PHYS-E055101 Low Temperature Physics: Nanoelectronics

Superconducting qubits

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I. THE JOSEPHSON RELATIONS (REVIEW)

In the previous lecture we derived the two fundamental Josephson relations,

Josephson voltage-phase relation:

$$\boxed{\frac{d\varphi(t)}{dt} = \frac{2e}{\hbar}V(t)}. \tag{1}$$

The Josephson current-phase relation

$$\boxed{I_J = I_0 \cdot \sin\varphi}. \tag{2}$$

II. SQUID DEVICES

The simplest useful device that one can make consists of two Josephson junctions in parallel, as in Fig. 1. This is called a SQUID (superconducting quantum interference device).

Consider now the following sample: a single Josephson junction (two electrodes separated by an insulator) and let us tale two points L and R on the left and right electrodes. In the theory of superconductivity, when a magnetic field is present, the simple phase difference across the junction must be replaced by the gauge-invariant phase difference between the two points L (left) and R (right):

$$\varphi_{\mathrm{tot}} = \varphi_L - \varphi_R - \frac{2\pi}{\Phi_0}\int_L^R \vec{A}d\vec{l}, \tag{3}$$

where the flux quantum is

$$\Phi_0 = \frac{h}{2e} = 2.07 \times 10^{-15} Tm^2. \tag{4}$$

To understand where this is coming from (and without going into the full theoretical details) imagine that we would like to write a Schrödinger equation for the superconducting order parameter (which is the same as a wavefunction, but normalized to some superfluid density $n_s(\vec{r}, t)$),

$$\Psi(\vec{r}, t) = \sqrt{n_s(\vec{r}, t)}e^{i\theta(\vec{r}, t)}. \tag{5}$$

The equation satisfied by the order parameter should be like the usual single-particle Schrödinger equation, but with some mass m and a charge q corresponding to Cooper pairs.

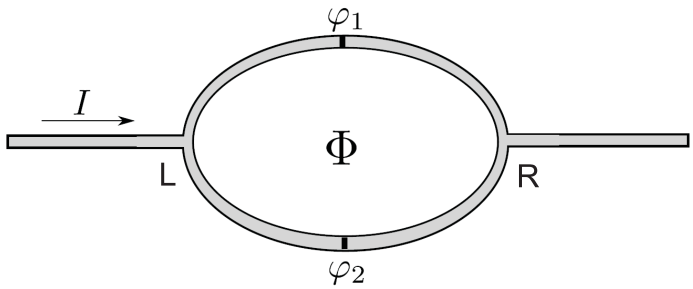


FIG. 1. Schematic of a dc-SQUID. Figure from Ref. [1].

So

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - q\vec{A}(\vec{r})\right)^2\Psi(\vec{r},t). \tag{6}$$

This yields a superfluid current (with the replacement $q = -2e$),

$$\vec{J}_s = q\mathrm{Re}\left[\Psi^*(\vec{r},t)\left(-i\frac{\hbar}{m}\vec{\nabla} - \frac{q}{m}\vec{A}(\vec{r})\right)\Psi(\vec{r},t)\right] \tag{7}$$

$$= \frac{\hbar q}{m}n_s(\vec{r},t)\left[\vec{\nabla}\theta(\vec{r},t) + \frac{2\pi}{\Phi_0}\vec{A}(\vec{r})\right]. \tag{8}$$

Now notice that we have a rather curious situation: neither θ nor $\vec{A}$ are directly experimentally measurable, yet the current is! If we make a change of gauge in $\vec{A}$, $\vec{A}' = \vec{A} + \nabla\chi$ (note that $\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\vec{A}' = \vec{B}'$), the supercurrent can be written as well in the same form

$$\vec{J}_s = \frac{\hbar q}{m}n_s(\vec{r},t)\left[\vec{\nabla}\theta'(\vec{r},t) + \frac{2\pi}{\Phi_0}\vec{A}'(\vec{r})\right]. \tag{9}$$

provided that we redefine θ' as $\theta' = \theta - \frac{2\pi}{\Phi_0}\chi$. Thus a change of gauge yields a change in both the vector potential and the phase, such that the supercurrent remains the same. This means that one can define a gauge-invariant phase as

$$\varphi(t) = \theta + \frac{2\pi}{\Phi_0}\int \vec{A}d\vec{l}, \tag{10}$$

which yields Eq. (3) when taken across a junction.

Let us look now at the device shown in Fig. 1 and evaluate the total gauge-invariant phase difference around the SQUID ring

$$\varphi_{\mathrm{SQUID}} = -\varphi_1 - \frac{2\pi}{\Phi_0}\int_L^R \vec{A}_1 d\vec{l} + \varphi_2 - \frac{2\pi}{\Phi_0}\int_R^L \vec{A}_2 d\vec{l} \tag{11}$$

$$= \varphi_2 - \varphi_1 + \frac{2\pi}{\Phi_0}\oint \vec{A}d\vec{l} \tag{12}$$

$$= \varphi_2 - \varphi_1 + \frac{2\pi\Phi}{\Phi_0}. \tag{13}$$

Here by $\vec{A}_1$ and $\vec{A}_2$ we mean the value of $\vec{A}$ evaluated on the SQUID branch containing the junction 1 and respectively 2.

Now, the total phase should be single valued, that is $\varphi_{\rm SQUID} = 2\pi \times$ (integer), that is

$$\varphi_1 - \varphi_2 = \frac{2\pi\Phi}{\Phi_0}(\mathrm{mod} 2\pi). \tag{14}$$

The total current through the ring (assuming identical junctions) is

$$I = I_0 \sin\varphi_1 + I_0 \sin\left(\varphi_1 - \frac{2\pi\Phi}{\Phi_0}\right), \tag{15}$$

therefore the current through the SQUID is

$$I = 2I_0 \cos\left(\frac{\pi\Phi}{\Phi_0}\right) \sin\left(\varphi_1 - \frac{\pi\Phi}{\Phi_0}\right). \tag{16}$$

Another, more symmetric form, can be obtained if we define the average phase $\varphi = (\varphi_1 + \varphi_2)/2$,

$$I = 2I_0 \cos\left(\frac{\pi\Phi}{\Phi_0}\right) \sin\varphi. \tag{17}$$

This means that the maximum current through the SQUID is $2I_0 \cos\left(\frac{\pi\Phi}{\Phi_0}\right)$, depending on the magnetic field. Note that this is an interference effect - as you change the magnetic field, there are oscillations in the current, from a maximum value $2I_0$ to destructive interference (zero current). Therefore by measuring the maximum current through the SQUID we can find the flux (and knowing the SQUID area, also the field magnetic field B). Also this relation shows that a SQUID is equivalent to a single junction with $E_{\rm J}$ tunable by magnetic field, $E_{J\ \rm SQUID}(\Phi) = 2E_J \cos\left(\frac{\pi\Phi}{\Phi_0}\right)$.

III. CLASSICAL THEORY OF JOSEPHSON CIRCUIT ELEMENTS: ENERGIES AND THE CONSTRUCTION OF THE SYSTEM LAGRANGIAN AND HAMILTONIAN

From now on we will proceed in a systematic way. The aim is to build the Hamiltonian of a Josephson junction as the basic circuit component. In the process, we will also identify the canonically conjugate variables of the system.

