Linear Programming Modelling

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Optimization modelling

- The problem is defined in *implicitly* terms of
  - an **Objective function** to minimize of maximize
  - by choosing optimal values for **decision variables**
  - subject to **constraints**
- Optimization software solves the problem automatically
  - This approach is a dramatically different from *explicit* (simulation) models where the result is obtained by applying some formulas in given order
- Most common optimization model types:
  - Linear Programming (LP) problem
  - Mixed Integer Linear Programming (MILP) problem

Optimization problem example

- Sample problem with two variables

$$\min x_1^2 + x_2^2$$

s.t.

$$x_1 + x_2 \geq 3$$

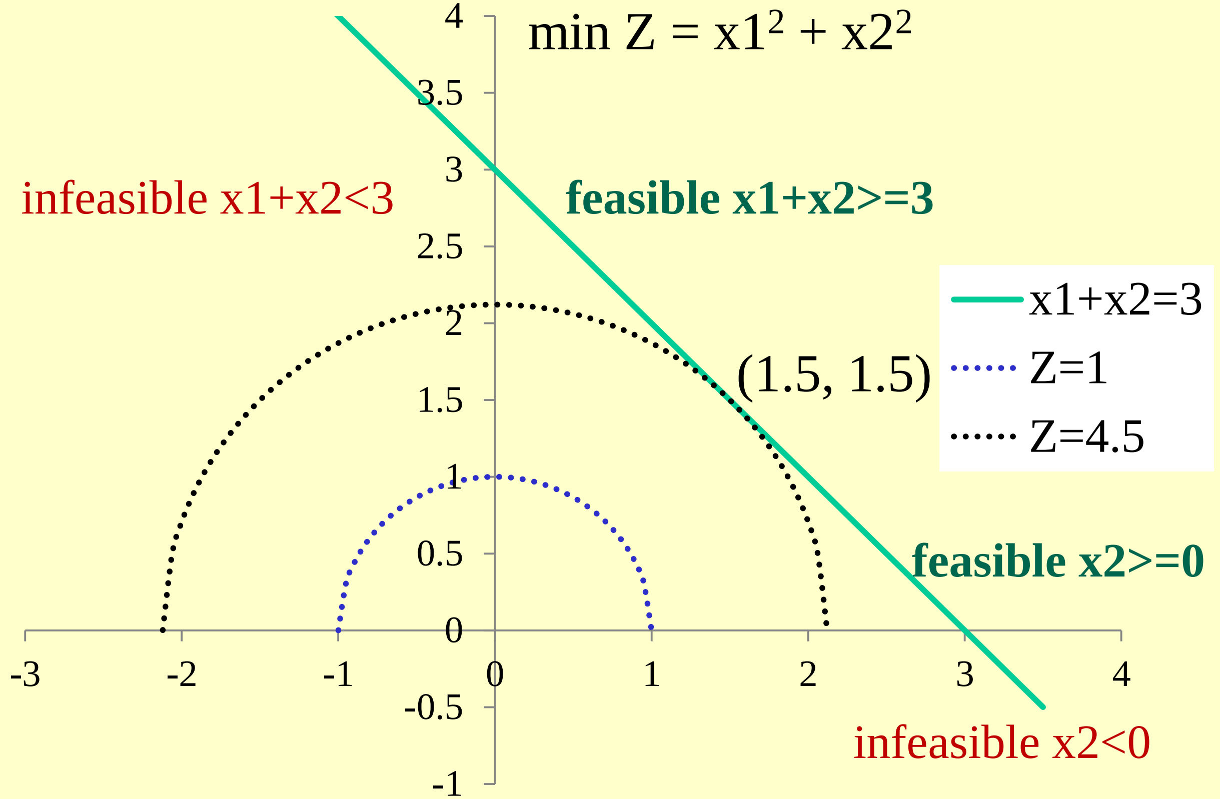
$$x_2 \geq 0$$

$$x_1, x_2 \in R$$

- In a two-dimensional case the problem can be illustrated and solved graphically
  - Constraints define the feasible region in the plane
  - Level curves of the objective function show the *height of the terrain*

Optimization problem example

- **Constraints** define the feasible region in the plane
- **Level curves** of the objective function show the height of the terrain



General mathematical optimization problem

Find values for ***decision variables*** **x** that minimize or maximize the ***objective function*** f(**x**) subject to ***constraints***:

$\min f(\mathbf{x})$	(objective function)
subject to	
$\mathbf{h}(\mathbf{x}) = \mathbf{0}$	(vector of equality constraints)
$\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$	(vector of inequality constraints)
$\mathbf{x} \in R^n$ (or $\mathbf{x} \in N^n$)	(vector of decision variables)

If domain of **x** is R^n, it is a ***continuous optimization problem***

If all x_i are integers, it is an ***integer optimization problem***

A ***mixed integer problem*** contains both integer and real x_i

Properties of optimization problems

- Consider general problem min (max) f($\mathbf{x}$) s.t. $\mathbf{x} \in X$
- A particular solution $\mathbf{x} = \mathbf{x}^*$ is
  - ***feasible*** if it satisfies all constraints (i.e. $\mathbf{x}^* \in X$)
  - ***infeasible*** if it does not satisfy all constraints
  - ***optimal*** if it is feasible and minimizes (maximizes) f($\mathbf{x}$)
- The ***problem*** is
  - ***feasible*** if at least one feasible solution exists
  - ***infeasible*** if no feasible solution exists
  - ***unbounded*** if infinitely good feasible solutions exist
- The problem can have
  - no optimal solutions: when the problem is infeasible or unbounded
  - one unique optimal solution
  - multiple (equally good) optimal solutions

Optimization model transformations

- Transformations
  - $\min f(\mathbf{x}) = -\max -f(\mathbf{x})$;
  - $\max f(\mathbf{x}) = -\min -f(\mathbf{x})$
  - $g(\mathbf{x}) \le 0 \Leftrightarrow -g(\mathbf{x}) \ge 0$
  - $g(\mathbf{x}) = 0 \Leftrightarrow g(\mathbf{x}) \le 0 \wedge g(\mathbf{x}) \ge 0$
  - $g(\mathbf{x}) \le 0 \Leftrightarrow g(\mathbf{x}) + s^2 = 0$ where s is an unconstrained real variable
- Constrained problem can be transformed into unconstrained by augmenting objective with a penalty term, i.e. a ***barrier function***
  - $\min f(\mathbf{x})$ s.t. $g(\mathbf{x}) \le 0 \Leftrightarrow \min f(\mathbf{x})+M\cdot\max\{\mathbf{g}(\mathbf{x}),0\}$
    - M is a big positive number

Optimization model types

- Depending on the structure of the objective function and constraints, optimization models can be classified in different ways
  - Single variable and multiple variables
  - Continuous, discrete or mixed integer problems
    - Decision variables are continuous, binary (0/1), general integers, or mixed
    - Integer programming, mixed integer programming
  - Unconstrained and constrained problems
  - Convex and non-convex problems
    - Linear, quadratic and nonlinear problems
  - Single objective and multi-objective problems
    - $\mathbf{f}(\mathbf{x})$ is a vector of objective functions

