ELEC-E4130

Lecture 13: Multilayer (Stratified Media)
One layer, normal incidence

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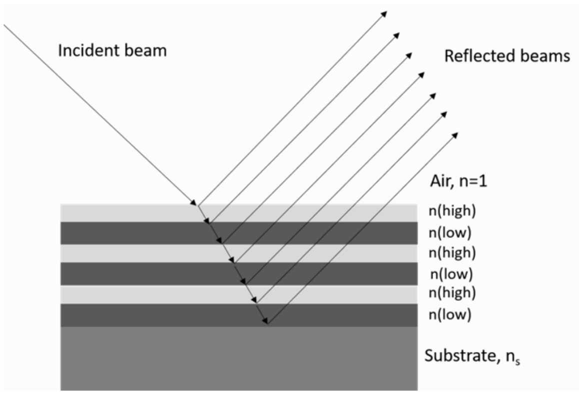
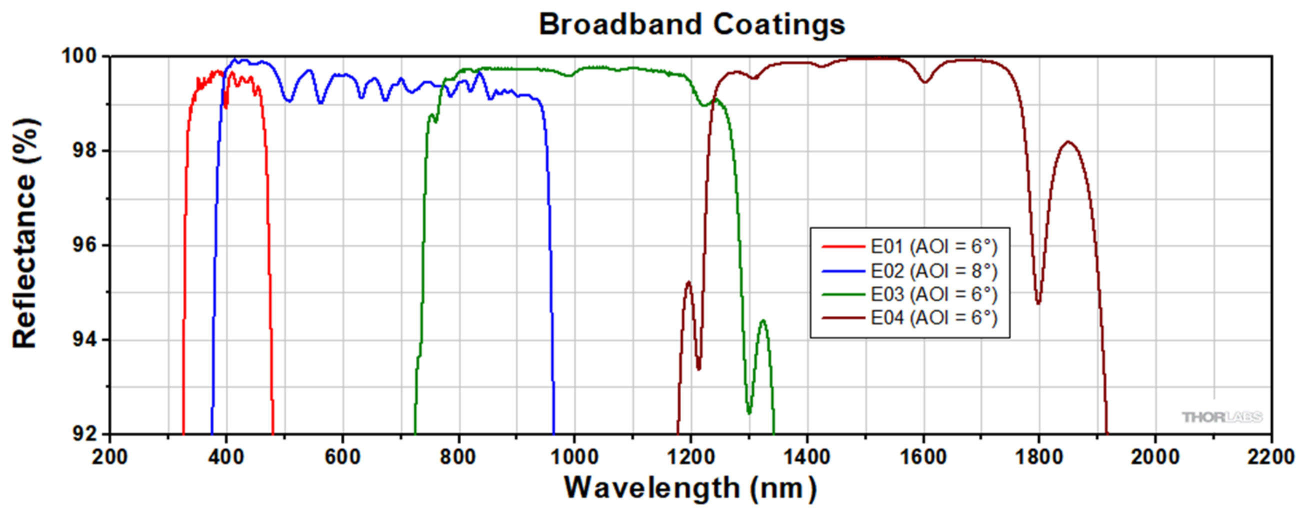
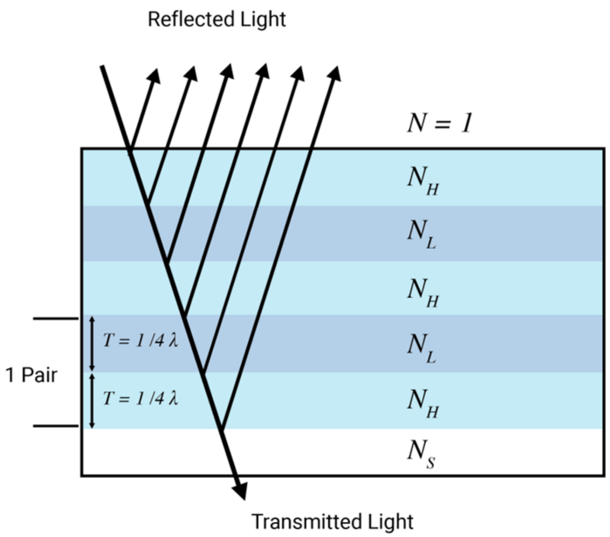
Multiple layers mirrors



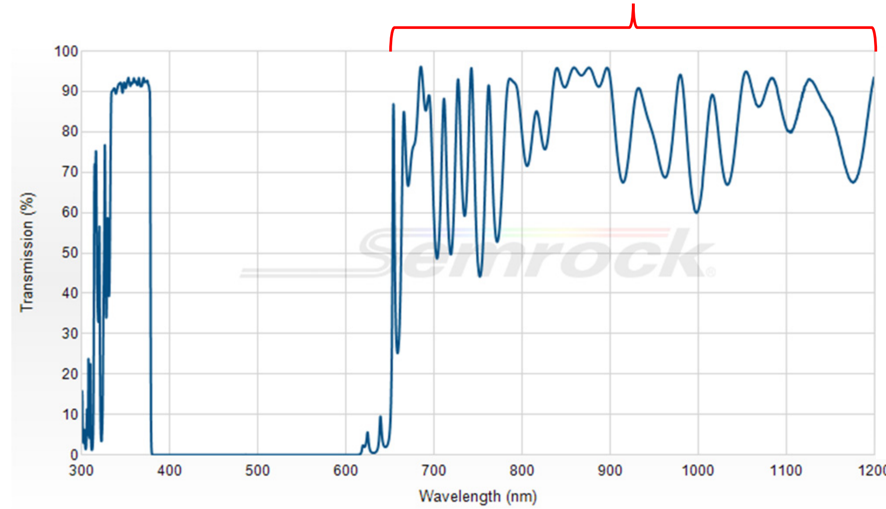


- Tune layer permittivity and thickness to enhance **constructive interference** of multiple reflections at the air dielectric interface
- Dielectric surface quality superior to metallic surface quality