What we know so far are the two Josephson relations,

$$\begin{cases} I_J = I_0 \sin\varphi, \\ \frac{d\varphi}{dt} = \frac{2e}{\hbar} V \text{ or } V = \frac{\Phi_0}{2\pi}\frac{d\varphi}{dt}. \end{cases} \tag{18}$$

These are the constitutive relations for this effect: this is all about it, once you know them you should be able to calculate any current, voltage, or superconducting phase in circuits containing Josephson junctions.

To construct Lagrangians and Hamiltonians we need first to understand the relevant energies that are associated with these systems.

A. The energy associated with the Josephson effect

This energy is simply

$$U_J = \int I_J \cdot V \cdot dt = \int I_0 \sin\varphi \frac{\Phi_0}{2\pi}\frac{d\varphi}{dt} dt = -E_J \cos\varphi, \tag{19}$$

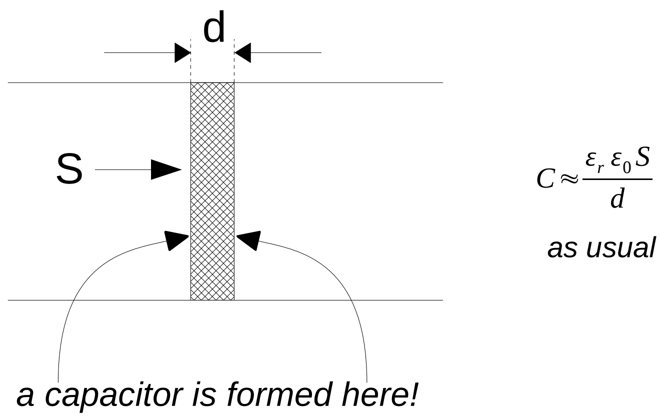
or

$$\boxed{U_J(\varphi) = -E_J \cos\varphi}. \tag{20}$$

Let us look at what this formula tells us. Suppose the phase $\varphi = 0$, then we have $U_J = -E_J$. The meaning of this is as follows: when we connect two superconductors via a weak link (thus allowing for tunneling) the ground state energy is lowered by E_J with respect to the uncoupled situation. E_J is here the tunneling energy. This situation is general whenever you have a tunneling barrier. Now, imagine that we "twist" the superconducting phase, thus allowing for a $\varphi \neq 0$. The potential energy increases with respect to the "untwisted" ($\varphi = 0$) case. So a change in phase results in the system accumulating the energy $E_J(1 - \cos\varphi)$.

B. The capacitive energy

Is this the only energy in the system? Actually not. We have not said anything about the fact that any two pieces of metal separated by an insulator behave as a capacitor of capacitance C.



This capacitor is at a potential $V = \frac{\Phi_0}{2\pi}\dot{\varphi}$, therefore it has the energy

$$\boxed{K(\dot{\varphi}) = \frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{\Phi_0}{2\pi}\right)^2 C\dot{\varphi}^2}. \tag{21}$$

Note the form of this expression and imagine that φ is like a coordinate: as will soon become clear, K is the analog of a kinetic energy, while U_J is the analog of a potential energy.

C. The Lagrangian

Now we can construct a Lagrangian in the canonical way,

$$\mathcal{L}(\varphi, \dot{\varphi}) = K(\dot{\varphi}) - U_J(\varphi) = \frac{1}{2}\left(\frac{\Phi_0}{2\pi}\right)^2 C\dot{\varphi}^2 + E_J \cos\varphi. \tag{22}$$

The Lagrange equations are

$$\frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{\varphi}}\right) - \frac{\delta\mathcal{L}}{\delta\varphi} = 0, \tag{23}$$

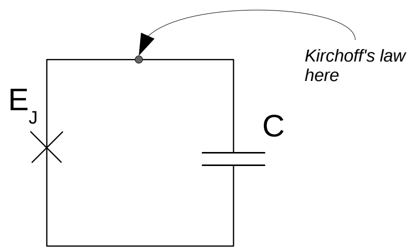
i.e.

$$\frac{\Phi_0}{2\pi}C\ddot{\varphi} + I_0 \sin\varphi = 0. \tag{24}$$

Note that

$$\frac{\Phi_0}{2\pi}C\ddot{\varphi} = \frac{d}{dt}(CV) = \frac{dQ}{dt} = I_{\text{capacitor}} \tag{25}$$

is the current through the capacitor.



This means that the Lagrange equations Eq. (23) produce precisely the Kirchoff's law for the addition of currents.

D. The Hamiltonian

Next: construct the Hamiltonian

$$H\left(p,\varphi\right)=p\dot{\varphi}-\mathcal{L}, \tag{26}$$

with

$$p=\frac{\delta\mathcal{L}}{\delta\dot{\varphi}}=\left(\frac{\Phi_0}{2\pi}\right)^2C\dot{\varphi}, \tag{27}$$

as the canonical momentum, associated with the coordinate φ (remember that $V=\frac{\Phi_0}{2\pi}\dot{\varphi}$).

Interpretation: $Q=CV$ is the charge on the capacitor, therefore the canonical momentum is

$$p=\frac{\Phi_0}{2\pi}Q=\frac{\hbar}{2e}Q=\hbar n, \tag{28}$$

where n is the number of Cooper pairs on the capacitor, which produces a charge $Q=(2e)n$. We can also write the Hamiltonian in the alternative forms

$$H\left(n,\varphi\right)=4E_Cn^2-E_J\cos\varphi, \tag{29}$$

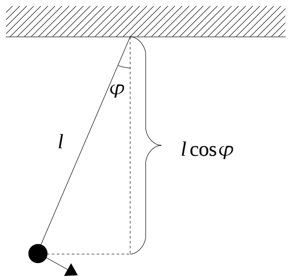
where $4E_C=\frac{(2e)^2}{2C}$ is charging energy corresponding to a single Cooper pair.

$$H\left(Q,\varphi\right)=\frac{p^2}{2C\left(\frac{\Phi_0}{2\pi}\right)^2}-E_J\cos\varphi, \tag{30}$$

or

$$\boxed{H\left(p,\varphi\right)=\frac{Q^2}{2C}-E_J\cos\varphi}. \tag{31}$$

What is the closest mechanical analog of this system? It is a pendulum!



In fact, it is called a Josephson pendulum in the literature.

EXERCISE: write down the (classical) Hamilton equations of motion.

IV. SUPERCONDUCTING QUBITS: MOTIVATION

We are searching for a physical realization of the concept of quantum computing based on Josephson junctions. We would need a system that has two discrete energy states, well-separated from the rest, so that we can selectively address them by applying a resonant field. We should be able to prepare the system in any state, let it evolve, then measure it. To be able to perform 2-qubit gates, we should be able to couple the system with another one in an externally controllable way. During the quantum gate operations, the system should not be subject to other uncontrolled influences (decoherence). The measurement apparatus should interact with the system only during the measurement (and not during the time when the gates are applied). These are very tough requirements: for example, we want a system that interacts selectively with its electromagnetic environment: it interacts only with those degrees of freedom used for producing gates, and very little with the rest of the electromagnetic degrees of freedom.