Optimization model types – Exampes

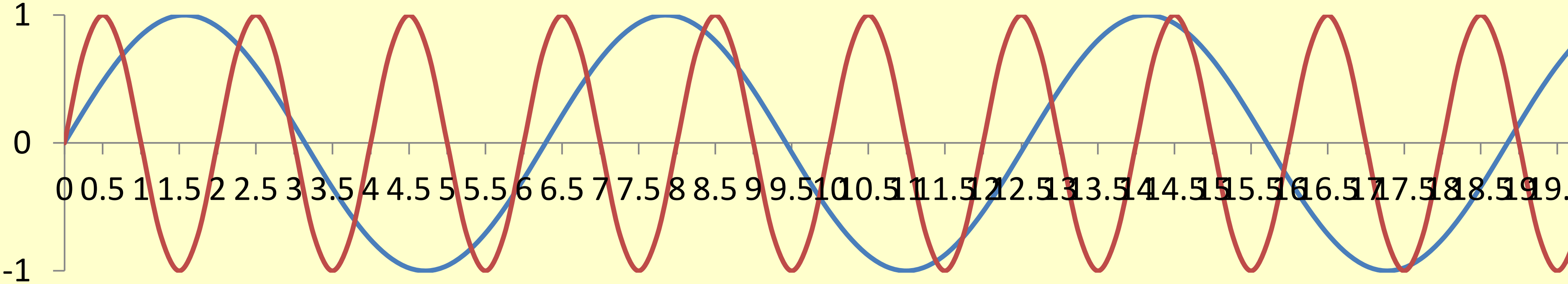
- Sizing of ground source heat pump
  - Single objective (minimize life-cycle costs)
  - Single continuous variable (size of pump)
  - Constrained non-linear convex problem
- Unit commitment of power plants
  - Single objective (maximize profit)
  - Multiple variables of mixed types
  - Constrained non-convex problem
- Investment in new production technology
  - Multiple objectives (economic, environmental, policy, …)
  - Multiple discrete (binary) variables
  - Constrained or unconstrained problem

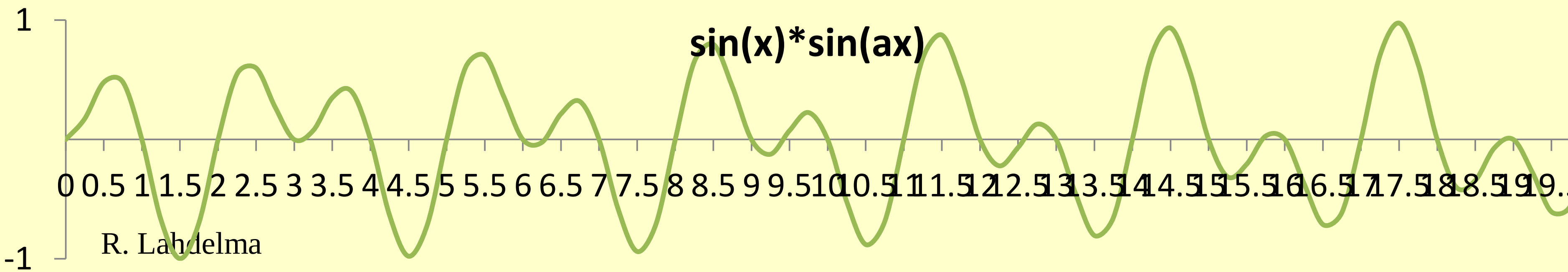
Solving optimization problems

- To solve problems it is necessary to understand the different problem types and their properties
  - There is no universal way to find the optimum or even a feasible solution to an arbitrary problem
  - Different solution algorithms are required for different problem types
- **Most important is to determine if the optimization problem is convex or not!**
  - Convex problem = minimize convex objective function in a convex region
  - Convex problem: relatively easy
  - Non-convex problem: potentially very hard

Impossible to solve non-convex model

- Consider max/min f(x)=sin(x)*sin(ax)
  - Each factor has max/min at +1/-1
  - If the peaks and valleys coincide then f(x)=+1 or -1
  - If a is chosen properly, peaks and valleys never meet
  - No optimum, values approaching +1/-1





Real-life optimization problems

- A real-life model differs from theoretical models in several aspects
  - Normally the problem is never unbounded
  - The existence of a feasible solution can often be verified intuitively
  - Often many model parameters are uncertain or imprecise
  - It is not necessary to find the true optimum – a near-optimal solution and sometimes even a reasonably good solution may suffice

LP and MILP modelling

- ***Linear Programming*** and ***Mixed Integer Linear Programming*** are most commonly used approaches for practical problems because
  - the modelling techniques are very versatile and flexible
  - efficient and reliable solvers exist for these problems
- Arbitrary convex optimization problems can be approximated by LP models
- Many non-convex optimization problems can be approximated by MILP models

LP modelling

- By far, the most commonly used optimization modelling technique
  - Applicable for a wide class of different problems
  - Easy to formulate
  - Easy to understand
  - Very large models can be solved efficiently
  - Interpretation of results and various sensitivity analyses are (relatively) easy
- Many energy optimization problems can be represented as LP models
  - Why can LP modelling not always be used?

Applicability of LP models

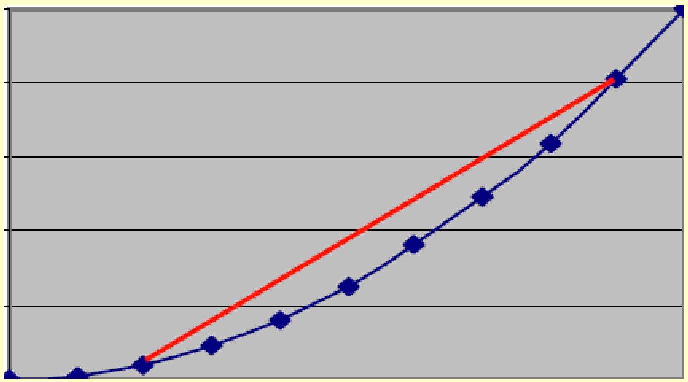
- LP models work only in convex problems
  - The **minimization problem is convex** when:
    - The minimized objective function is **convex**
    - The feasible region is **convex**
  - The **maximization problem is convex** when:
    - The maximized objective function is **concave**
    - The feasible region is **convex**
  - An LP model is a piecewise linear convex model
- How can non-convex problems be modelled?

Convex optimization problem

- A convex optimization problem is of form
  min f(x); s.t. $\mathbf{x} \in X$
  - where f() is a convex function and
  - X is a convex set
- Similarly max f($\mathbf{x}$) s.t. $\mathbf{x} \in X$ where f() is a concave function is a convex optimization problem
- The feasible region X is a convex set when
  - functions in inequality constraints $g(\mathbf{x}) \leq 0$ are convex and
  - functions in equality constraints $h(\mathbf{x}) = 0$ are linear.

