$

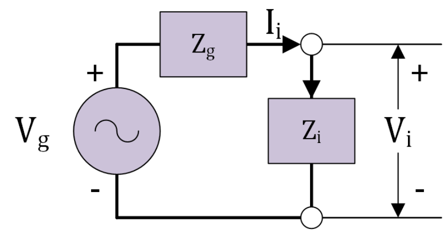
algebra

$$V(z) = \frac{I_L}{2}\left((Z_L + Z_0) e^{\gamma(\ell - z)} + (Z_L - Z_0) e^{-\gamma(\ell - z)}\right)$$

$$I(z) = \frac{I_L}{2Z_0}\left((Z_L + Z_0) e^{\gamma(\ell - z)} - (Z_L - Z_0) e^{-\gamma(\ell - z)}\right)$$

Loaded, Finite Transmission line





$$V(z) = \frac{I_L}{2}\left((Z_L + Z_0)e^{\gamma(\ell - z)} + (Z_L - Z_0)e^{-\gamma(\ell - z)}\right)$$

$$I(z) = \frac{I_L}{2Z_0}\left((Z_L + Z_0)e^{\gamma(\ell - z)} - (Z_L - Z_0)e^{-\gamma(\ell - z)}\right)$$

z' = ℓ - z

$$V(z') = \frac{I_L}{2}\left((Z_L + Z_0)e^{\gamma z'} + (Z_L - Z_0)e^{-\gamma z'}\right)$$

$$I(z') = \frac{I_L}{2Z_0}\left((Z_L + Z_0)e^{\gamma z'} - (Z_L - Z_0)e^{-\gamma z'}\right)$$

hyperbolic functions

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

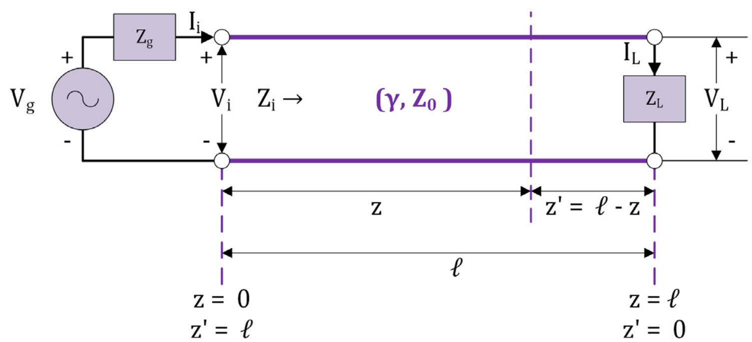
$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

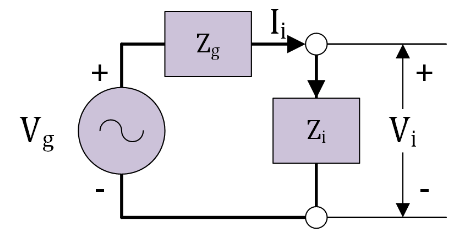
Algebra

$$V(z') = I_L(Z_L \cosh(\gamma z') + Z_L \sinh(\gamma z'))$$

$$I(z') = \frac{I_L}{2Z_0}(Z_L \sinh(\gamma z') + Z_0 \cosh(\gamma z'))$$

Loaded, Finite Transmission line





$$V(z') = I_L(Z_L \cosh(\gamma z') + Z_0 \sinh(\gamma z'))$$

$$I(z') = \frac{I_L}{Z_0}(Z_L \sinh(\gamma z') + Z_0 \cosh(\gamma z'))$$

$$Z(z') = \frac{V(z')}{I(z')} = Z_0 \frac{Z_L \cosh(\gamma z') + Z_0 \sinh(\gamma z')}{Z_L \sinh(\gamma z') + Z_0 \cosh(\gamma z')}$$