It is possible to design circuits made of superconducting components to do exactly this. The qubits will be called superconducting qubits. But to build a quantum processor, we would need a few more things. How do we actually measure the state of the qubit? We will not discuss this here, but rest assured that it is possible. Then, we would need as well two-qubit gates. That is, we need some way of having the qubits interact for some time and then be able to decouple them. This is also possible, and the mathematical tools needed to analyze such a system are similar to the ones introduced in this lecture.

V. QUANTIZATION OF JOSEPHSON CIRCUIT ELEMENTS

Until now, φ and n were classical variables. Let us do something crazy: let's quantize them. This means that the phase and number of particles will become operators. We can write them as $\hat{\varphi}$ and $\hat{n}$. Since they are conjugate variables, they satisfy

$$[\hat{\varphi}, \hat{p}] = i\hbar, \tag{32}$$

or

$$[\hat{\varphi}, \hat{n}] = i. \tag{33}$$

We can even write, by analogy with the "real" coordinate and momentum,

$$\hat{p} = -i\hbar \frac{\partial}{\partial \varphi}, \tag{34}$$

$$\hat{n} = -i \frac{\partial}{\partial \varphi}, \tag{35}$$

i.e. we are using here the coordinate representation.

Now we have a quantum problem: we have a Hamiltonian

$$\hat{H} = 4E_C \hat{n}^2 - E_J \cos \hat{\varphi}, \tag{36}$$

and therefore we can write the corresponding Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi (\varphi, t) = \hat{H} \psi (\varphi, t) . \tag{37}$$

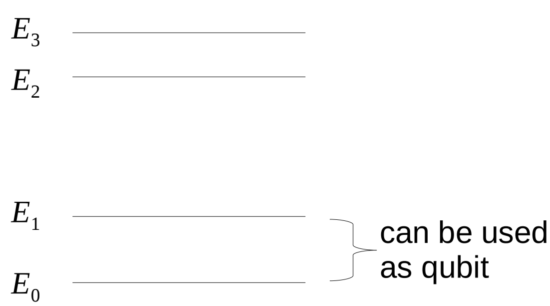
The ψ appearing here is simply the wavefunction of a fictional particle with coordinate φ evolving under the Hamiltonian Eq. (36). Its modulus squared value gives the density probability for the particle to have the phase φ as usual in quantum mechanics. Note however that all the quantities entering the Schrödinger equation refer to a macroscopic system: the phase ϕ for example is the superconducting phase associated with the collective motion of an enormous amount of electrons composing the superconductor. The wavefunction ψ is sometimes referred to as "macroscopic wavefunction".

If we now solve for the eigenvalues E_k satisfying

$$\hat{H} \psi_k (\varphi, t) = E_k \psi_k (\varphi, t) , \tag{38}$$

we find the energy levels $E_0, E_1, E_2 \ldots$.

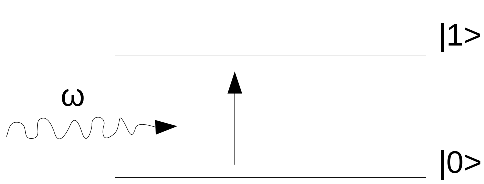
They look like this



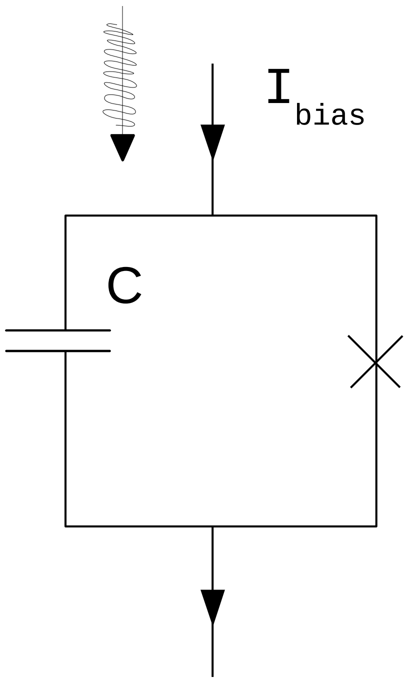
This is an anharmonic oscillator. Note that the energy levels are not equally separated, thus we could in principle use only two of them as a qubit. So, we have a qubit, isn't it so? Indeed. The only "minor" problem is that we have to manipulate it (*i.e.* to produce gates). How do we do it?

VI. STUDY CASE I: THE PHASE QUBIT

We certainly need some externally-controlled parameter or some way of coupling radiation so that we can excite (controllably) the qubit. How can we do this?



Here is one possibility: we can try to excite the system by using an external current bias I. The resulting device is called a "phase qubit". The real device is a bit more complicated (see the images at the end of the lecture). This bias current typically has a high-frequency r.f. component on top of a constant current. When this high-frequency component is resonant to the qubit frequency, Rabi oscillations are produced and the qubit can be excited from the ground state to the first excited state.



Let us see how this will work. The first question is how to include the bias current in our Hamiltonian description of the circuit. The existence of this current will add another energy

$$\boxed{U_I = \int (-I) V\, dt = \int -I \frac{\Phi_0}{2\pi}\frac{d\varphi}{dt} dt = -I\frac{\Phi_0}{2\pi}\varphi} \tag{39}$$

(Here I took the current I with a minus sign. It doesn't matter so much if you put a plus instead; then you will get the potential below tilted from right to left but nothing significant will change in the physics.) Then we simply have this additional potential energy in the Lagrangian,

$$\mathcal{L}(\varphi, \dot{\varphi}) = K(\dot{\varphi}) - U_J(\varphi) - U_I(\varphi) \tag{40}$$

$$= \frac{1}{2}\left(\frac{\Phi_0}{2\pi}\right)^2 C\dot{\varphi}^2 + E_J \cos\varphi + \frac{I\Phi_0}{2\pi}\varphi. \tag{41}$$

Then, as before, the Lagrange equations

$$\frac{d}{dt}\left(\frac{\delta \mathcal{L}}{\delta \dot{\varphi}}\right) - \frac{\delta \mathcal{L}}{\delta \varphi} = 0, \tag{42}$$

will yield Kirkhhoff's current law for electrical circuits,

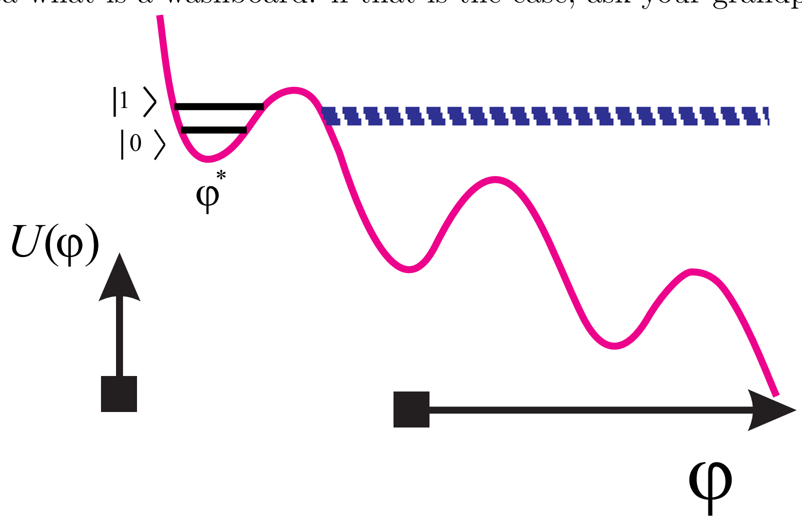
$$\frac{\Phi_0}{2\pi}C\ddot{\varphi} + I_0 \sin\varphi = I. \tag{43}$$