Convex and concave functions

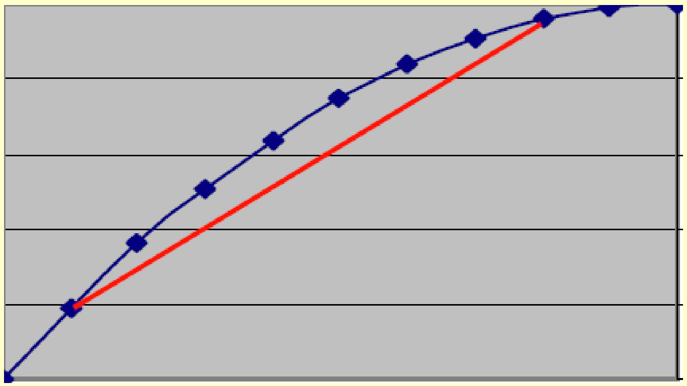
- A function f(x) is **convex** if linear interpolation between any two points x and y does not yield a lower value than the function



- Mathematically

$$f(\alpha x+(1-\alpha)y) \leq \alpha f(x)+(1-\alpha)f(y)$$
$$\text{for all } x, y \text{ and } \alpha \in [0,1]$$

- A function f(x) is **concave** if linear interpolation between any two points x and y does not yield a higher value than the function

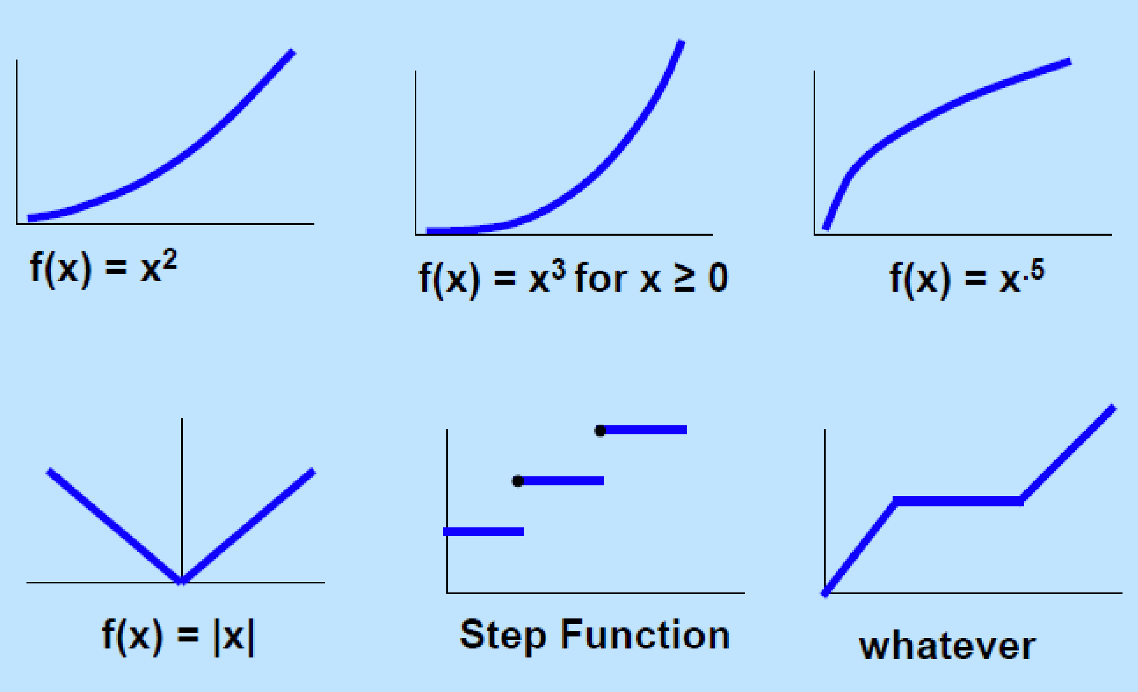


- Mathematically

$$f(\alpha x+(1-\alpha)y) \geq \alpha f(x)+(1-\alpha)f(y)$$
$$\text{for all } x, y \text{ and } \alpha \in [0,1]$$

Convex and concave functions

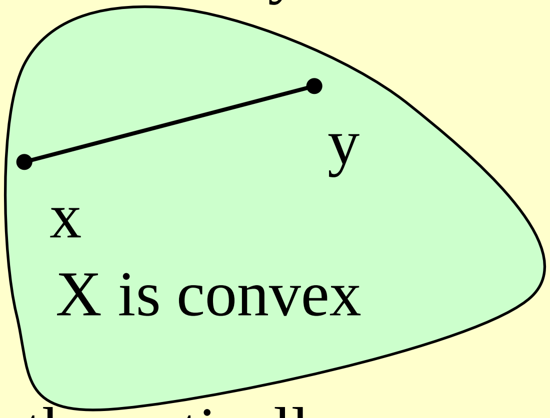
- Which functions are convex and which are concave?

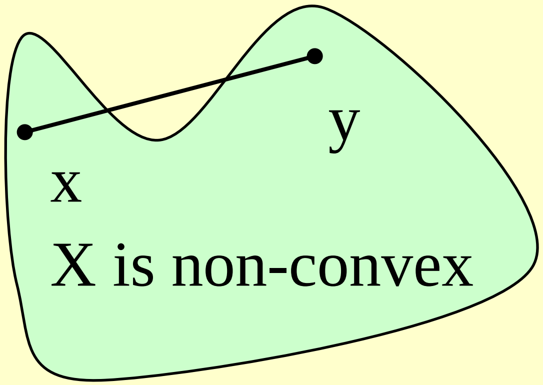


  - Some functions are neither convex nor concave
  - If f(x) is convex, then –f(x) is concave and vice versa
  - **Only linear functions are both convex and concave**

Convex set

- A set X is **convex** if the line segment connecting any two points x and y of the set is in the set





- Mathematically
  - If $\mathbf{x},\mathbf{y} \in X$, then $\alpha\mathbf{x}+(1-\alpha)\mathbf{y} \in X$ for all $\alpha \in [0,1]$
- A constraint $g(\mathbf{x}) \leq 0$ defines a convex set if $g(\mathbf{x})$ is a convex function.
- The intersection of convex sets is a convex set
  - Thus multiple constraints $g_i(\mathbf{x}) \leq 0$ with convex functions $g_i(\mathbf{x})$ define a convex set

Convex optimization problems

- Convex optimization problems are relatively easy to solve because
  - A local optimum is also a global optimum
  - They can be solved using hill-climbing strategy: starting from any feasible point move in a direction where f(x) improves while maintaining feasibility
  - If the functions f(), **g**(), **h**() are smooth (first derivatives are continuous), various gradient-based methods can be used to identify improving directions
- Non-convex problems are difficult, because a local optimum is not in the general case a global optimum

Linear programming (LP) models

- An LP model has a linear objective function f(**x**) and linear constraints $g_i(\mathbf{x})$:

$$\min\ (\max)\ c_1x_1 + c_2\, x_2 + \ldots\, c_nx_n$$

s.t.

$$a_{11}x_1 + a_{12}x_2 + \ldots\, a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\, a_{2n}x_n \leq b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\, a_{mn}x_n \leq b_m$$

- Typical matrix representation:

$$\min\ (\max)\ \mathbf{cx}$$

s.t.

$$A\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \geq 0$$ // traditionally variables are non-negative

Linear programming (LP) models

- Special case of convex problems
  - f(**x**), **g**(**x**) and **h**(**x**) are linear functions of **x**
  - The constraints are (hyper-) planes in *n* dimensions
  - The feasible area is an *n*-dimensional polyhedron
  - The optimum is at a corner point at the intersection between some constraint planes
- Very efficient solution algorithms for LP models exist
  - The **Simplex algorithm** can solve LP models with millions of variables and constraints
- Non-linear convex problems can be approximated by LP models with arbitrarily good accuracy
- Non-convex problems cannot be represented as LP models