Multiple layer filters

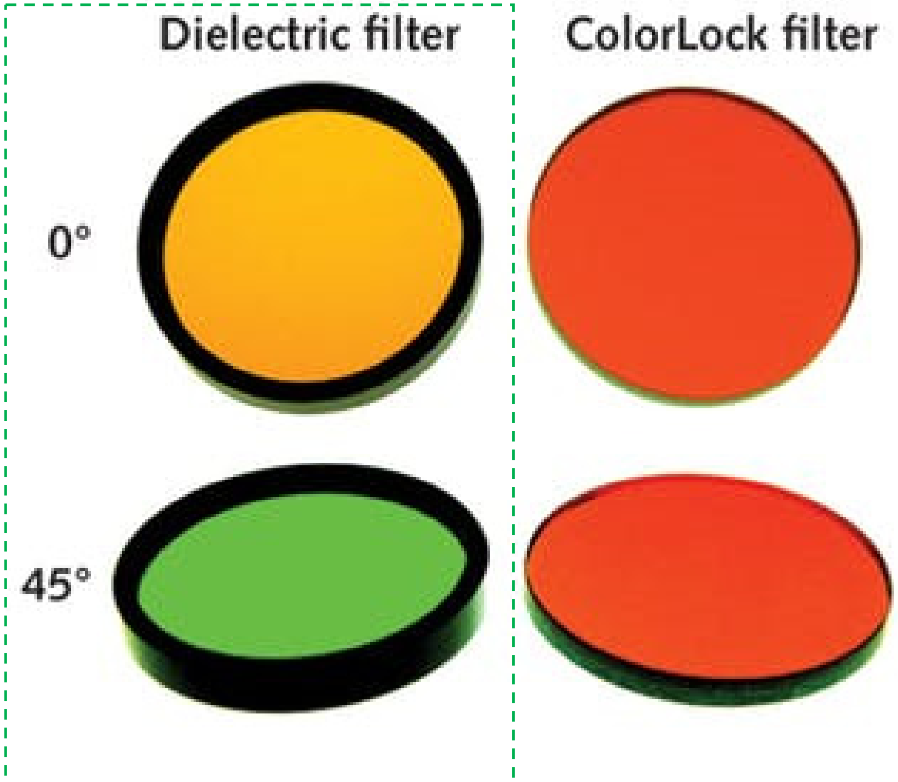


Sometimes impossible to suppress natural periodic behavior



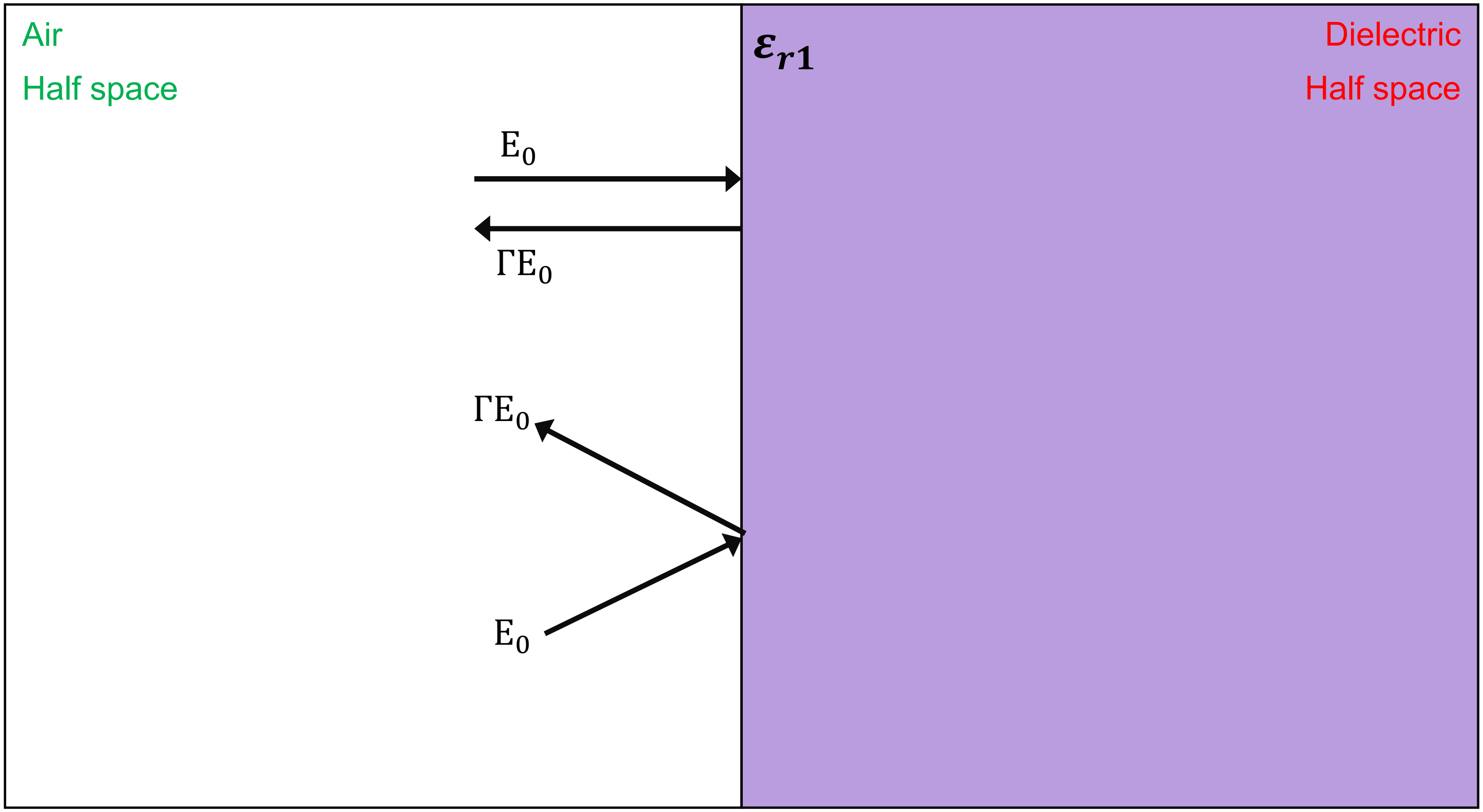
- Tune layer permittivity and thickness to enhance **destructive interference** of multiple reflections at input the air-dielectric interface
- This enhances **constructive interference** at the output dielectric-air interface

Angle dependent behavior

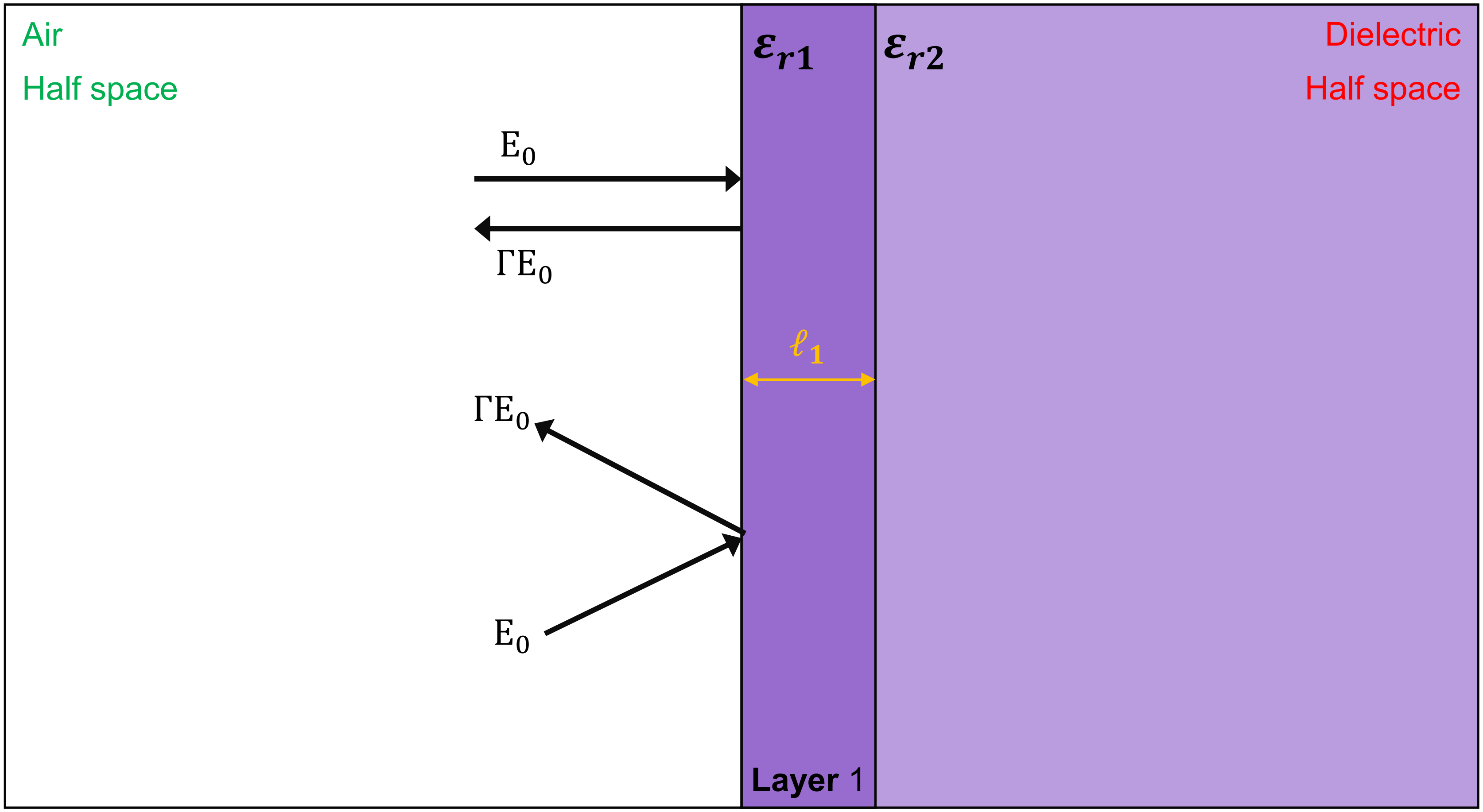


- Transmission and reflection are defined by steady state constructive and destructive interference
- Interference controlled by optical path length within the dielectric
- Optical path is increased going from normal to obliques incidence angle