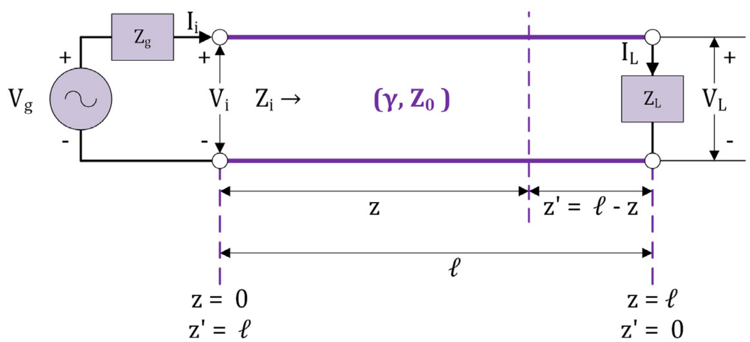
hyperbolic identities

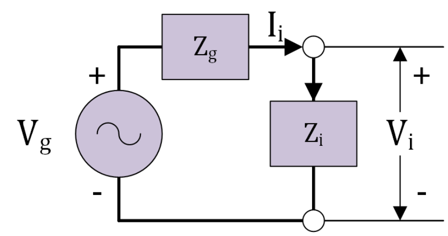
$$Z(z') = \frac{V(z')}{I(z')} = Z_0 \frac{Z_L + Z_0 \tanh(\gamma z')}{Z_0 + Z_L \tanh(\gamma z')}$$

$$Z_i = Z(z' = \ell) = Z(z = 0)$$

$$\boxed{Z_i = Z_0 \frac{Z_L + Z_0 \tanh(\gamma \ell)}{Z_0 + Z_L \tanh(\gamma \ell)}}$$

Loaded, Finite Transmission line





Voltage divider relations

$$V_i = \frac{Z_i}{Z_g + Z_i} V_g$$

$$I_i = \frac{1}{Z_g + Z_i} V_g$$

Matched load $Z_L = Z_0$

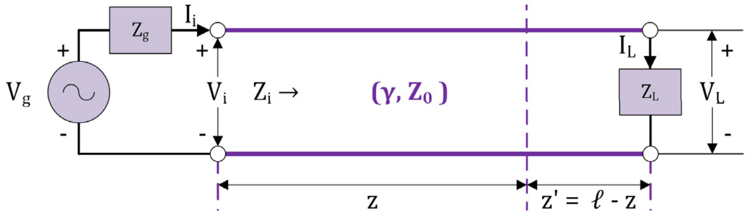
$$Z(z') = \frac{V(z')}{I(z')} = Z_0 \frac{Z_0 + Z_0 \tanh(\gamma z')}{Z_0 + Z_0 \tanh(\gamma z')} = Z_0$$

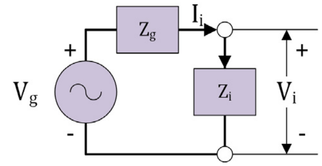
$$V(z) = V_i e^{-\gamma z}$$

$$I(z) = I_i e^{-\gamma z}$$

- Behave as if the line is infinite
  - No reverse traveling wave
  - No reflection

Loaded, Finite Transmission line





$$\boxed{Z_i = Z_0 \frac{Z_L + Z_0 \tanh(\gamma\ell)}{Z_0 + Z_L \tanh(\gamma\ell)}}$$