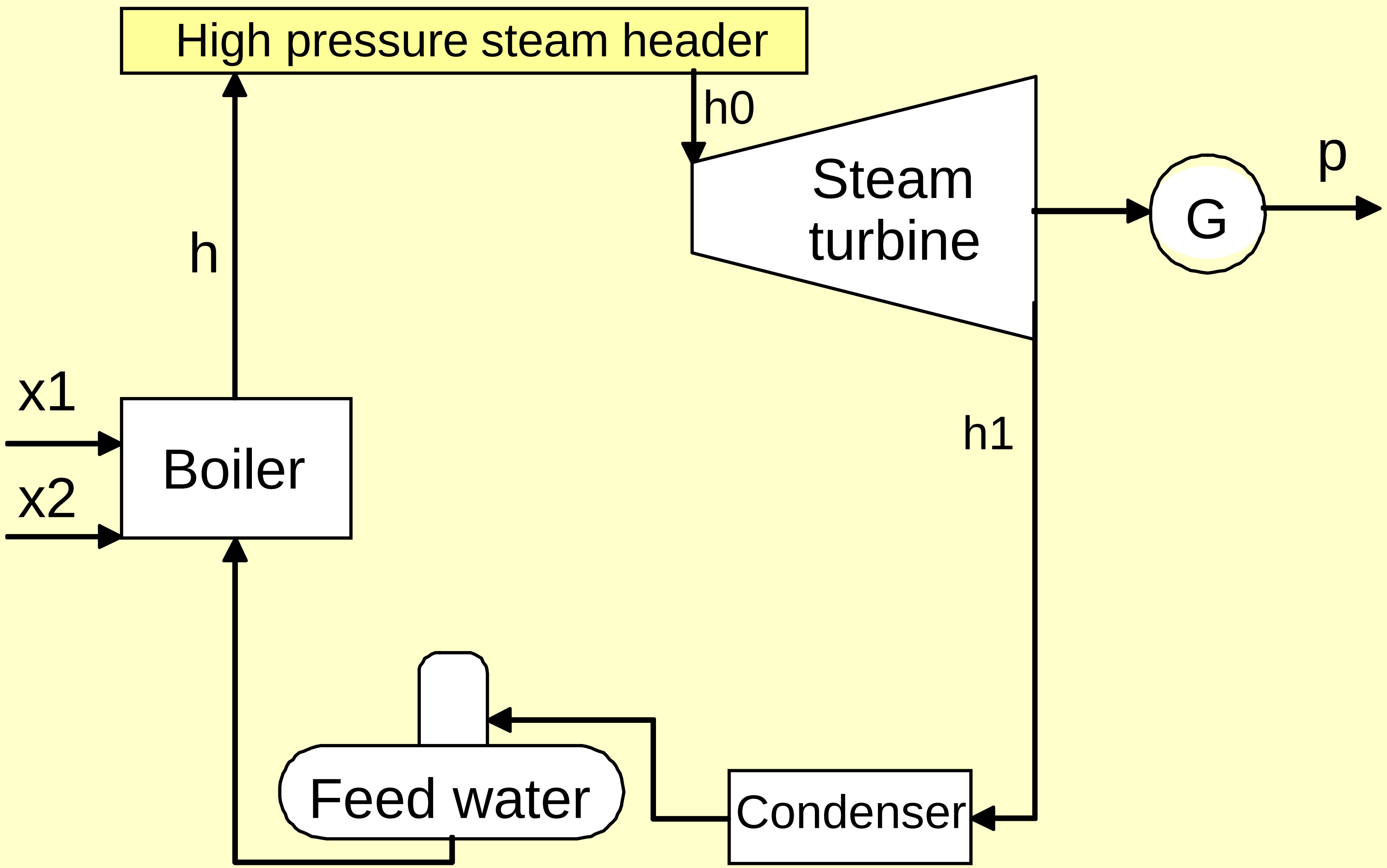
How to define an LP model?

1. Write down a verbal explanation of what is the goal or purpose of the model
   – E.g. to minimize costs or maximize profit from some specific operation or activity
2. Define the **decision variables** (and parameters)
   – Use as descriptive or generic names as you like: x1, x2, fuel, …
   – Give short description for them
   – Also specify the unit (MWh, GJ, €/kg, m3/s, …)
3. Define the **objective function** to minimize or maximize as a *linear function* of the variables
4. Define the **constraints** as *linear* inequality or equality constraints for the variables

LP example: Dual fuel condensing power plant

- Boiler can use two different fuels simultaneously in any proportion
- Boiler produces high pressure steam for a turbine driving a generator to produce electricity
- After turbine, steam is condensed back into water
- Fuels have different prices and consumption ratios
- Produced power is sold to market
- Typical objective is to **maximize profit** = revenue from selling power minus fuel costs

Dual fuel condensing power plant



LP example: Dual fuel condensing power plant

- Maximize profit during each hour of operation
- Decision variables
  - x1, x2 fuel consumption (MWh)
  - p power output (MWh)
- Parameters
  - r1, r2 consumption ratios for fuels (1)
  - c1, c2, c price for fuels and power (€/MWh)
  - $x1^{max}$, $x2^{max}$ upper bounds for fuel consumption (MWh)
  - b hourly maximal production capacity (MWh)

LP example: Dual fuel condensing power plant

- Objective function

max $c*p - c1*x1 - c2*x2$ // power sales minus fuel cost

- Constraints

$p = x1/r1 + x2/r2$ // power depends production

$p \le b$ // capacity limit

$x1 \le x1^{max}$, $x2 \le x2^{max}$, $x1, x2 \ge 0$

- Substitute expression for p to eliminate third variable

max $(c/r1-c1)*x1 + (c/r2 - c2)*x2$

$x1/r1 + x2/r2 \le b$ // capacity limit

$x1 \le x1^{max}$, $x2 \le x2^{max}$, $x1, x2 \ge 0$

LP example: Dual fuel condensing power plant, numerical

- Parameters
  - Fuel consumption ratios $(r1, r2) = (3.33, 2.5)$
  - Fuel & power prices $(c1, c2, c) = (20, 25, 80)$ €/MWh
  - Upper bounds for fuels $(x1^{max}, x2^{max}) = (150,100)$ MWh
  - Production capacity $b = 60$ MWh

$$\max\ (80/3.33-20)*x1 + (80/2.5-25)*x2 = 4*x1 + 7*x2$$

$$0.3*x1 + 0.4*x2 \leq 60$$

$$x1 \leq 150$$

$$x2 \leq 100$$

$$x1, x2 \geq 0$$

Graphical representation of LP models

- Models with two variables can be represented and solved graphically
  - Linear constraints are drawn as lines
    - The feasible region appears as a polygon
    - The feasible region may be unbounded in some direction
    - If the constraints are contradictory, the feasible region is empty and the model is infeasible
  - Level curves of objective function f(x) = K = constant are draw as dotted lines
    - Optimum is where a level curve touches the feasible region with with maximal or minimal K
    - This happens at some corner
    - If two corners yield optimal value, all points on the connecting edge are optimal (infinite number of optima)