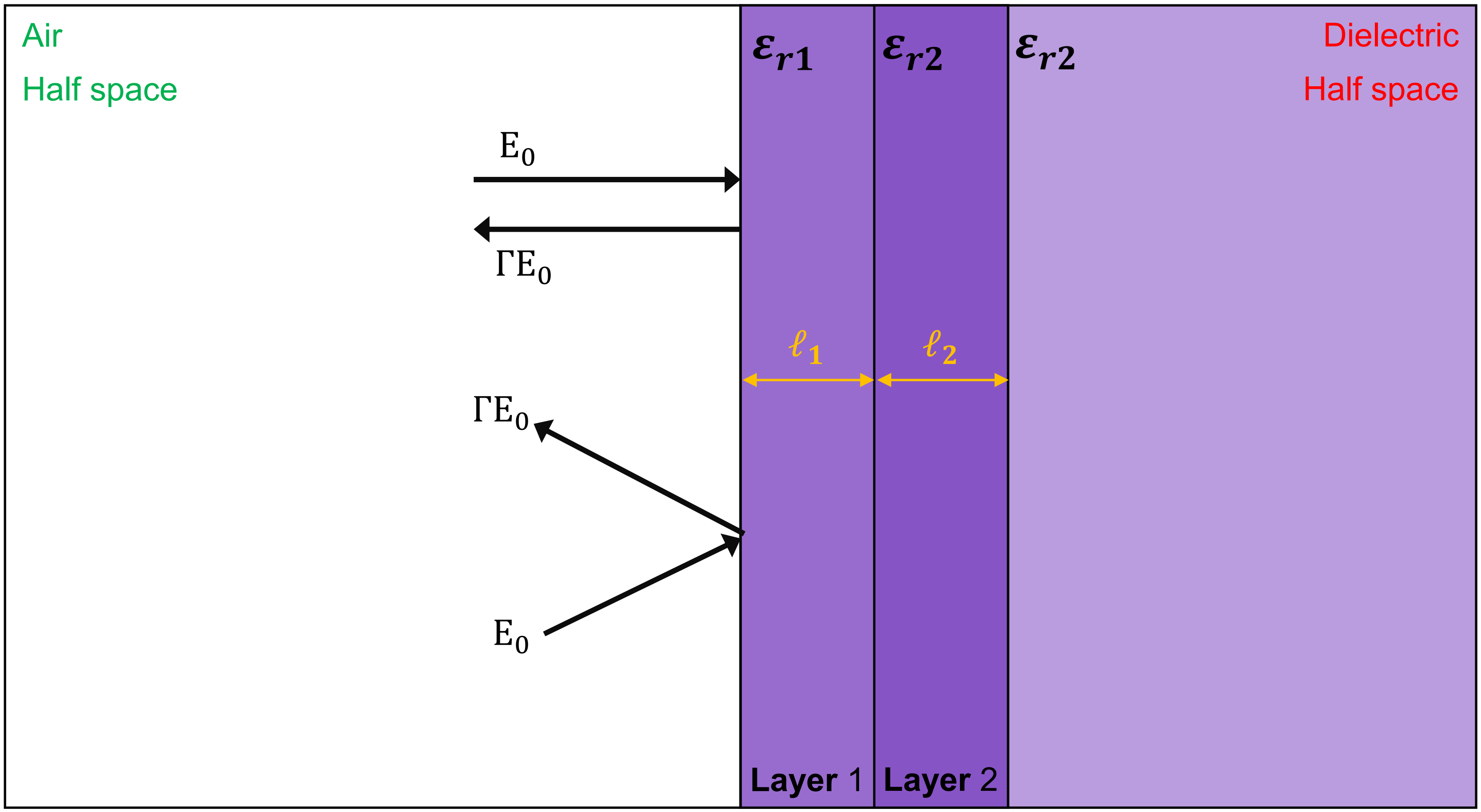
Single interface



Finite thickness layer

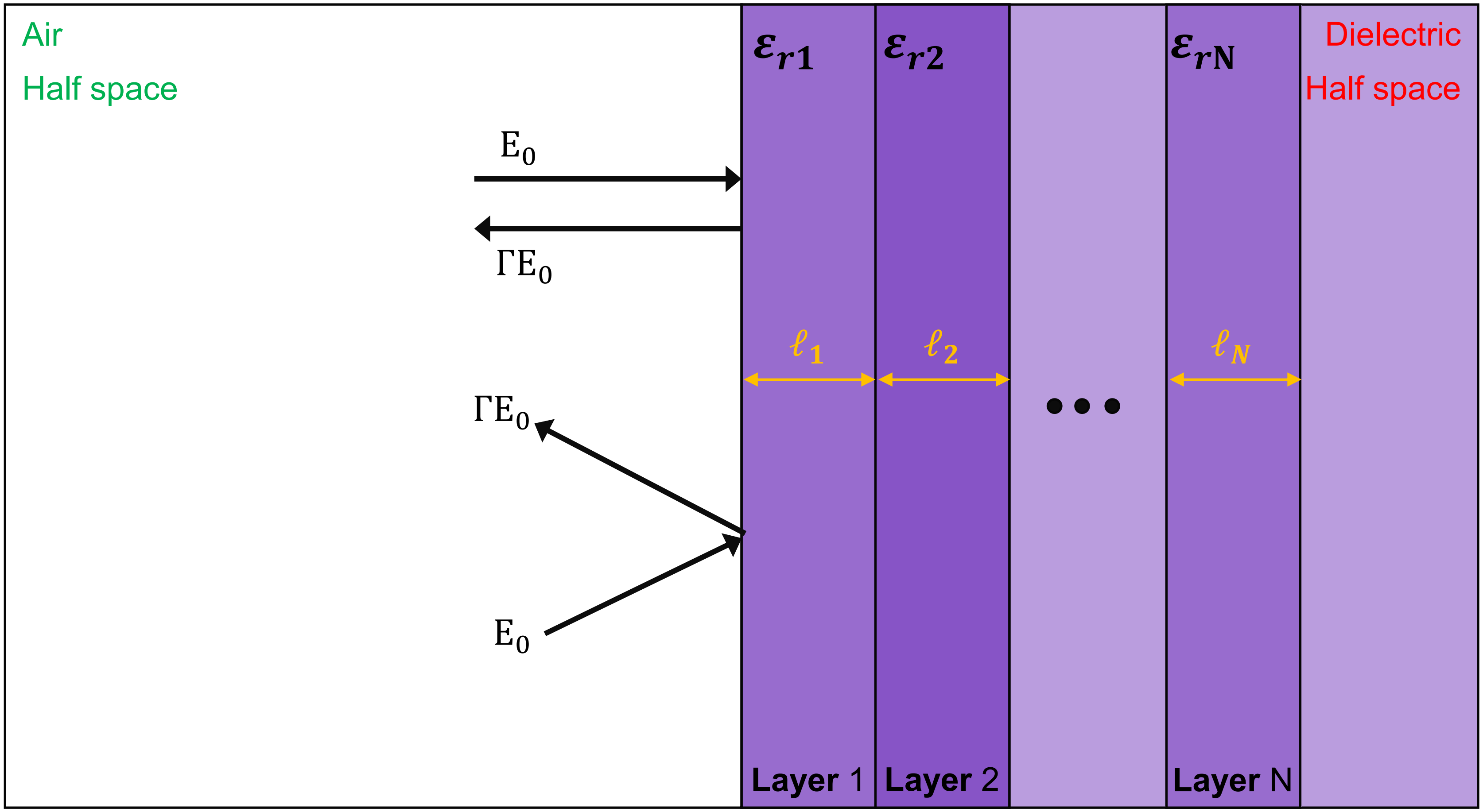


Two layers

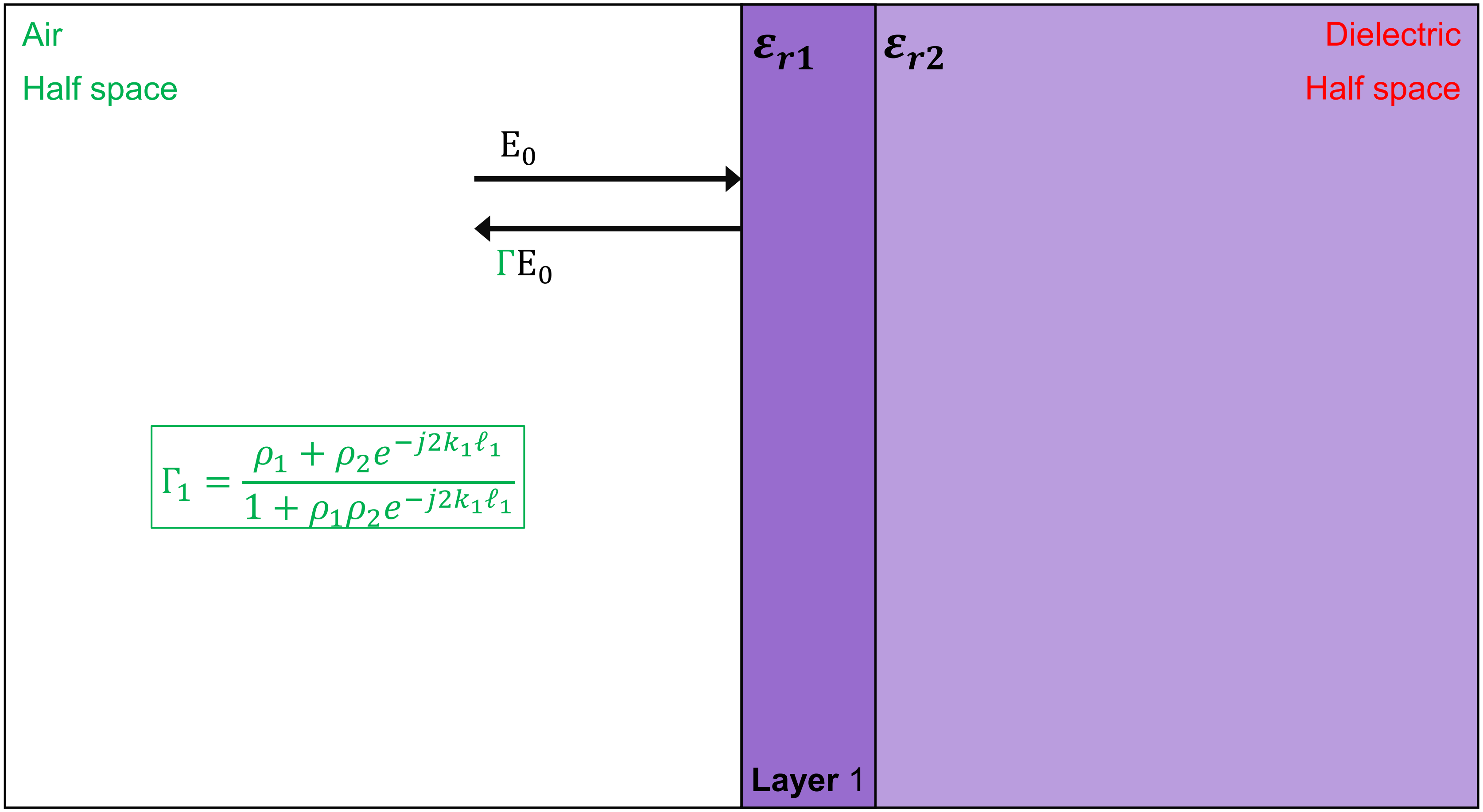


Air
Half space
E_0
ΓE_0
ε_{r1}
ε_{r2}
ε_{r2}
Dielectric
Half space
ℓ_1
ℓ_2
ΓE_0
E_0
Layer 1
Layer 2

N Layers


Air
Half space
E_0
ΓE_0
ε_{r1}
ε_{r2}
ε_{rN}
Dielectric
Half space
ℓ_1
ℓ_2
ℓ_N
ΓE_0
E_0
Layer 1
Layer 2
Layer N

Today: Normal incidence, single layer



$$\boxed{\Gamma_1 = \frac{\rho_1 + \rho_2 e^{-j2k_1\ell_1}}{1 + \rho_1\rho_2 e^{-j2k_1\ell_1}}}$$

Field relations

Total E-field

$$E(z) = E_{0+}e^{-jkz} + E_{0-}e^{jkz} = E_+(z) + E_-(z)$$

Total H-field

$$H(z) = \frac{1}{\eta}\left[E_{0+}e^{-jkz} - E_{0-}e^{jkz}\right] = \frac{1}{\eta}\left[E_+(z) - E_-(z)\right]$$

Right hand rule

Forward traveling E-field

$$E_+(z) = E_{0+}e^{-jkz}$$

Reverse traveling E-field

$$E_-(z) = E_{0-}e^{jkz}$$

Forward E-field @ z = 0

$$E_{0+}$$

Reverse E-field @ z = 0

$$E_{0-}$$

Algebra

$$E_+(z) = \frac{1}{2}\left[E(z) + \eta H(z)\right]$$

$$E_-(z) = \frac{1}{2}\left[E(z) - \eta H(z)\right]$$

Matrix vector Multiplication