Open Circuit Load: $Z_L \rightarrow \infty$

$$Z_{io} = \frac{Z_0}{\tanh(\gamma\ell)}$$

Lossless line

$$Z_{io} = \frac{R_0}{j\tan(\beta\ell)}$$

$$Z_{io} = -jR_0 \cot(\beta\ell)$$

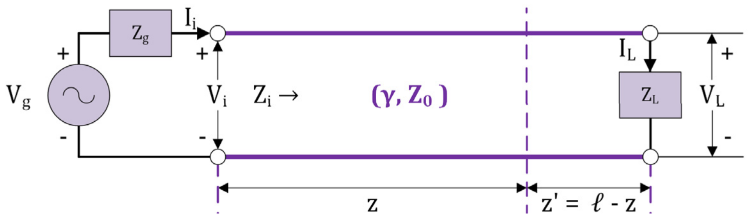
Short Circuit Load: $Z_L = 0$

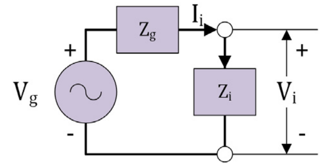
$$Z_{is} = Z_0 \tanh(\gamma\ell)$$

Lossless line

$$Z_{is} = jR_0 \tan(\beta\ell)$$

Loaded, Finite Transmission line





$$\boxed{Z_i = Z_0 \frac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)}}$$

Assume lossless transmission line

$$\gamma = j\beta$$

Quarter wave section

$$\ell = (2n-1)\frac{\lambda}{4}$$

$$\beta = \frac{2\pi}{\lambda}$$

$$\tan(\beta\ell) = \tan\left((2n-1)\frac{\pi}{2}\right) \rightarrow \pm\infty$$

$$\boxed{Z_i = \frac{Z_0^2}{Z_L}}$$

Half wave section

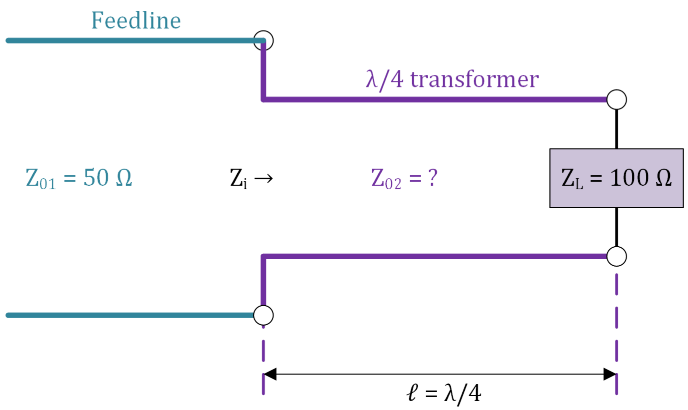
$$\ell = n\frac{\lambda}{2}$$

$$\beta = \frac{2\pi}{\lambda}$$

$$\tan(\beta\ell) = \tan(n\pi) = 0$$

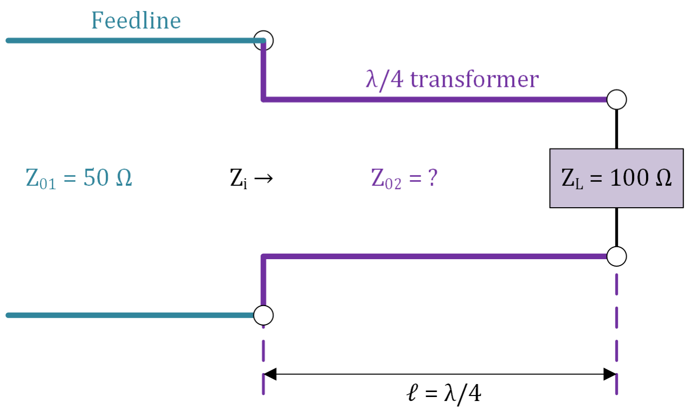
$$\boxed{Z_i = Z_L}$$

In Class Exercise 1



A 50 Ω lossless transmission line is to be matched to a resistive load impedance with Z_L = 100 Ω via a quarter wave section as shown in the figure thereby eliminating reflections along the feedline. Find the characteristic impedance of the quarter wave transformer

In Class Exercise 1, Solution



A 50 Ω lossless transmission line is to be matched to a resistive load impedance with Z_L = 100 Ω via a quarter wave section as shown in the figure thereby eliminating reflections along the feedline. Find the characteristic impedance of the quarter wave transformer