The Hamiltonian is then

$$H = p\dot{\varphi} - \mathcal{L}(\varphi, \dot{\varphi}) \tag{44}$$

$$= \frac{Q^2}{2C} - E_J \cos\varphi - I\frac{\Phi_0}{2\pi}\varphi. \tag{45}$$

The quantity $U(\varphi) = -E_J \cos\varphi - I\left(\frac{\Phi_0}{2\pi}\right)\varphi$ is the new "potential energy". It is also called the "washboard potential", due to its characteristic shape (perhaps many of you have no idea what is a washboard: if that is the case, ask your grandparents :-)).



A side note here: the phase variable φ behaves like any quantum mechanical variable, only that it characterizes a macroscopic system (the entire electrical circuit). Looking at the washboard potential, the "particle" with position φ can tunnel from one well to the next one, a process referred to as macroscopic quantum tunneling. The tunneling probability can be calculated using the WKB theory. At finite temperature, thermal activation across the barrier occurs, with probability given by the Arrhenius' law.

Getting back to our task: as before, Q and $\frac{\Phi_0}{2\pi}\varphi$ are the canonically conjugate variables, and we can quantize the system,

$$[\frac{\Phi_0}{2\pi}\hat{\varphi}, \hat{Q}] = i\hbar. \tag{46}$$

Or, in terms of the number n of Cooper pairs, $\hat{Q} = (2e)\hat{n}$, and $[\hat{\varphi}, \hat{n}] = i$. Sure enough, in the "coordinate" representation (here the "coordinate" is the phase φ) we would write $\hat{n} = -i\frac{d}{d\varphi}$.

But let us come back to the problem of single gates. Suppose that we bias with a current $I = I_{\text{const}} + I_x \cos \omega t$. We can separate in the Hamiltonian the time-dependent part from the constant part $\hat{H}_{\text{const.}}$,

$$\hat{H}_{\text{const}} = \frac{\hat{Q}^2}{2C} - E_J \cos \hat{\varphi} - I_{\text{const}} \left(\frac{\Phi_0}{2\pi} \right) \hat{\varphi}. \tag{47}$$

We will study first the Hamiltonian $\hat{H}_{\text{const}}$, and attempt to solve the Schrödinger equation for a particle in the washboard potential $U_{\text{const}}(\varphi) = -E_J \cos \varphi - I_{\text{const}} \left(\frac{\Phi_0}{2\pi} \right) \varphi$ appearing in H_{const}. If we do not insist of being too precise, the easiest way to do this is to approximate this potential with a quadratic one by expanding it around the position of a local minimum φ^*, such as

$$U_{\text{const}}(\varphi) \approx U_{\text{const}}(\varphi^*) + \frac{1}{2} \frac{d^2 U_{\text{const}}}{d\varphi^2} |_{\varphi=\varphi^*} (\varphi - \varphi^*)^2. \tag{48}$$

We can now simply write the time-independent Schrödinger equation thinking about φ as the "coordinate". That is, $Q = -i(2e)\frac{d}{d\varphi}$ and

$$E_k \psi_k(\varphi) = \left[-\frac{(2e)^2}{2C} \frac{d^2}{d\varphi^2} - E_J \cos \varphi - I_{\text{const}} \left(\frac{\Phi_0}{2\pi} \right) \varphi \right] \psi_k(\varphi), \tag{49}$$

where $k = 0, 1, \ldots$.. Let us imagine that we solved this equation and we found the first two eigenstates $\psi_0 = \langle \varphi | 0 \rangle$ and $\psi_1 = \langle \varphi | 1 \rangle$ with the eigenvalues E_0 and E_1 respectively. We can write

$$\hat{H}_{\text{const}} = E_0 |0\rangle\langle 0| + E_1 |1\rangle\langle 1| = \frac{E_0 + E_1}{2} I + \frac{E_1 - E_0}{2} \sigma_z, \tag{50}$$

where $I = |0\rangle\langle 0| + |1\rangle\langle 1|$ and $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$ is the Z – Pauli matrix.

Now we are ready to solve the full problem, including the time-dependent current. So we go back to the full Hamiltonian, which we write

$$\hat{H} = \hat{H}_{\text{const}} - I_x \frac{\Phi_0}{2\pi} \cos \omega t (\hat{\varphi} - \varphi^*) + \ldots, \tag{51}$$

where ... is $-I_x \frac{\Phi_0}{2\pi} \cos \omega t(\varphi^*)$ and can be ignored (it does not contain operators ...).

To understand the effect of this oscillatory term, we use first-order perturbation theory. We have to calculate the matrix elements of the operators $\varphi - \varphi^*$ in the basis $|0\rangle, |1\rangle$ as follows:

$$\langle 0 | \hat{\varphi} - \varphi^* | 0 \rangle = \langle 1 | \hat{\varphi} - \varphi^* | 1 \rangle = 0 \tag{52}$$

$$\langle 0 | \hat{\varphi} - \varphi^* | 1 \rangle = \langle 1 | \hat{\varphi} - \varphi^* | 0 \rangle = \epsilon, \tag{53}$$

therefore $\hat{\varphi} - \varphi^* = \epsilon(|0\rangle\langle 1| + |1\rangle\langle 0|) = \epsilon \cdot \hat{\sigma}_x$. Eq. (52) above results immediately by noticing that if the potential is approximated as in Eq. (48) then φ^* is in fact the average of φ on any state. For Eq. (53), we can simply use the results from the harmonic oscillator and find explicitly the value ϵ. We will not write the result here because the point is just to demonstrate that the driving field couples through the σ_x operator to the qubit. Note that this ϵ plays the same role as the electrical dipoles in the standard treatment of atoms interacting with electromagnetic radiation.