$$\begin{bmatrix} E(z) \\ H(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \eta^{-1} & -\eta^{-1} \end{bmatrix}\begin{bmatrix} E_+(z) \\ E_-(z) \end{bmatrix}$$

$$\begin{bmatrix} E_+(z) \\ E_-(z) \end{bmatrix} = \begin{bmatrix} 1 & \eta \\ 1 & -\eta \end{bmatrix}\begin{bmatrix} E(z) \\ H(z) \end{bmatrix}$$

Suppress z-dep.

$$\begin{bmatrix} E \\ H \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \eta^{-1} & -\eta^{-1} \end{bmatrix}\begin{bmatrix} E_+ \\ E_- \end{bmatrix}$$

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \begin{bmatrix} 1 & \eta \\ 1 & -\eta \end{bmatrix}\begin{bmatrix} E \\ H \end{bmatrix}$$

Wave impedance and reflection

z-dependent Wave impedance

$$Z(z) = \frac{E(z)}{H(z)} = \frac{E}{H} = \frac{E_+ + E_-}{\frac{1}{\eta}[E_+ - E_-]} = \eta \frac{1 + \frac{E_-}{E_+}}{1 - \frac{E_-}{E_+}} = \eta \frac{1 + \Gamma}{1 - \Gamma}$$

z-dependent Reflection coefficient

$$\Gamma(z) = \frac{E_-(z)}{E_+(z)} = \frac{E_-}{E_+} = \frac{\frac{1}{2}[E(z) + \eta H(z)]}{\frac{1}{2}[E(z) - \eta H(z)]} = \frac{\frac{E}{H} - \eta}{\frac{E}{H} - \eta} = \frac{Z - \eta}{Z - \eta}$$