LP example:
Power plant model, graphical representation

max 4*x1 + 7*x2

0.3*x1 + 0.4*x2 ≤ 60

x1 ≤ 150

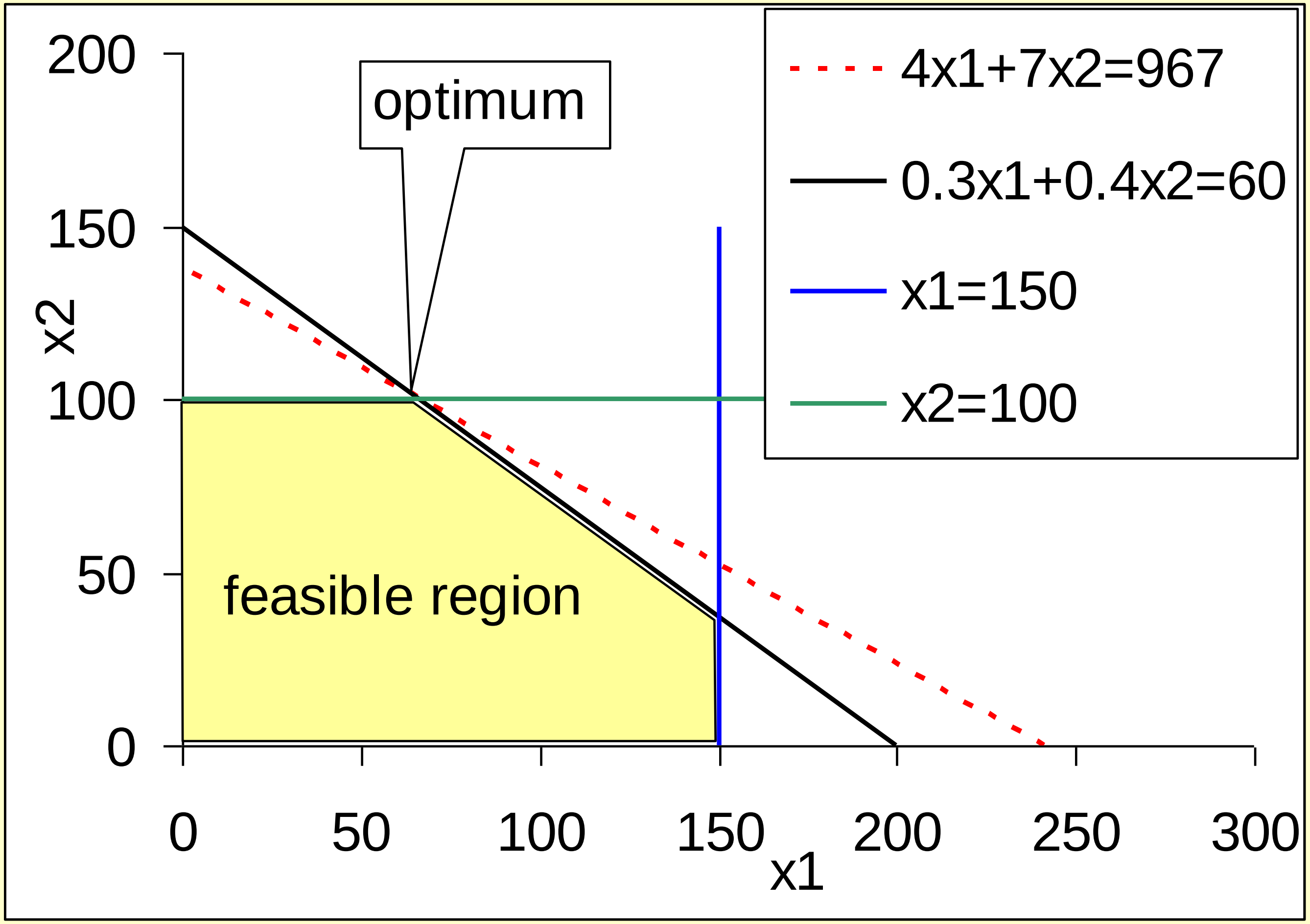
x2 ≤ 100

x1, x2 ≥ 0

Optimum at

x2=100

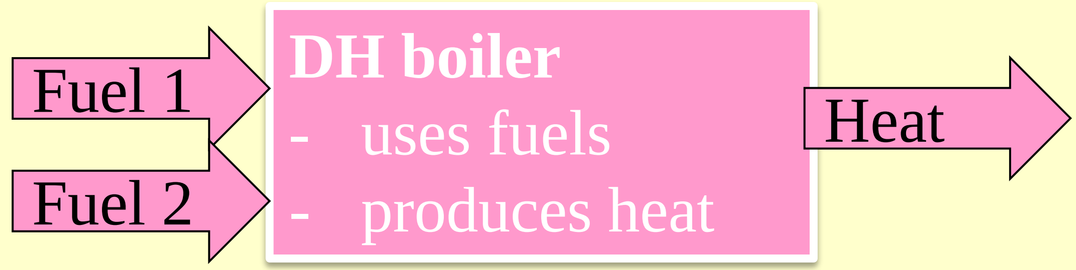
x1=66.7



Properties of LP models

- Similar to a general optimization problem, an LP problem can be
  - **Feasible**, if one or more feasible solutions exist
  - **Infeasible**, if no feasible solutions exist, i.e. constraints are conflicting
    - Example: min 0 s.t. $x \geq 1$, $x \leq 0$
  - **Unbounded**, if infinitely good solutions exist
    - Example: max x s.t. $x \geq 0$
- An LP problem has infinite number of optima if two or more corner solutions yield optimal value
  - Then all convex combinations of optimal corner solutions are optimal

LP example: DH boiler



- A dual fuel boiler to produce district heat
  - Goal to meet demand (MWh) as cheaply as possible
  - Decision variables
    - x1, x2 fuel consumption (MWh)
  - Parameters
    - r1, r2 consumption ratios for fuels (1)
    - c1, c2 prices for fuels (€/MWh)
    - $x1^{max}$, $x2^{max}$ upper bounds for fuel consumption (MWh)
    - b demand of heat

$\min c1*x1 + c2*x2$

$x1/r1 + x2/r2 \geq b$ // allowed to produce excess

$x1 \leq x1^{max}$, $x2 \leq x2^{max}$, $x1, x2 \geq 0$

LP example: DH boiler, numerical example

- Parameters
  - Fuel consumption ratios (r1, r2) = (1.25, 1.11)
  - Fuel prices (c1, c2) = (20, 25) €/MWh
  - Upper bounds for fuels ($x1^{max}$, $x2^{max}$) = (150,100) MWh
  - Heat demand b = 120

min 20*x1 + 25*x2;

0.8*x1 + 0.9*x2 ≥ 120;

x1 ≤ 150;

x2 ≤ 100;

x1, x2 ≥ 0;

Solving LP problems – canonical form

- The simplex algorithm for LP problems is based on solving linear equation systems
  - First the problem is reformulated into *canonical form*, where all constraints are of equality type

min (max) **cx**		min (max) **cx**
s.t.	→	s.t.
$A\mathbf{x} \leq \mathbf{b}$		$A\mathbf{x} + \mathbf{s} = \mathbf{b}$
$\mathbf{x} \geq 0$		$\mathbf{x},\mathbf{s} \geq 0$

  - $\mathbf{s} = [s_1, s_2, ..., s_m]^T$ is a vector of *slack* variables
  - Greater than –type equations get *surplus* variables

$A\mathbf{x} \geq \mathbf{b} \quad \rightarrow \quad A\mathbf{x} - \mathbf{s} = \mathbf{b} \quad \rightarrow \quad -A\mathbf{x} + \mathbf{s} = -\mathbf{b}$

Solving LP problems – canonical form

– In canonical form, the LP problem can be rewritten as

$\min (\max)\ \mathbf{cx}$

s.t.