To conclude, we find that

$$\hat{H} = \frac{E_1 - E_0}{2}\hat{\sigma}_z + I_x \epsilon \frac{\Phi_0}{2\pi}\hat{\sigma}_x \cos\omega t, \tag{54}$$

or, with the notations $\nu = E_1 - E_0$, $\hbar\Omega = I_x \epsilon \frac{\Phi_0}{2\pi}$ we get

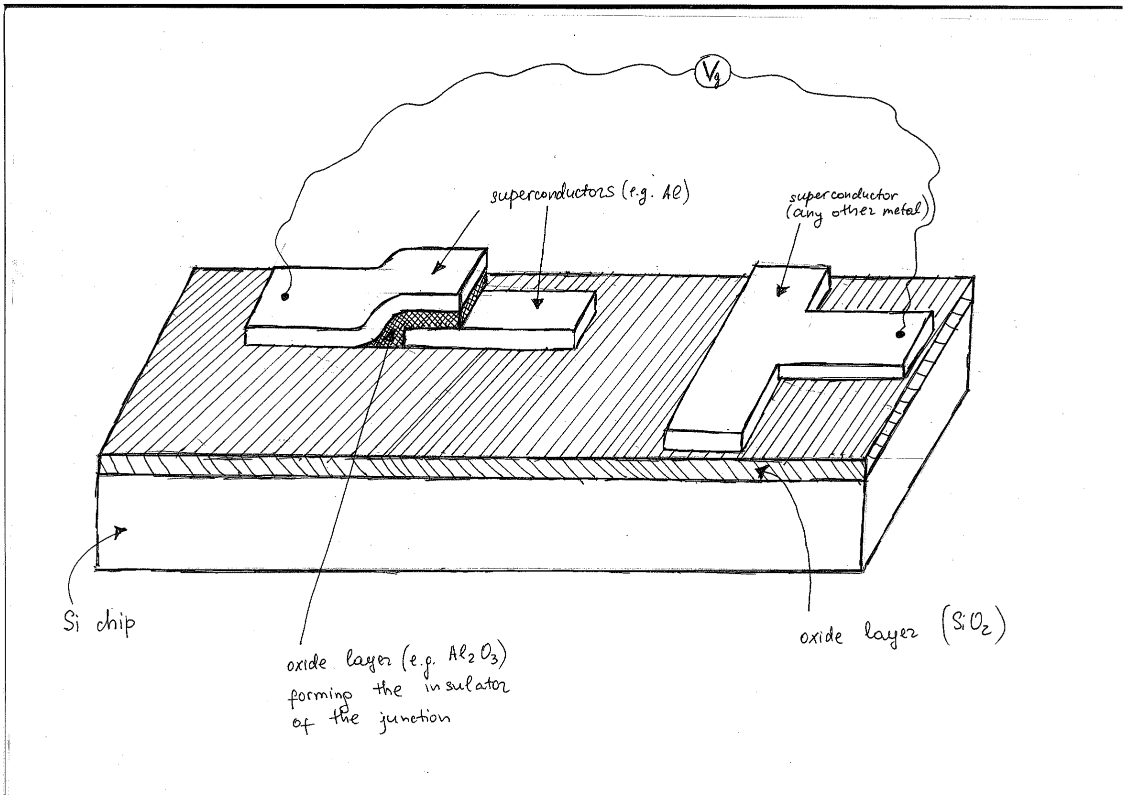
$$\hat{H} = \frac{\nu}{2}\hat{\sigma}_z + \hbar\Omega\hat{\sigma}_x \cos\omega t. \tag{55}$$

This is it! Now we have a Hamiltonian that has exactly the same form as H_{total} of Eq. (65) and that can be used to produce Rabi oscillations, single-qubit gates, *etc.*.

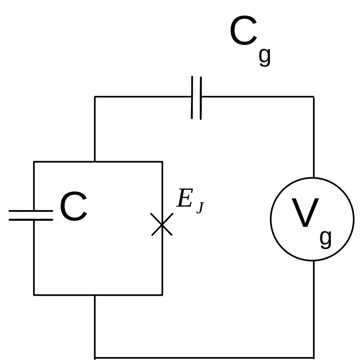
VII. STUDY CASE II: THE CHARGE QUBIT (THE COOPER PAIR BOX)

Here is yet another example: the Cooper pair box. Idea: what if, instead of pushing a current through the junction, we use the charge stored in the capacitance to manipulate the state of the junction with an external voltage V_g(V_g - gate voltage).

How would the sample look like?



Equivalent circuit



C_g – gate capacitance

We now go through the same procedure:

- find the "kinetic" energy: the electrostatic energy stored in the two capacitors is:

$$K = \frac{CV^2}{2} + \frac{C_g\left(V_g - V\right)^2}{2}, \tag{56}$$

where

$$V = \frac{\hbar}{2e}\dot{\varphi} = \frac{\Phi_0}{2\pi}\dot{\varphi}. \tag{57}$$

So,

$$K(\dot{\varphi}) = \frac{C_\Sigma}{2}\left(\frac{\Phi_0}{2\pi}\dot{\varphi} - \frac{C_g}{C_\Sigma}V_g\right)^2 + \frac{C_g V_g^2}{2}, \tag{58}$$

Now $\frac{C_g V_g^2}{2}$ is constant, let us neglect it. And also we write $C_\Sigma = C_g + C$.

We then have

$$\mathcal{L}(\varphi,\dot{\varphi}) = K(\dot{\varphi}) - U_J(\varphi) = \frac{C_\Sigma}{2}\left(\frac{\Phi_0}{2\pi}\dot{\varphi} - \frac{C_g}{C_\Sigma}V_g\right)^2 + E_J\cos\varphi. \tag{59}$$

$$p = \frac{\delta\mathcal{L}}{\delta\dot{\varphi}} = \frac{\Phi_0}{2\pi}C_\Sigma\left(\frac{\Phi_0}{2\pi}\dot{\varphi} - \frac{C_g}{C_\Sigma}V_g\right) = n\hbar. \tag{60}$$

Also

$$\frac{\Phi_0}{2\pi}\dot{\varphi} = \frac{(2e)\,n}{C_\Sigma} + \frac{C_g}{C_\Sigma}V_g. \tag{61}$$

We now can construct the Hamiltonian,

$$H(p,\varphi) = p\dot{\varphi} - \mathcal{L} = (n\hbar)\frac{2\pi}{\Phi_0}\left(\frac{(2e)\,n}{C_\Sigma} + \frac{C_g}{C_\Sigma}V_g\right) - \frac{C_\Sigma}{2}\frac{(n\hbar)^2}{C_\Sigma^2}\left(\frac{2\pi}{\Phi_0}\right)^2 - E_J\cos\varphi, \tag{62}$$

but $\frac{C_\Sigma}{2}\frac{(n\hbar)^2}{C_\Sigma^2}\left(\frac{2\pi}{\Phi_0}\right)^2 = \frac{(2e)^2}{2}\frac{n^2}{C_\Sigma}$ therefore we get

$$\boxed{H = \frac{(2e)^2}{2C_\Sigma}\left(n + \frac{C_g V_g}{2e}\right)^2 - E_J\cos\varphi}. \tag{63}$$

Quantize: again $[\hat{\varphi},\hat{n}] = i$ and $[\hat{\varphi},\hat{p}] = i\hbar$ and in the "coordinate" representation of $\hat{p} = -i\hbar\frac{\partial}{\partial\varphi}$, $\hat{n} = -i\frac{\partial}{\partial\varphi}$.

So:

$$\hat{H} = \frac{(2e)^2}{2C_\Sigma}\left(\hat{n} + \frac{C_g V_g}{2e}\right)^2 - E_J\cos\hat{\varphi}. \tag{64}$$

Then we can use this as before, namely solving $\hat{H}|\psi\rangle = E|\psi\rangle$ etc.