Summarize

$$Z(z) = \eta \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

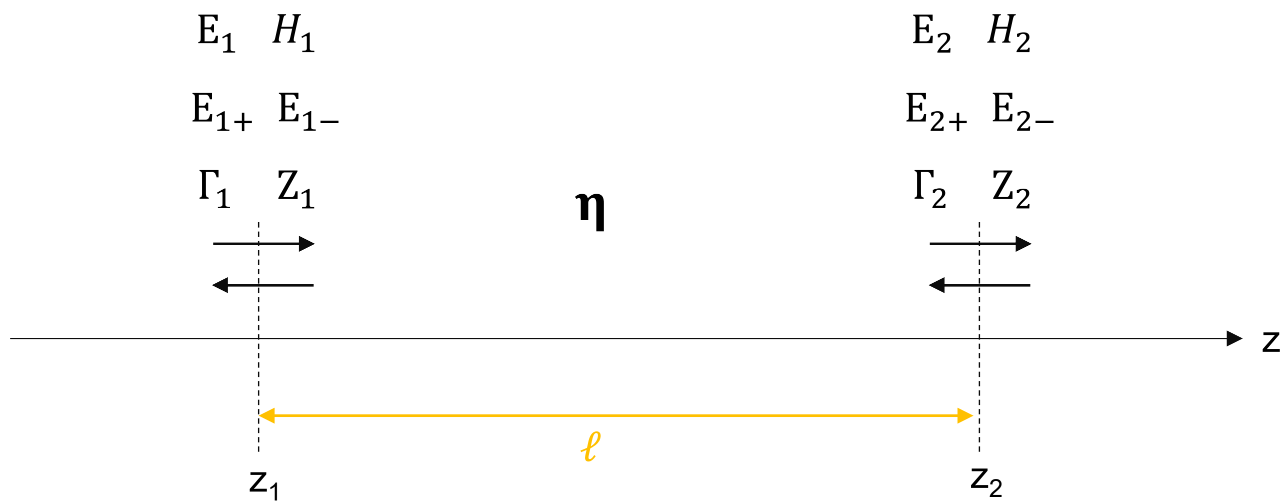
$$\Gamma(z) = \frac{Z(z) - \eta}{Z(z) + \eta}$$

Propagation of reflection coefficient

$$\Gamma(z) = \frac{E_{-}(z)}{E_{+}(z)} = \frac{E_{0+}e^{-jkz}}{E_{0-}e^{jkz}} = \frac{E_{0+}}{E_{0-}}e^{-j2kz} = \Gamma(0)e^{-j2kz}$$

- The reflection coefficient at any point along z is the reflection coefficient at z = 0 modified by twice the propagation phase kz
- Reflection coefficient magnitude is unchanged
- Only reflection coefficient phase is modified

Propagation matrices: two points on z-axis



"Reverse" propagation

$$E_{1+} = e^{jk\ell} E_{2+}$$

"Forward" propagation

$$E_{1-} = e^{-jk\ell} E_{2-}$$

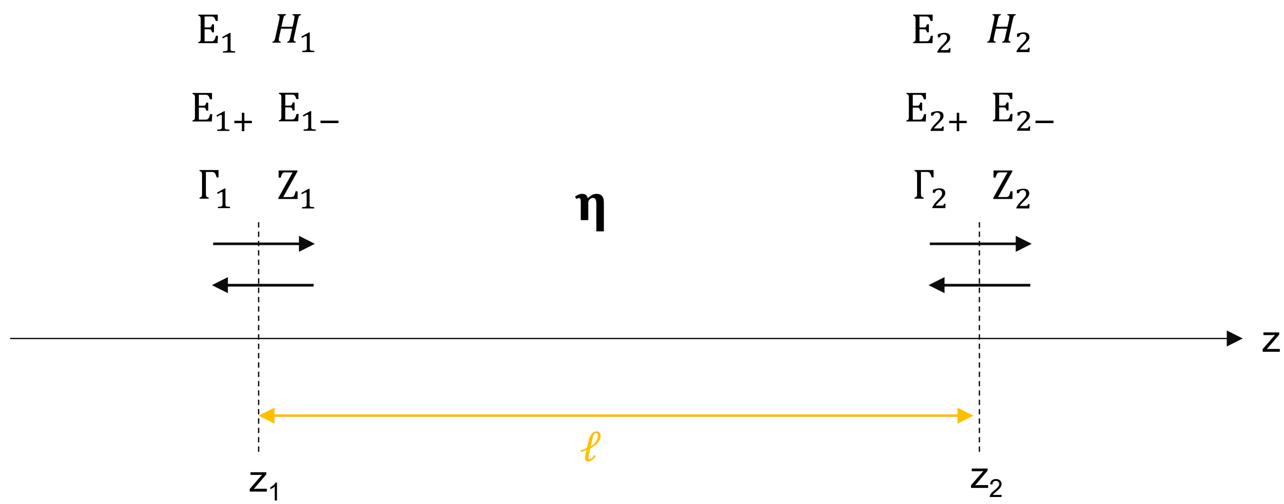
"Forward" propagation

$$E_{2+} = e^{-jk\ell} E_{1+}$$

"Reverse" propagation

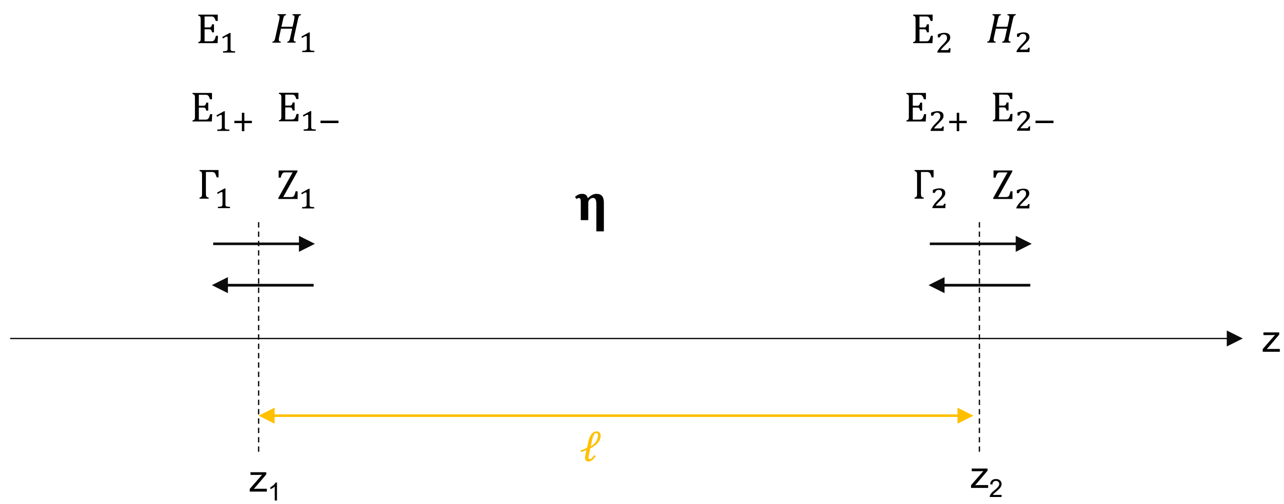
$$E_{2-} = e^{jk\ell} E_{1-}$$

Propagation matrices: two points on z-axis



$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \begin{bmatrix} e^{jk\ell} & 0 \\ 0 & e^{-jk\ell} \end{bmatrix} \begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix} \quad \rightarrow \quad \boxed{\Gamma_1 = \frac{E_{1-}}{E_{1+}} = \frac{E_{2-}e^{-jkz}}{E_{2+}e^{jkz}} = \Gamma_2 e^{-j2kz}}$$