$A\mathbf{x} = \mathbf{b}$

$\mathbf{x} \geq 0$

– The new A-matrix contains the original A and an identity matrix $A = [A|I]$

– The new **x**-vector contains the original decision variables and the slacks $\mathbf{x}^T = [\mathbf{x}^T|\mathbf{s}^T]$

– The new **c**-vector contains the original **c** and zeros as cost coefficients for the slacks

Solving LP problems – canonical form

- The original problem contained *m* constraints and *n* variables
  - In canonical form the problem contains *m* constraints and *m*+*n* variables
    - *n* original decision variables and *m* slacks
    - The A-matrix is more wide than tall
  - Thus, there are more variables than equations
- Such an *underdetermined system* has in general an infinite number of solutions
  - The idea is to fix *n* of the variables to zero (their lower bounds) and solve the remaining *m* variables from the *m* equations

Solving LP problems – basic solutions

- The Simplex algorithm explores ***basic solutions*** of the equation system
  - A **basis** is a set of *m* linearly independent columns of the A-matrix
  - We partition A = [B|N] where B is the basis and N is the non-basic part
  - **x** is partitioned similarly into basic variables $\mathbf{x}^B$ and non-basic $\mathbf{x}^N$
  - **c** is partitioned similarly into $\mathbf{c}^B$ and $\mathbf{c}^N$
- The problem is rewritten as

min (max) $\mathbf{c}^B\mathbf{x}^B + \mathbf{c}^N\mathbf{x}^N$

s.t.

$B\mathbf{x}^B + N\mathbf{x}^N = \mathbf{b}$

$\mathbf{x}^B, \mathbf{x}^N \geq 0$

Solving LP problems – basic solutions

$\min (\max)\ \mathbf{c}^B\mathbf{x}^B + \mathbf{c}^N\mathbf{x}^N$

s.t.

$B\mathbf{x}^B + N\mathbf{x}^N = \mathbf{b}$

$\mathbf{x}^B, \mathbf{x}^N \geq 0$

- A basic solution is obtained by setting $\mathbf{x}^N = 0$ and solving

$\mathbf{x}^B = B^{-1}(\mathbf{b} - N\mathbf{x}^N) = B^{-1}\mathbf{b}$

- Basic solutions correspond to ***corner points***, i.e. intersections between constraint equations
  - When a slack is non-basic (zero) the constraint is active (equality holds)
  - When a slack is non-zero, the constraint is inactive (strict inequality)
  - A basic solution is *feasible* if (and only if) $\mathbf{x}^B \geq 0$

Solving LP problems – basic solution example

- The power plant problem in canonical form

$$\begin{aligned} \max\ & 4*x1 + 7*x2; \\ & 0.3*x1 + 0.4*x2 \le 60 \\ & x1 \le 150 \\ & x2 \le 100 \\ & x1, x2 \ge 0 \end{aligned} \Rightarrow \begin{aligned} \max\ & 4*x1 + 7*x2; \\ & 0.3*x1 + 0.4*x2 + s1 = 60 \\ & x1 + s2 = 150 \\ & x2 + s3 = 100 \\ & x1, x2, s1, s2, s3 \ge 0 \end{aligned}$$

A = [0.3 0.4 1 0 0;
1 0 0 1 0;
0 1 0 0 1]

– Select x1,x2, s2 as basic and s1=s3=0 non-basic

$$\begin{aligned} 0.3*x1 + 0.4*x2 &= 60 \\ x1 + s2 &= 150 \\ x2 &= 100 \end{aligned}$$

$\Rightarrow$ x2 = 100; x1 = (60-40)/0.3 = 67; s2 = 150-67 = 83

Objective = 4*67 + 7*100 = 967 (this happens to be optimum, see graph)

Solving LP problems – basic solutions

- In principle LP problems could be solved by
  - computing all basic solutions and
  - selecting among the feasible ones the one with optimal objective function value
  - But the number of basic solutions is potentially

$$\binom{m+n}{m} = \frac{(m+n)!}{m!n!}$$

    - m = number of constrains, n = number of decision variables
  - Already with m=n=20 there are 137 846 528 820 combinations

Solving LP problems – Simplex algorithm

- The Simplex algorithm searches the optimum among the basic solutions
  - It starts with some basic solutions such as slack-basis
  - It moves to an adjacent basic solution so that the solution improves
    - In an adjacent basic solution exactly one variable is replaced in the basis
    - graphically it means moving between corners along an edge
  - This is repeated until optimum is found
- Theoretically the Simplex algorithm may explore an exponential number of basic solutions
  - In practice the algorithm is fast and polynomial in complexity