To eliminate reflections, the input impedance looking into the quarter-wave line should be equal to Z_{01}, the characteristic impedance of the feedline: Z_{in} = 50 Ω

$$Z_i = Z_{01} = \frac{Z_{02}^2}{Z_L}$$

$$Z_i = Z_{02} = \sqrt{Z_{02} Z_L} = \sqrt{50 \cdot 100} = 70.7\ \Omega$$

Lossless Lines with resistive termination: VSWR

$$V(z') = \frac{I_L}{2}\left(\underbrace{(Z_L + Z_0)e^{\gamma z'}}_{\text{Forward traveling wave}} + \underbrace{(Z_L - Z_0)e^{-\gamma z'}}_{\text{Reverse traveling wave}}\right)$$

Factor out forward traveling wave

$$V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left(1 + \frac{\overbrace{(Z_L - Z_0)e^{-\gamma z'}}^{\text{Reverse}}}{\underbrace{(Z_L + Z_0)e^{\gamma z'}}_{\text{Forward}}}\right)$$

$$V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left(1 + \frac{(Z_L - Z_0)}{(Z_L + Z_0)}e^{-2\gamma z'}\right)$$

$$V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left(1 + \Gamma e^{-2\gamma z'}\right)$$

$$\boxed{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma|e^{j\theta_\Gamma}}$$

$\gamma = j\beta$

$$V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{j\beta z'}\left(1 + \Gamma e^{-2j\beta z'}\right)$$

$$V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{j\beta z'}\left(1 + |\Gamma|e^{j(\theta_\Gamma - 2\beta z')}\right)$$

$$V(z') = |V_{max}| = 1 + |\Gamma|$$
$$\text{when } \theta_\Gamma - 2\beta z' = \pm 2n\pi$$

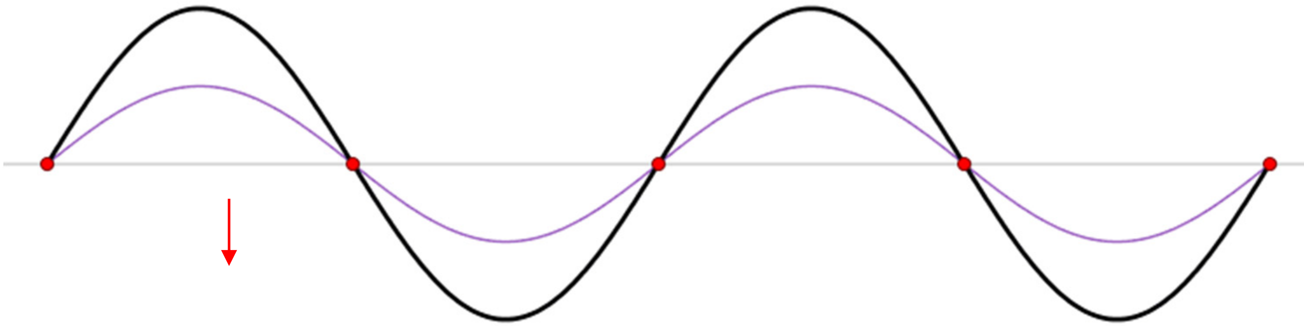
$$V(z') = |V_{min}| = 1 - |\Gamma|$$
$$\text{when } \theta_\Gamma - 2\beta z' = \pm(2n + 1)\pi$$

Voltage Standing Wave Ratio (VSWR)

$$\boxed{S = \frac{|V_{max}|}{|V_{min}|} = \frac{1 + |\Gamma|}{1 - |\Gamma|}}$$

$$\boxed{|\Gamma| = \frac{1 - S}{1 + S}}$$

VSWR alternative view



* wikipedia

Maximum **constructive** interference between forward and reverse traveling waves

$$|V_{max}| = |V_0^+| + |V_0^-| = |V_0^+| + |\Gamma V_0^+| = |V_0^+|(1 + |\Gamma|)$$

Maximum **destructive** interference between forward and reverse traveling waves

$$|V_{min}| = |V_0^+| - |V_0^-| = |V_0^+| - |\Gamma V_0^+| = |V_0^+|(1 - |\Gamma|)$$

$$\boxed{S = \frac{|V_{max}|}{|V_{min}|} = \frac{|V_0^+|(1 + |\Gamma|)}{|V_0^+|(1 - |\Gamma|)} = \frac{1 + |\Gamma|}{1 - |\Gamma|}}$$

$$\boxed{|\Gamma| = \frac{1 - S}{1 + S}}$$

VSWR: where can I see it?