Propagation matrices: two points on z-axis

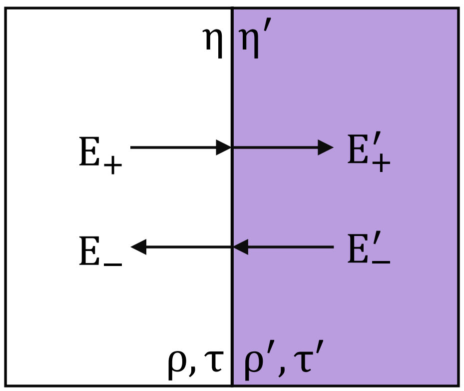


$$\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \eta^{-1} & -\eta^{-1} \end{bmatrix} \begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \eta^{-1} & -\eta^{-1} \end{bmatrix} \begin{bmatrix} e^{jk\ell} & 0 \\ 0 & e^{-jk\ell} \end{bmatrix} \begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix}$$

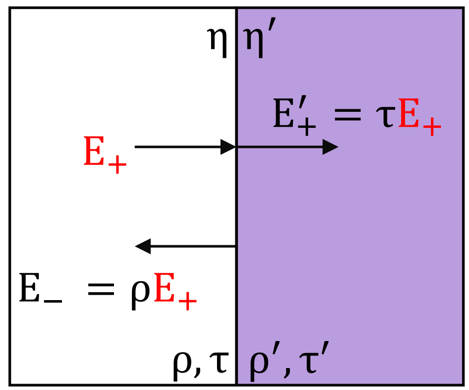
$$\boxed{\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \eta^{-1} & -\eta^{-1} \end{bmatrix} \begin{bmatrix} e^{jk\ell} & 0 \\ 0 & e^{-jk\ell} \end{bmatrix} \begin{bmatrix} 1 & \eta \\ 1 & -\eta \end{bmatrix} \begin{bmatrix} E_2 \\ H_2 \end{bmatrix}}$$

Matching matrices across interfaces

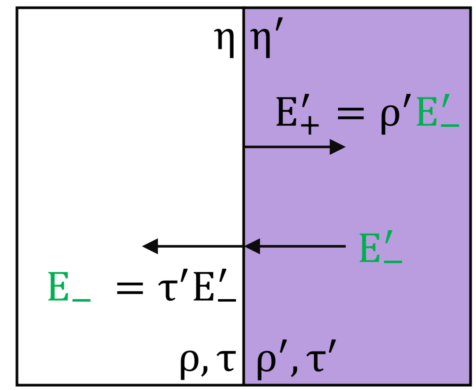
Total fields



+ traveling fields



- traveling fields



Total fields

$$E = E_+ + E_- \qquad E' = E'_+ + E'_-$$

$$H = H_+ + H_- \qquad H' = H'_+ + H'_-$$

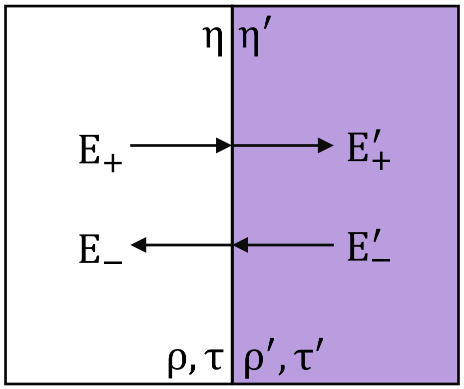
Boundary Conditions

$$\boxed{\begin{aligned} E &= E' \\ H &= H' \end{aligned}}$$

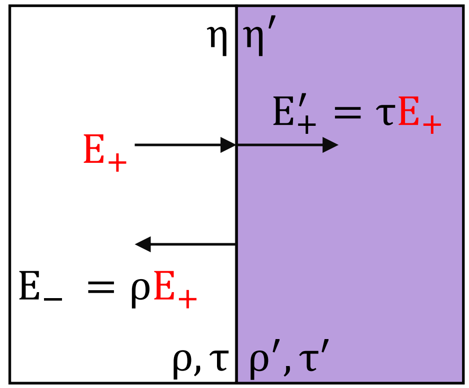
- Lossless dielectric
- No free charge
- TEM wave at normal incidence
  - All fields tangential
- Tangential field continuity

Matching matrices across interfaces

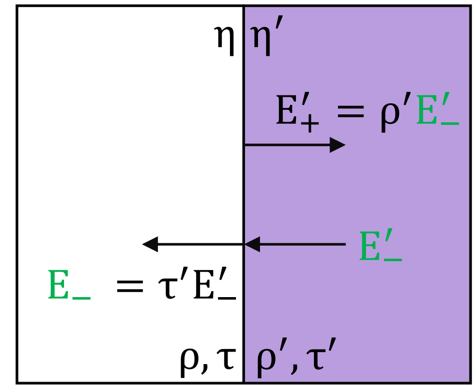
Total fields



+ traveling fields



- traveling fields



$$E_+ + E_- = E'_+ + E'_-$$

$$H_+ + H_- = H'_+ + H'_-$$

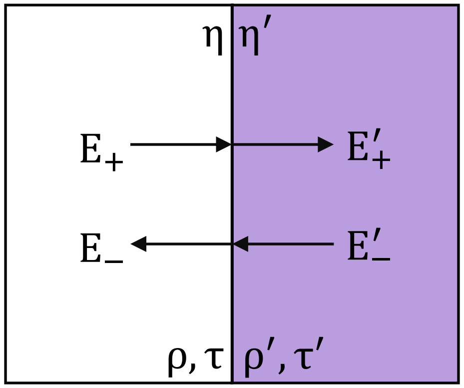
Boundary Conditions

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \frac{1}{\tau} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} E'_+ \\ E'_- \end{bmatrix}$$

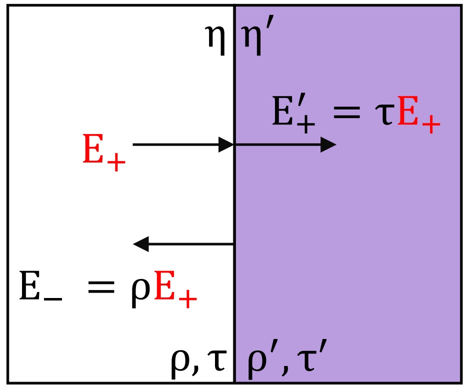
$$\begin{bmatrix} E'_+ \\ E'_- \end{bmatrix} = \frac{1}{\tau'} \begin{bmatrix} 1 & \rho' \\ \rho' & 1 \end{bmatrix} \begin{bmatrix} E_+ \\ E_- \end{bmatrix}$$

Matching matrices across interfaces

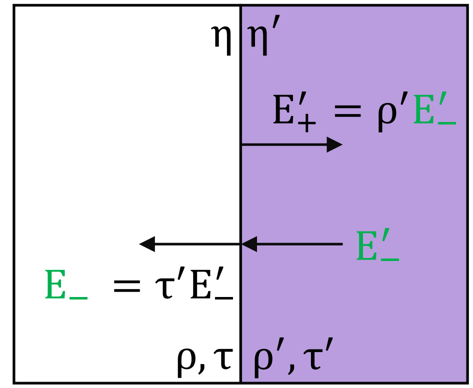
Total fields



+ traveling fields



- traveling fields



$\eta \rightarrow \eta'$

$$\rho = \frac{\eta' - \eta}{\eta' + \eta}$$

$$\tau = \frac{2\eta'}{\eta' + \eta}$$

$\eta' \rightarrow \eta$

$$\rho' = \frac{\eta - \eta'}{\eta + \eta'}$$

$$\tau' = \frac{2\eta}{\eta + \eta'}$$

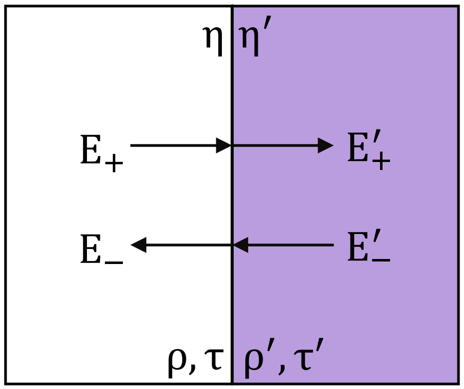
Relations

$$\tau = 1 + \rho \qquad \rho = -\rho'$$

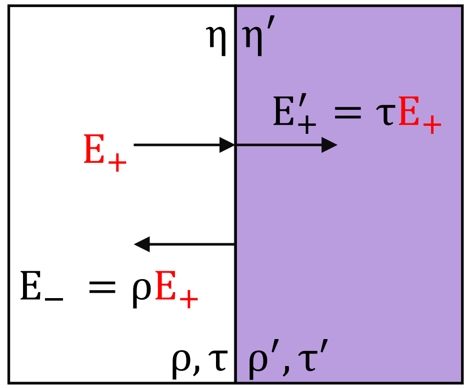
$$\tau' = 1 + \rho' \qquad \tau\tau' = 1 - \rho^2$$