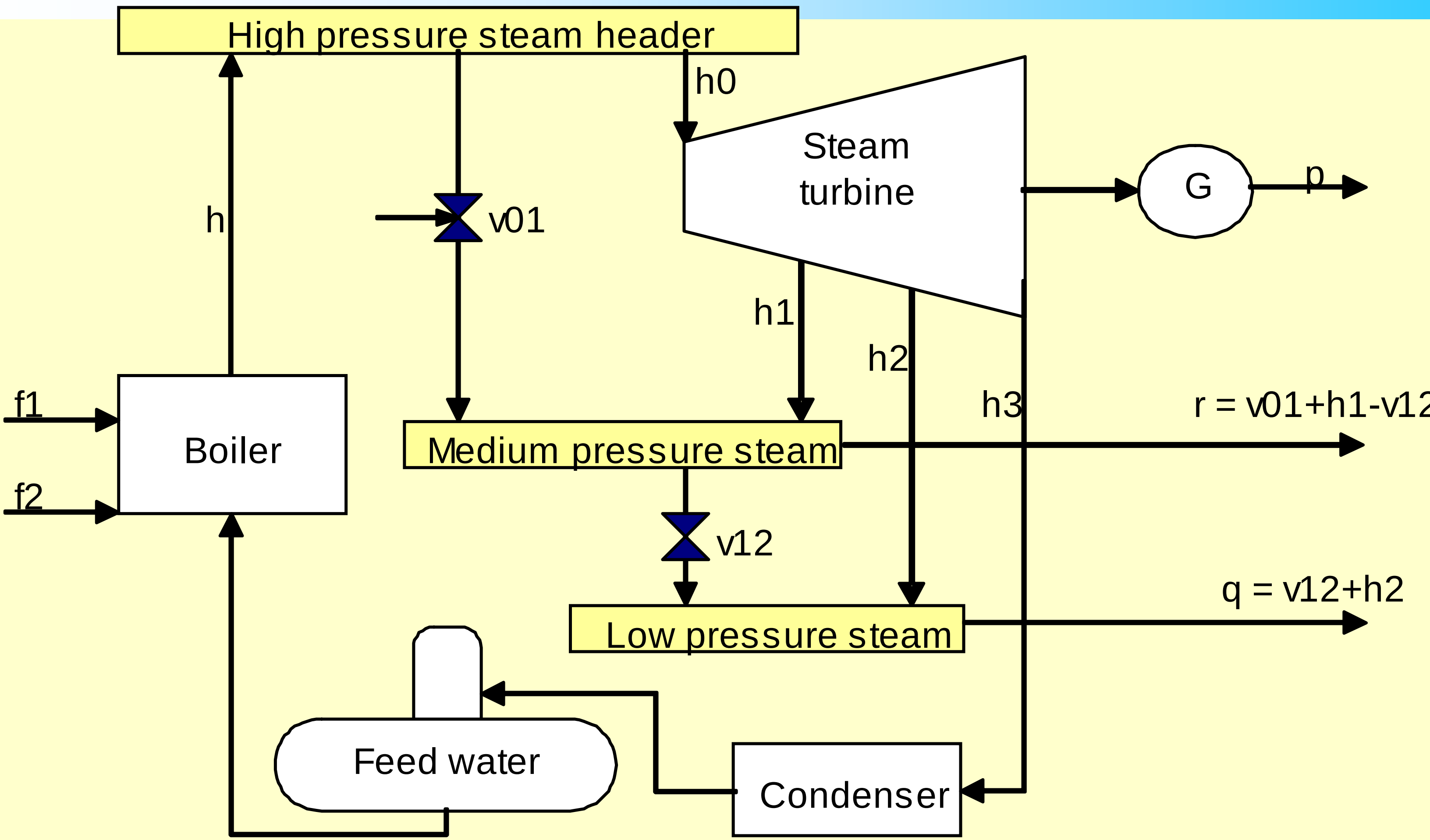
CHP – Combined Heat and Power

- Cogeneration means production of two or more energy products together in an integrated process
  - CHP = combined heat and power generation
  - Trigeneration:
    - district heating + cooling + power
    - high pressure process heat + low pressure heat + power
  - Technologies: backpressure turbines, combined steam&gas turbines, combustion engine with excess heat utilization …
  - Much more efficient than producing the products separately – **over 90% efficiency possible**
  - Cost-efficient way to reduce CO_2 emissions

CHP planning

- Objective is to maximize profit s.t. production constraints
- Hourly production of the different products must be planned together
  - Production of heat & cooling must meet the demand (natural monopoly)
  - Power production is planned to maximize the profit from sales to the spot market (free market)
- A long-term model consists of many hourly models in sequence
  - E.g. an annual model consists of 8760 hourly models
  - Hourly forecasts for demand and power price
- Various advanced analyses, e.g. risk analysis require solving many long-term models
  - Solution must be fast!

Sample backpressure/bleeder turbine plant

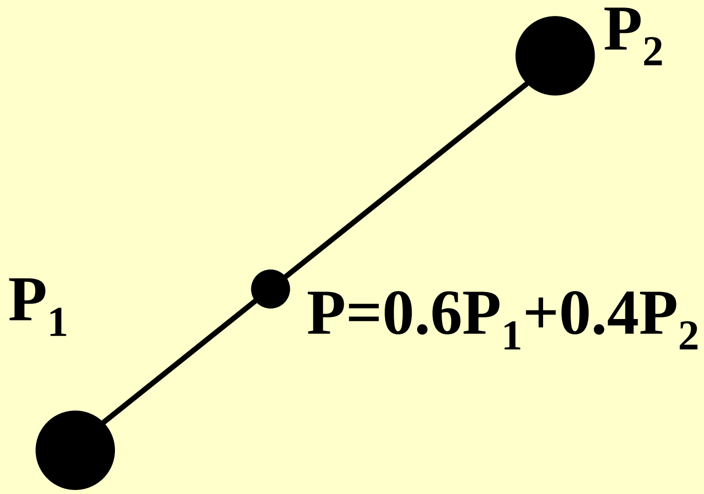


CHP modelling

1. Modelling as generic LP problem
   – Each component (boiler, turbine, generator, reduction valves) is modelled through linear constraints
   – Component models are combined with balance equations for energy and material flows
   – Model is solved using generic LP software
2. Modelling using special extreme point formulation
   – Extreme points of plant characteristic can be obtained
     - By analyzing LP model
     - Experimentally by running the plant in different modes
     - By computing theoretically
   – Model can be solved by generic LP or using very efficient specialized algorithm **Power Simplex**

LP modelling technique using convex combination

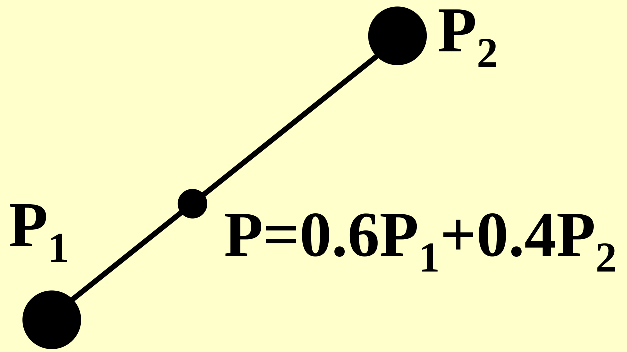
- Weighted average = linear interpolation between two points
  - $P = xP_1 + (1-x)P_2$ with $x \in [0,1]$ is a convex combination of coordinates P_1 and P_2.



LP modelling technique using convex combination

- Equivalent formulation with two weights x_1& x_2

$$P = x_1 P_1 + x_2 P_2 \text{ where } x_1 + x_2 = 1,\ x_1, x_2 \geq 0.$$



- More generally for any number of points P_j

$$P = \Sigma_j\, x_j P_j \text{ where } \Sigma_j\, x_j = 1,\ x_j \geq 0.$$

- Expressions are linear with respect to x_j
  - Points can be scalars (1-dimensional case) or
  - Points can be vectors (multiple dimensions)

LP model for CHP plant – LP-encoding of convex characteristic operating region

- The power plant characteristic defines the feasible operating area in the 3D space (c,p,q)
  - p = power production, q = heat production, c = cost
- Encode model as a convex combination of extreme (corner) points

max $c^p p$ – C // c^p is power price

s.t.

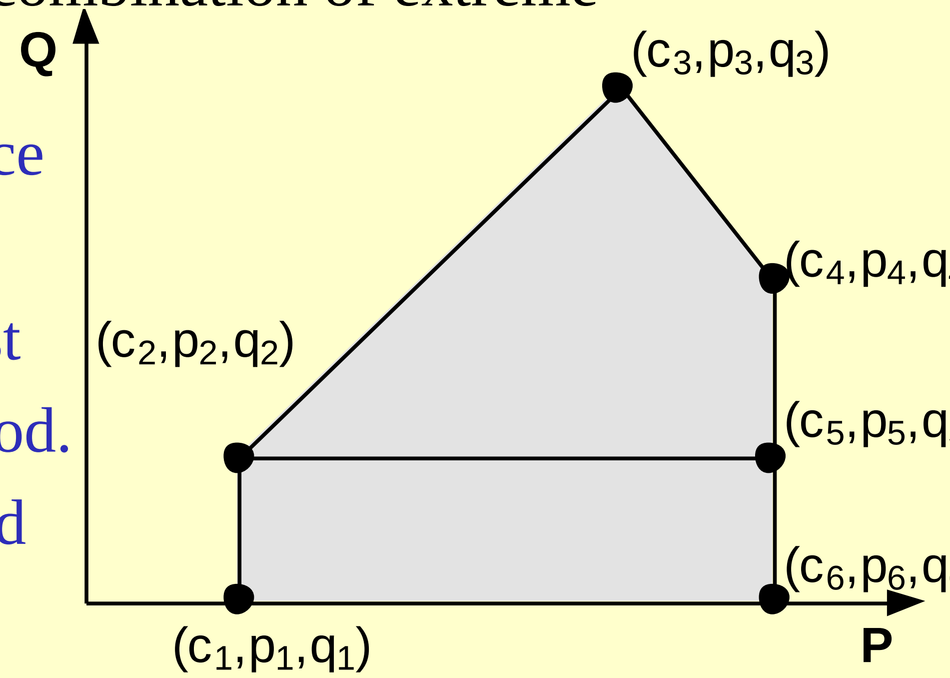
$\Sigma_j c_j x_j$ = C // variable prod. cost

$\Sigma_j p_j x_j$ = P // variable power prod.