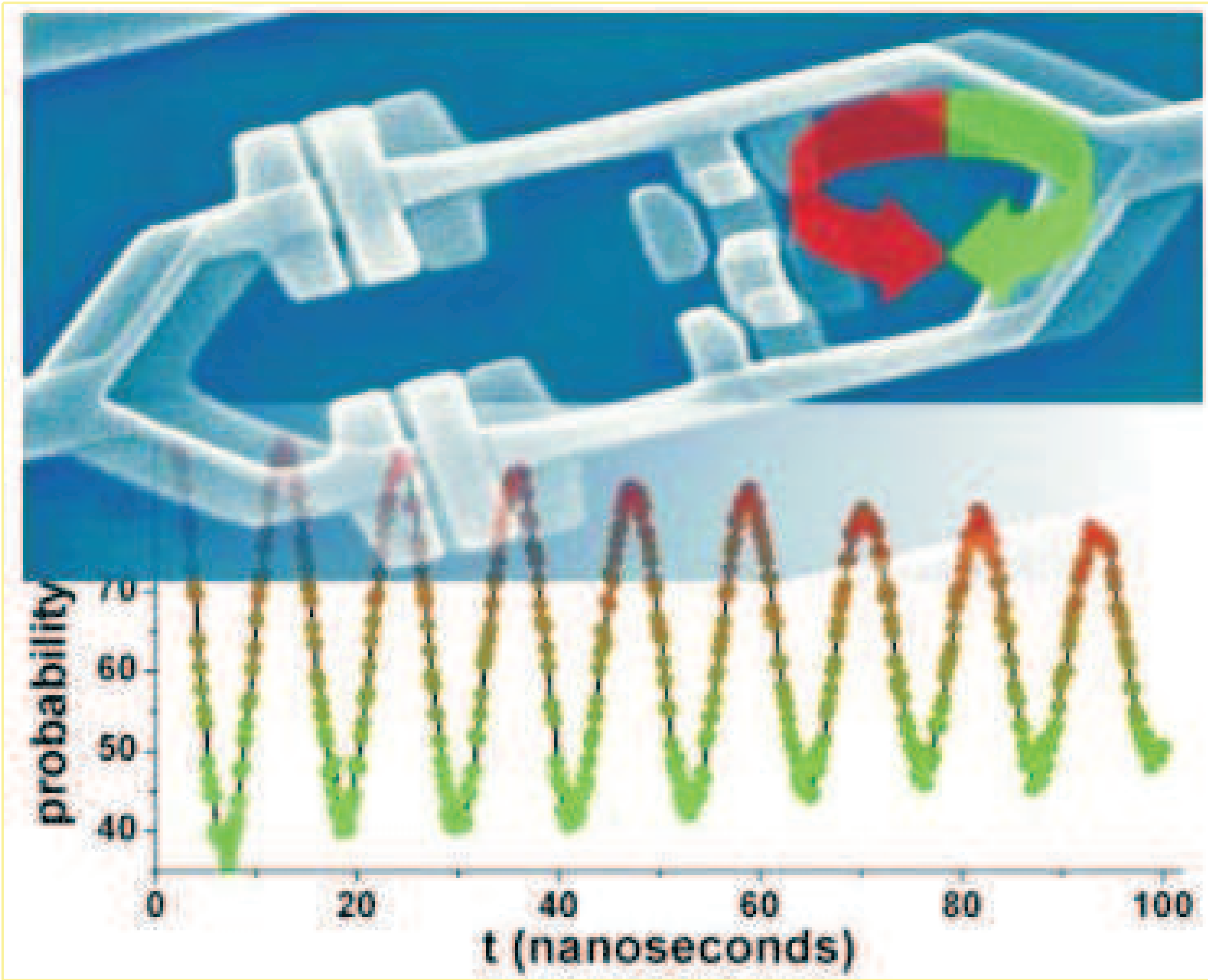
There is one very subtle point though. You might notice that if we attempt to solve $i\hbar\frac{\partial}{\partial t}|\psi\rangle = E|\psi\rangle$ the effect of $\frac{C_g V_g}{2e}$ can be eliminated by a gauge transformation $|\psi(t)\rangle \to e^{-i\frac{C_g V_g}{2e}\varphi}|\psi(t)\rangle$. It looks like nothing except an irrelevant phase will change if we play externally with V_g! A more careful analysis (that lead to the invention of the so-called transmon) shows that this gauge transformation is not possible if one considers the effects of the boundary conditions $\psi(\varphi = 0) = \psi(\varphi = 2\pi)$. In this case, driving through the gate n_g is still feasible.

Finally, we should notice that all our calculations above have assumed $E_{\mathrm{C}} \ll E_J$, in which case the number of particles can be considered as a continuous variable and we can write $\hat{n} = -i\frac{\partial}{\partial\varphi}$. In this limit, the fluctuation of the number of particles operator are large compared to 1 (while the fluctutions of the phase are small). The opposite regime exists as well (called charge regime), where $E_C \gg E_J$. In this situation we cannot write $\hat{n} = -i\frac{\partial}{\partial\varphi}$ because the discrete character of n (namely that n is an integer, $n = 0, 1, 2\ldots$) becomes important: we cannot treat n as a continuous variable anymore. But this type of sample can be used as well to define a "charge-qubit", in which the two qubit states are characterised by the presence or absence of a single Cooper pair in the Cooper pair box.

VIII. A FEW EXAMPLES OF SUPERCONDUCTING QUBITS REALIZED EXPERIMENTALLY

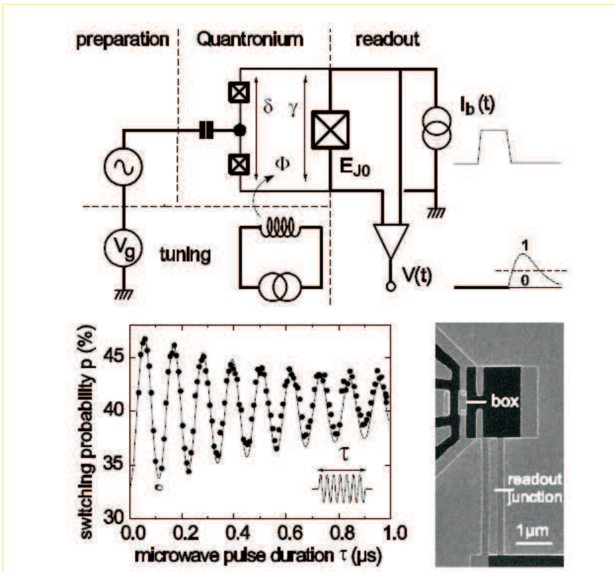
Here we show some examples of superconducting qubits together with their Rabi oscillations.

Below is a flux qubit and the corresponding Rabi oscillations. This qubit has been developed at TU Delft.