Matching matrices across interfaces

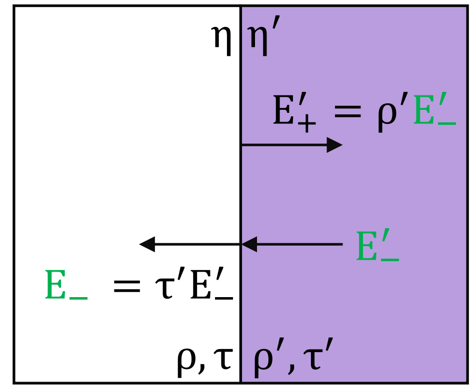
Total fields



+ traveling fields



- traveling fields



Impedance continuity

$$Z = \frac{E}{H} = \frac{E'}{H'} = Z' \quad \xrightarrow{\text{Slide 12}} \quad \eta \frac{1+\Gamma}{1-\Gamma} = \eta' \frac{1+\Gamma'}{1-\Gamma'}$$

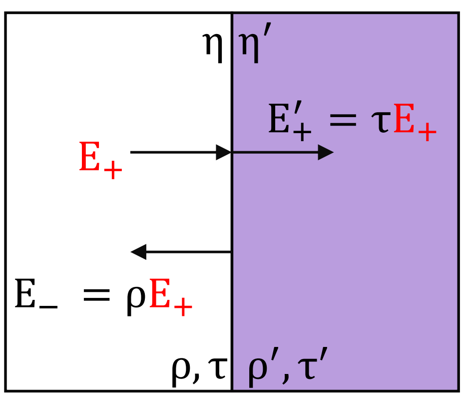
Algebra

$$\Gamma = \frac{\rho + \Gamma'}{1 + \rho \Gamma'}$$

$$\Gamma' = \frac{\rho + \Gamma}{1 + \rho' \Gamma}$$

What if there only an incident field from left?

+ traveling fields



$$E'_- = 0$$

$$\Gamma' = \frac{E'_-}{E'_+} = 0$$

$$Z' = \eta' \frac{1+0}{1-0} = \eta'$$

$$Z = Z' = \eta'$$

$$\Gamma = \frac{\rho + 0}{1 + \rho 0} = \rho$$

Plug it in

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \frac{1}{\tau} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} E'_+ \\ 0 \end{bmatrix}$$

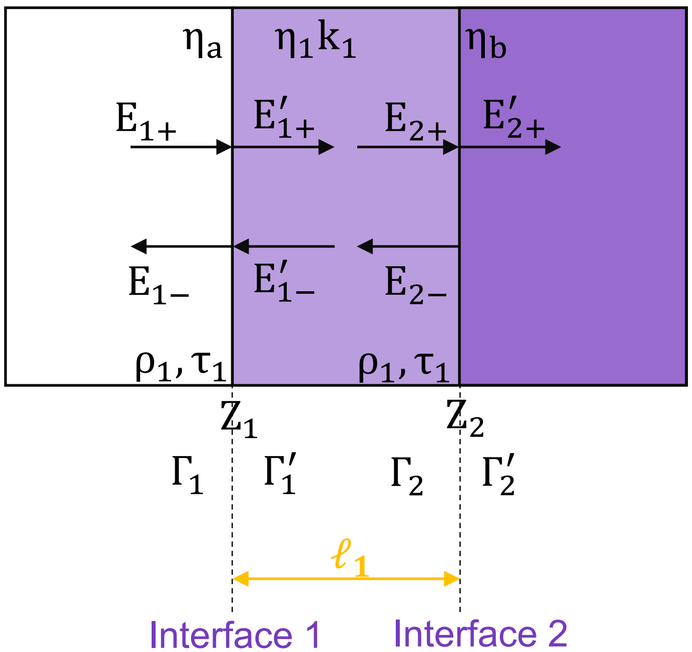
solve

$$E_- = \rho E_+$$

$$E'_+ = \tau E_+$$

- This is the final condition for the dielectric slab
- Once energy enters the half space it does not reflect back

Single Dielectric Slab

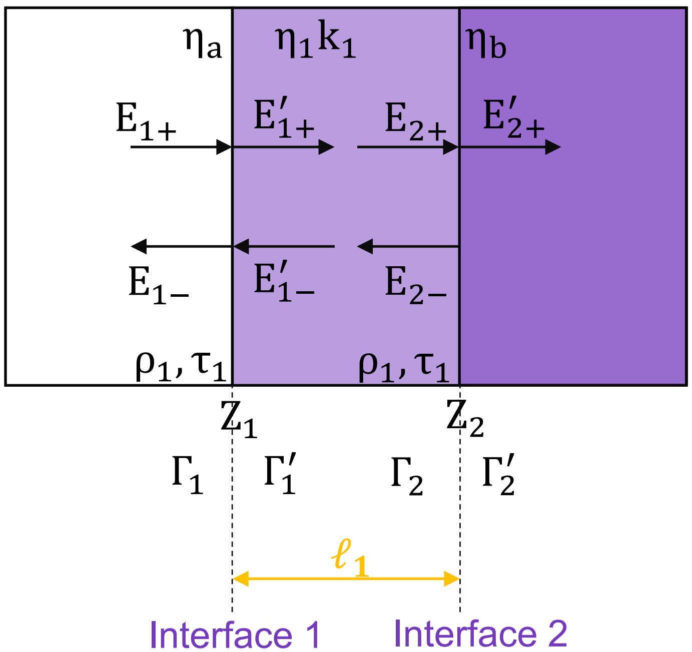


$$\rho_1 = \frac{\eta_1 - \eta_a}{\eta_1 + \eta_a} \qquad \tau_1 = 1 + \rho_1$$

$$\rho_2 = \frac{\eta_b - \eta_1}{\eta_b + \eta_1} \qquad \tau_2 = 1 + \rho_2$$

- Half space of η_a and half space of η_b separated by a slab of dielectric, ℓ_1 thick with η_1
- Field incident from the right
- Find total reflection: $\Gamma_1 = \frac{E_{1-}}{E_{1+}}$