Connectorized

Amplifier

ZX60-3018G+

50Ω   20 MHz to 3 GHz

CASE STYLE: GC957

Connectors	Model
SMA	ZX60-3018G-S+

+RoHS Compliant
The +Suffix identifies RoHS Compliance. See our web site for RoHS Compliance methodologies and qualifications

Measuring a standing wave is often easier than measuring an impedance

Features

- Wide bandwidth, 20 MHz to 3 GHz
- Low noise figure, 2.7 dB typ.
- Output power up to 12.8 dBm typ.
- Protected by US patent 6,790,049

Applications

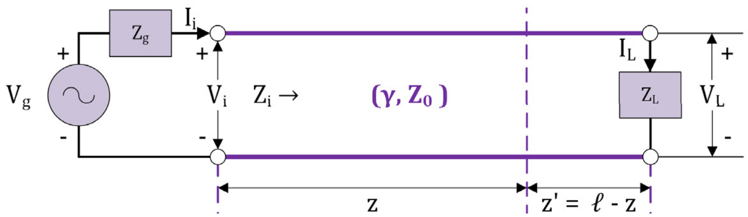
- Buffer amplifier
- Cellular
- PCS
- Lab
- Instrumentation
- Test equipment

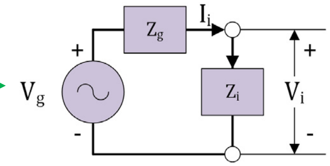
Electrical Specifications at T_{AMB} = 25°C

w.r.t 50 Ω

MODEL NO.	FREQ. (GHz)	DC VOLTAGE @ Pin V+ (V)	GAIN over frequency in GHz Typ (dB)					MAXIMUM POWER (dBm) Output (1 dB Comp.) Typ.		DYNAMIC RANGE		VSWR (:1) Typ.		ACTIVE DIRECTIVITY (dB) Isolation-Gain	DC OPERATING CURRENT @ Pin V+ (mA)	
	f_L - f_U		0.1	1.0	2.0	3.0	Min.at 2 GHz	f_L	f_U	NF (dB) Typ.	IP3 (dBm) Typ.	In	Out	Typ.	Typ.	Max.
ZX60-3018G+	0.02 - 3	12.0	22.8	21.9	20.3	18.8	18.0	12.8	10.2	2.7	25.0	1.3	1.4	2-6	34	45

Full Transmission Line Circuit





$$\boxed{Z_i = Z_0 \frac{Z_L + Z_0 \tanh(\gamma \ell)}{Z_0 + Z_L \tanh(\gamma \ell)}}$$

Add the generator voltage and impedance

Kirchhoff nodal analysis @ z = 0, z' = ℓ

$$V_i = V_g - I_i Z_g$$

Voltage and current phasors, arb. Line

$$V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left(1 + \Gamma e^{-2\gamma z'}\right)$$

$$I(z') = \frac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma z'}\left(1 - \Gamma e^{-2\gamma z'}\right)$$