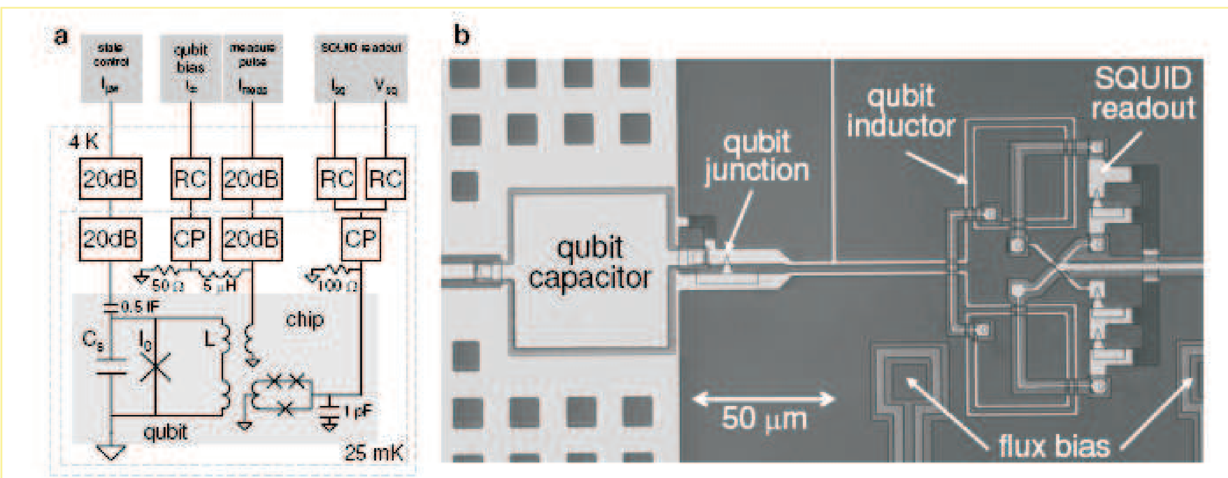


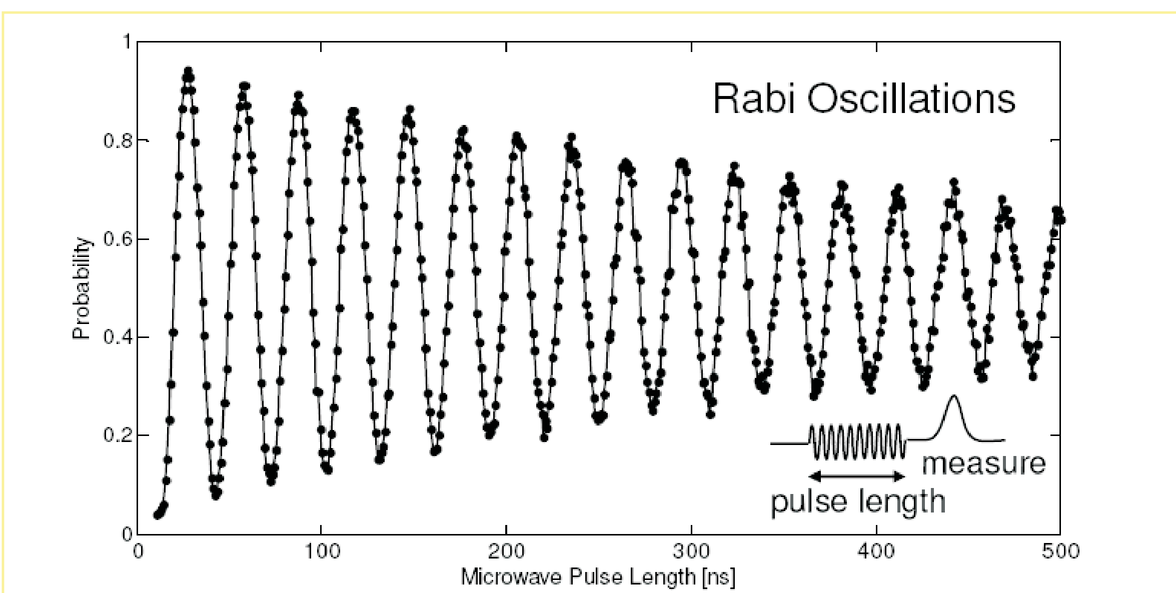
probability
70
60
50
40
0
20
40
60
80
100
t (nanoseconds)

Next figure shows the schematic of a charge-phase qubit developed in Saclay (upper drawing) and the real sample (lower-right). Rabi oscillations are shown in the lower-left figure. From D. Esteve and D. Vion, arXiv:cond-mat/0505676.



Finally, a phase qubit. (a) shows the schematic of a phase qubit developed at NIST and UCSB. The real sample is shown in (b). The lower figure shows the Rabi oscillations. From J. Martinis: Superconducting Phase Qubits, Quantum Information Processing **8**, 81 (2009).



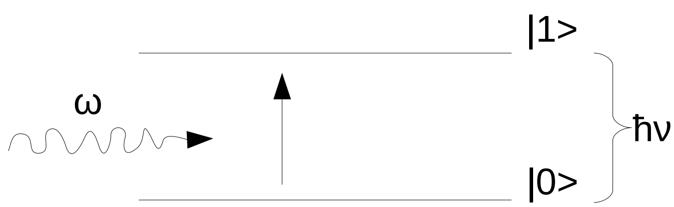


IX. APPENDIX: RABI OSCILLATIONS

To motivate and understand the use of superconducting circuits based on Josephson devices as qubits, let us review here the physics of Rabi oscillations. It is important to understand well this phenomenon, which is relevant for many fields in physics. Let us start with the generic Hamiltonian:

$$\hat{H}_{\text{total}} = \frac{\hbar\nu}{2}\hat{\sigma}_z + \hbar\Omega\cos\omega t\hat{\sigma}_x, \tag{65}$$

where $\Omega\cos\omega t$ is an **externally controllable** classical field.



The quantity Ω is called Rabi frequency. Return now to equation (65). Take $\nu = \omega$ (resonance) and write

$$|\psi\rangle = a\left(t\right)e^{-i\frac{\nu t}{2}}\left|1\right\rangle + b\left(t\right)e^{i\frac{\nu t}{2}}\left|0\right\rangle, \tag{66}$$

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = H_{\text{total}}\left|\psi\right\rangle, \tag{67}$$

$$\Rightarrow$$

$$\begin{cases} i\hbar\dot{a} = \hbar\Omega\cos\left(\omega t\right)e^{i\nu t}b \\ i\hbar\dot{b} = \hbar\Omega\cos\left(\omega t\right)e^{-i\nu t}a \end{cases}. \tag{68}$$

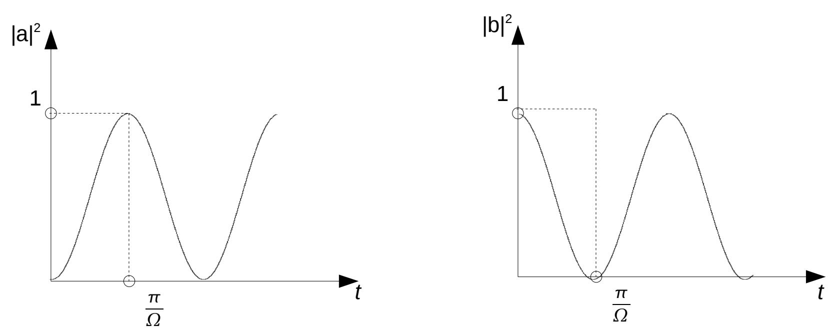
But $\nu = \omega$ and $\cos\left(\omega t\right) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$ so

$$\begin{cases} i\dot{a} = \frac{\Omega}{2}\left(1+e^{2i\omega t}\right)b \\ i\dot{b} = \frac{\Omega}{2}\left(1+e^{-2i\omega t}\right)a \end{cases} . \tag{69}$$