Reflection coefficient propagation



From slide 19

$$\Gamma_1 = \frac{\rho_1 + \Gamma'_1}{1 + \rho_1 \Gamma'_1}$$

From slide 14

$$\Gamma'_1 = \Gamma_2 e^{-j2k_1\ell_1}$$

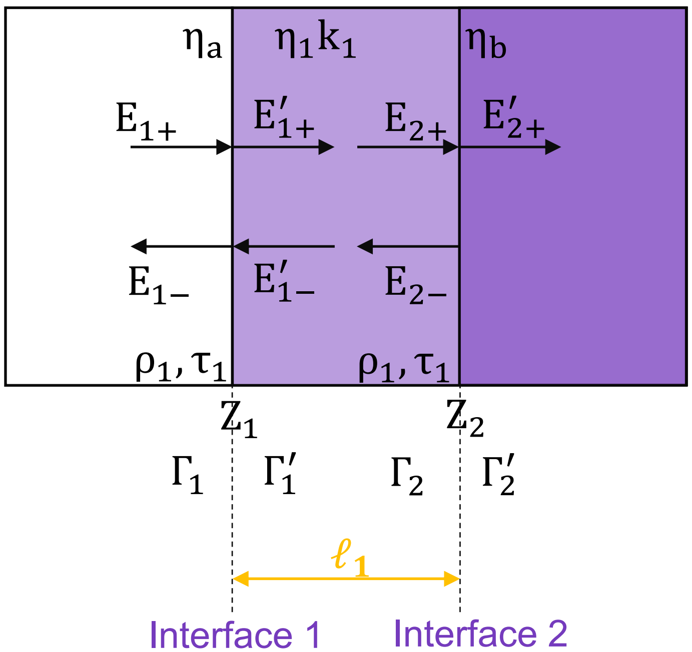
$$\Gamma_1 = \frac{\rho_1 + \Gamma_2 e^{-j2k_1\ell_1}}{1 + \rho_1 \Gamma_2 e^{-j2k_1\ell_1}}$$

From slide 20

$$\Gamma_2 = \rho_2$$

$$\boxed{\Gamma_1 = \frac{\rho_1 + \rho_2 e^{-j2k_1\ell_1}}{1 + \rho_1 \rho_2 e^{-j2k_1\ell_1}}}$$

Matching and propagation matrices (1/2)



From slide 17

$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \frac{1}{\tau_1} \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} E'_{1+} \\ E'_{1-} \end{bmatrix}$$

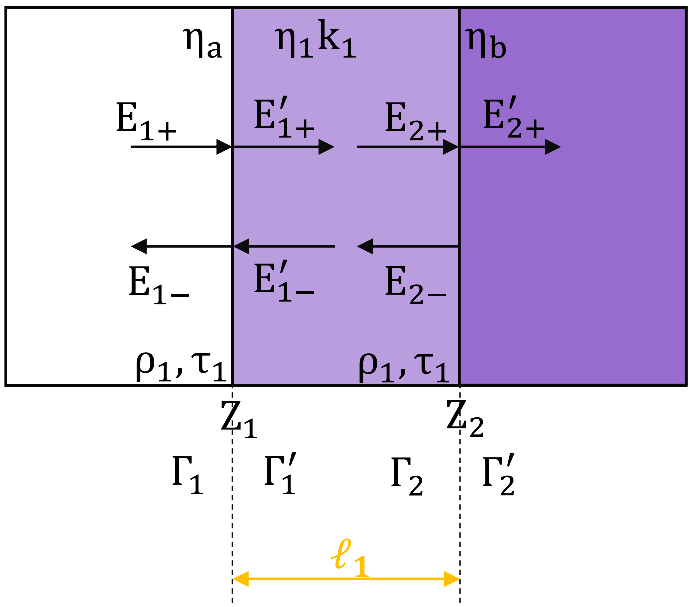
From slide 14

$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \frac{1}{\tau_1} \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} e^{jk_1\ell_1} & 0 \\ 0 & e^{-jk_1\ell_1} \end{bmatrix} \begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix}$$

From slide 17

$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \frac{1}{\tau_1} \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} e^{jk_1\ell_1} & 0 \\ 0 & e^{-jk_1\ell_1} \end{bmatrix} \frac{1}{\tau_2} \begin{bmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{bmatrix} \begin{bmatrix} E'_{2+} \\ E'_{2-} \end{bmatrix}$$

Matching and propagation matrices (2/2)



From slide 20: $\mathbf{E'_{2-} = 0}$

$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \frac{1}{\tau_1}\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}\begin{bmatrix} e^{jk_1\ell_1} & 0 \\ 0 & e^{-jk_1\ell_1} \end{bmatrix}\frac{1}{\tau_2}\begin{bmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{bmatrix}\begin{bmatrix} E'_{2+} \\ 0 \end{bmatrix}$$

Multiply through

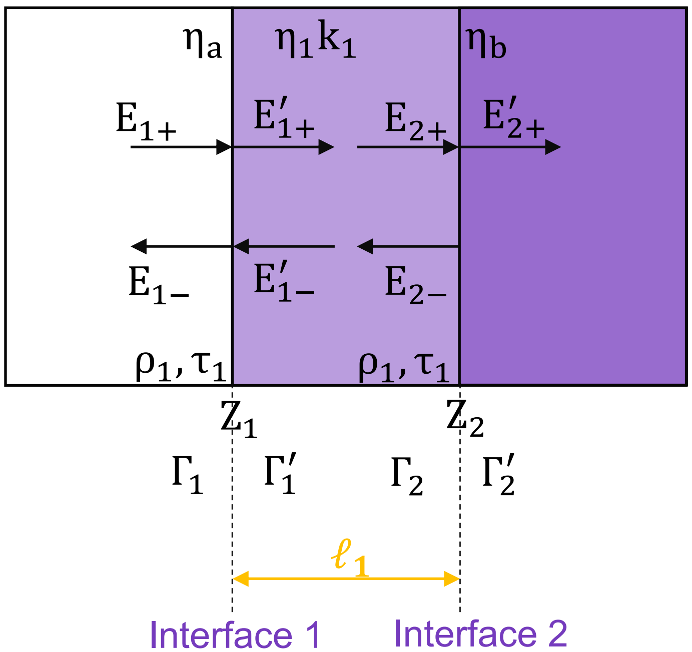
$$E_{1+} = \frac{e^{jk_1\ell_1}}{\tau_1\tau_2}\left(1 + \rho_1\rho_2 e^{-j2k_1\ell_1}\right)E'_{2+}$$

ratio

$$E_{1-} = \frac{e^{jk_1\ell_1}}{\tau_1\tau_2}\left(\rho_1 + \rho_2 e^{-j2k_1\ell_1}\right)E'_{2+}$$

$$\boxed{\Gamma_1 = \frac{E_{1-}}{E_{1+}} = \frac{\rho_1 + \rho_2 e^{-j2k_1\ell_1}}{1 + \rho_1\rho_2 e^{-j2k_1\ell_1}}}$$

Transmission



From slide 21: $\mathbf{E}'_{2-} = \mathbf{0}$

$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \frac{1}{\tau_1} \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} e^{jk_1\ell_1} & 0 \\ 0 & e^{-jk_1\ell_1} \end{bmatrix} \frac{1}{\tau_2} \begin{bmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{bmatrix} \begin{bmatrix} E'_{2+} \\ 0 \end{bmatrix}$$

$$E_{1+} = \frac{e^{jk_1\ell_1}}{\tau_1\tau_2}\left(1 + \rho_1\rho_2 e^{-j2k_1\ell_1}\right)E'_{2+}$$

$$\boxed{T_1 = \frac{E'_{2+}}{E_{1+}} = \frac{\tau_1\tau_2 e^{-jk_1\ell_1}}{1 + \rho_1\rho_2 e^{-j2k_1\ell_1}}}$$

Conclusions and next time

- Boundary conditions (field continuities) allow for a solution to the steady state time harmonic total fields
- $E_+(z)$, $E_-(z)$, and $\Gamma(z)$ have **simple propagation matrices** but **complicated relations across the interface**
- $E(z)$, $E(z)$, and $Z(z)$ have **complicated propagation matrices** but **simple relations across the interface**
- The matrix from provides a natural form for recursion and enables solution of the total transmitted field
- Next time we'll
  - Solve for the reflection via geometric series (ray tracing)
  - Extrapolate to multiple layers
  - Explore oblique incidence angle