Input Voltage and current phasors

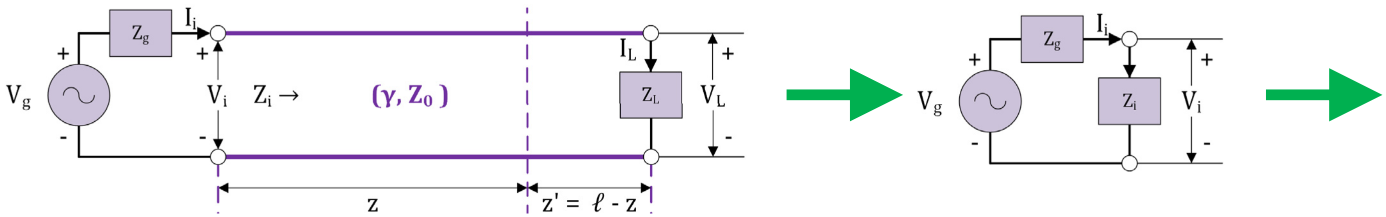
$$V_i = V(z' = \ell) = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma \ell}\left(1 + \Gamma e^{-2\gamma \ell}\right)$$

$$I_i = I(z' = \ell) = \frac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma \ell}\left(1 - \Gamma e^{-2\gamma \ell}\right)$$

$$\frac{I_L}{2}(Z_L + Z_0)e^{\gamma \ell} = V_g \frac{Z_0}{Z_0 + Z_g} \frac{1}{1 - \Gamma_g \Gamma e^{-2\gamma \ell}}$$

$$\boxed{\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}}$$

Full Transmission Line Circuit



$$\boxed{Z_i = Z_0 \frac{Z_L + Z_0 \tanh(\gamma\ell)}{Z_0 + Z_L \tanh(\gamma\ell)}}$$

Voltage and current phasors, arb. Line

$$V(z') = \boxed{\frac{I_L}{2}(Z_L + Z_0)} e^{\gamma z'}\left(1 + \Gamma e^{-2\gamma z'}\right)$$

$$I(z') = \frac{1}{Z_0}\boxed{\frac{I_L}{2}(Z_L + Z_0)} e^{\gamma z'}\left(1 - \Gamma e^{-2\gamma z'}\right)$$

$$\boxed{\frac{I_L}{2}(Z_L + Z_0)} = V_g e^{-\gamma\ell} \frac{Z_0}{Z_0 + Z_g} \frac{1}{1 - \Gamma_g \Gamma e^{-2\gamma\ell}}$$

$$\boxed{V(z') = V_g \frac{Z_0}{Z_0 + Z_g} e^{\gamma(z'-\ell)} \frac{1 + \Gamma e^{-2\gamma z'}}{1 - \Gamma_g \Gamma e^{-2\gamma\ell}}}$$

$$\boxed{I(z') = V_g \frac{1}{Z_0 + Z_g} e^{\gamma(z'-\ell)} \frac{1 + \Gamma e^{-2\gamma z'}}{1 - \Gamma_g \Gamma e^{-2\gamma\ell}}}$$

In Class Exercise 2

A 50 Ω transmission line is terminated in a load with Z_L = (100 + j50) Ω. Find the voltage reflection coefficient and the voltage standing wave ratio (VSWR)

In Class Exercise 2

A 50 Ω transmission line is terminated in a load with Z_L = (100 + j50) Ω. Find the voltage reflection coefficient and the voltage standing wave ratio (VSWR)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 + j50 - 50}{100 + j50 + 50} = \frac{50 + j50}{150 + j50} = 0.45e^{j0.15\pi}$$

$$S = \frac{1 + 0.45}{1 - 0.45} = 2.6$$

Summary

- Two conductor waveguides (TEM transmission lines) can be described by the characteristic impedance Z_0, propagation constant γ, and length ℓ
- Drive transmission line with a dimensionless generator with impedance Z_g
- Couple transmission line to load with impedance Z_L
- Use transmission line Z_0, γ and ℓ to define forward and reverse traveling voltage and current waves
- Use currents and voltages to define effective input impedance seen by the generator → Z_i
- Convert problem to lumped element
  - Transmission lines have physical length so matching occurs when
    - $Z_L = Z_0$ (**NO** reflection)
    - not when $Z_L = Z_0^*$ (**YES** reflection)