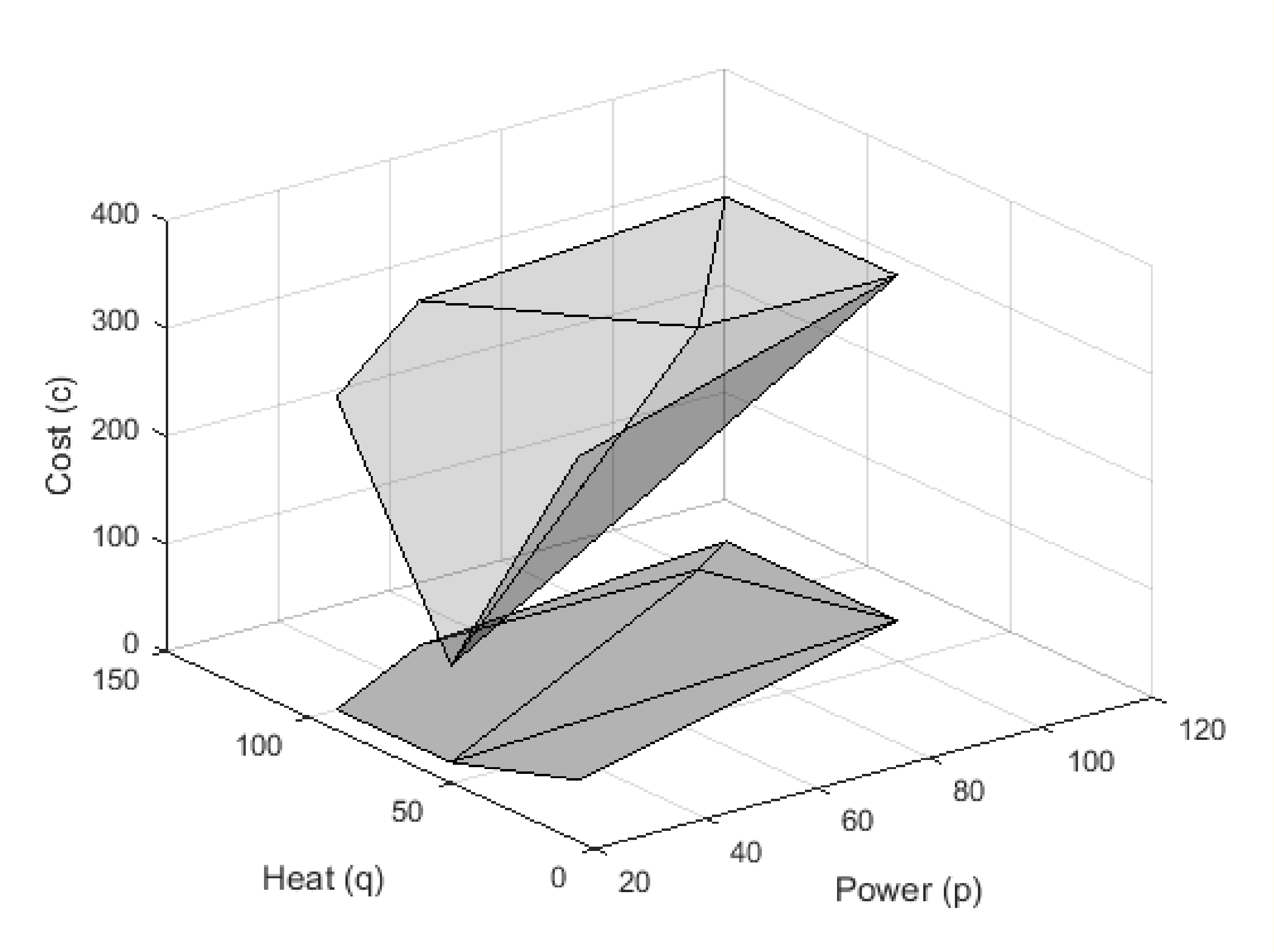
$\Sigma_j q_j x_j$ = Q // fixed heat demand

$\Sigma_j x_j$ = 1 // convex comb.

$x_j \geq 0$



LP model for CHP plant – LP-encoding of convex characteristic operating region



$$\min \sum_{j \in J_u} c_j x_j$$

$$\sum_{j \in J_u} x_j = 1$$

$$\sum_{j \in J} p_j x_j = P$$

$$\sum_{j \in J_u} q_j x_j = Q$$

$$x_j \geq 0,\ j \in J$$

Hourly trigeneration model

- Extreme point formlation with three commodities (p,q,r), multiple plants and multiple periods
  - Extreme points are in 4D space $(c_j^t, p_j^t, q_j^t, r_j^t)$
  - Index t for hour, index u for plant in set of plants U
  - J_u = set of extreme points of plant u

$$C_u^t = \sum_{j \in J_u} c_j^t x_j^t$$

$$P_u^t = \sum_{j \in J_u} p_j^t x_j^t$$

$$Q_u^t = \sum_{j \in J_u} q_j^t x_j^t$$

$$R_u^t = \sum_{j \in J_u} r_j^t x_j^t$$

$$\sum_{j \in J_u} x_j^t = 1$$

$$x_j^t \geq 0 \qquad u \in U$$

Review questions

- Please review lecture material and be prepared to answer review questions at next lecture
  1. Is the optimization problem max $x^2 + y^2$ s.t. $x,y \geq 0$ feasible, infeasible or unbounded? Why?
  2. Give a feasible solution to the above problem.
  3. How many optimal solutions does the problem max x^2 s.t. $-5 \leq x \leq 5$ have?
  4. Transform max x s.t. $x \leq 5$ replacing inequality constraint by equality constraint.
  5. Transform max x s.t. $x \leq 5$ into an unconstrained optimization problem.
  6. Why is classification of optimization problems important?
  7. Classify the following optimization problem: min $x^2 + y^2$ s.t. $x,y \geq 0$, $x \in R$, $y \in N$
  8. Why is LP modelling so common?
  9. Why are convex optimization problems relatively easy to solve?
  10. Give an example of an optimization problem which is difficult or impossible to solve.
  11. Does LP apply to non-linear problems? Why, or why not?
  12. When can an LP problem have infinite number of optimal solutions?
  13. Give an example of an infeasible LP problem
  14. Give an examle of a feasible LP problem without optimal solution
  15. Give an example of an LP problem with infinite number of optima