Note now that $e^{2i\omega t}$ and $e^{-2i\omega t}$ are fast-oscillation terms. They can be neglected, an approximation called the Rotating-Wave Approximation (RWA). So we can simply write

$$\begin{cases} i\dot{a} \simeq \frac{\Omega}{2}b \\ i\dot{b} \simeq \frac{\Omega}{2}a \end{cases} . \tag{70}$$

Let us assume we start in the ground state $|0\rangle$. Then $a(0) = 0$, $b(0) = 1$, and $a(t) = -i\sin\frac{\Omega t}{2}$, $b(t) = \cos\frac{\Omega t}{2}$. We then find that after a time $t = \frac{\pi}{\Omega}$ the qubit is flipped! (from $|0\rangle$ to $|1\rangle$). Similarly, if we start in $|1\rangle$, we discover that after a time $t = \frac{\pi}{\Omega}$ the qubit goes in the state $|0\rangle$! (up to a phase which we will not discuss here).

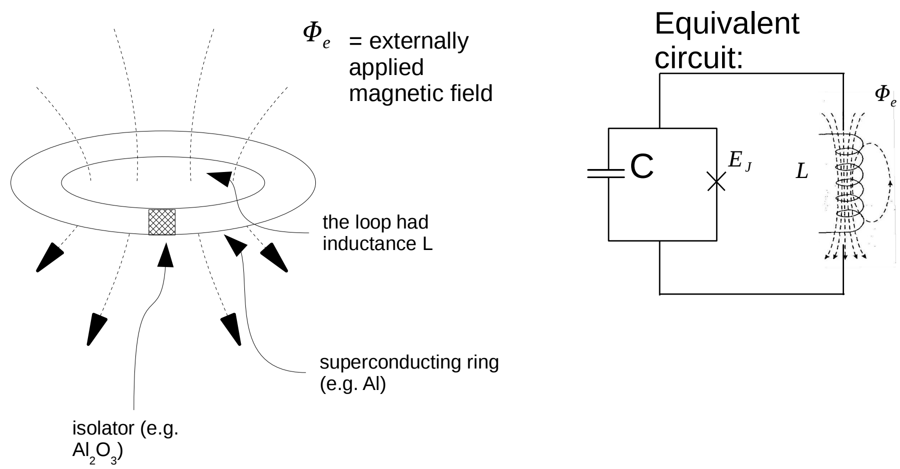


But now notice that this is precisely what X-gates do! Therefore, given an on-resonant field, applied for a time $\frac{\pi}{\Omega}$, we can achieve a flipping of the qubit (in general, rotations around x) between the states $|0\rangle$ and $|1\rangle$.

Note that here we considered the qubit as a quantum system but the radiation was treated classically. It is possible also to treat the radiation quantum-mechanically, by replacing $\Omega\cos(\omega t)\sigma_x$ with $\hbar\hat{a}^\dagger\hat{a} + \frac{g}{2}\hbar(\hat{a}^\dagger + \hat{a})\sigma_x$. This model can also be solved. The Rabi oscillation occurs in this case at a Rabi frequency $\Omega = \sqrt{n+1}g$. If $n \gg 1$ (classical field) then $\Omega = \sqrt{n}g$, therefore we recover the same result as above, namely that the Rabi frequency depends linearly on the amplitude of the driving field.

X. SUPPLEMENTARY MATERIAL (OPTIONAL): THE RF-SQUID QUBIT

Another example: the rf-SQUID qubit. In fact modern-design phase qubits have borrowed so many features from rf-SQUID qubits that it is sometimes dificult to really distinguish them. Idea: what if we use an external flux to bias the junction?



The current through the loop, I_L, can be found by writing:

$$\frac{\Phi_0}{2\pi}\varphi = \Phi_e + I_L \cdot L, \tag{71}$$

where $\frac{\Phi_0}{2\pi}\varphi$ is the change of flux felt by the junction.

So:

$$I_L = \frac{1}{L}\left(\frac{\Phi_0}{2\pi}\varphi - \Phi_e\right). \tag{72}$$

What is the energy associated with I_L?

$$U_L = \int I_L \cdot V \cdot dt = \int \frac{1}{L}\left(\frac{\Phi_0}{2\pi}\varphi - \Phi_e\right) \cdot \frac{\Phi_0}{2\pi}\dot{\varphi} \cdot dt, \tag{73}$$

or

$$U_L = \frac{1}{2L}\left(\frac{\Phi_0}{2\pi}\varphi - \Phi_e\right)^2. \tag{74}$$

The "kinetic" (electrostatic) energy is

$$K(\dot{\varphi}) = \frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{\Phi_0}{2\pi}\right)^2 C\dot{\varphi}^2, \tag{75}$$

the same as before.

The Josephson energy

$$U_J(\varphi) = -E_J \cos\varphi \tag{76}$$

is also similar.

The Lagrangian can be constructed as

$$\mathcal{L}(\varphi, \dot{\varphi}) = K(\dot{\varphi}) - U_J(\varphi) - U_L(\varphi) = \frac{1}{2}\left(\frac{\Phi_0}{2\pi}\right)^2 C\dot{\varphi}^2 + E_J\cos\varphi - \frac{1}{2L}\left(\frac{\Phi_0}{2\pi}\varphi - \Phi_e\right)^2, \tag{77}$$

where the canonical momentum is

$$p = \frac{\delta\mathcal{L}}{\delta\dot{\varphi}} = \left(\frac{\Phi_0}{2\pi}\right)^2 C\dot{\varphi} \tag{78}$$

- same as before. Again,

$$p = \hbar n = \frac{\hbar}{2e}Q \tag{79}$$

$$\Rightarrow$$

$$H(p, \varphi) = p\dot{\varphi} - \mathcal{L} = 4E_C n^2 - E_J\cos\varphi + \frac{1}{2L}\left(\frac{\Phi_0}{2\pi}\varphi - \Phi_e\right)^2 \tag{80}$$

Quantize:

$$[\hat{\varphi}, \hat{n}] = i, \tag{81}$$

or

$$[\hat{\varphi}, \hat{p}] = i.\hbar \tag{82}$$

"Coordinate" representation of operators:

$$\hat{p} = -i\hbar\frac{\partial}{\partial\varphi}, \tag{83}$$

$$\hat{n} = -i\frac{\partial}{\partial\varphi}. \tag{84}$$

For the Hamiltonian we can write

$$\boxed{\hat{H} = 4E_C\hat{n}^2 - E_J\cos\hat{\varphi} + \frac{1}{2L}\left(\frac{\Phi_0}{2\pi}\hat{\varphi} - \Phi_e\right)^2} \tag{85}$$

So now we simply have to solve the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi\left(\varphi, t\right) = \hat{H}\psi\left(\varphi, t\right), \tag{86}$$

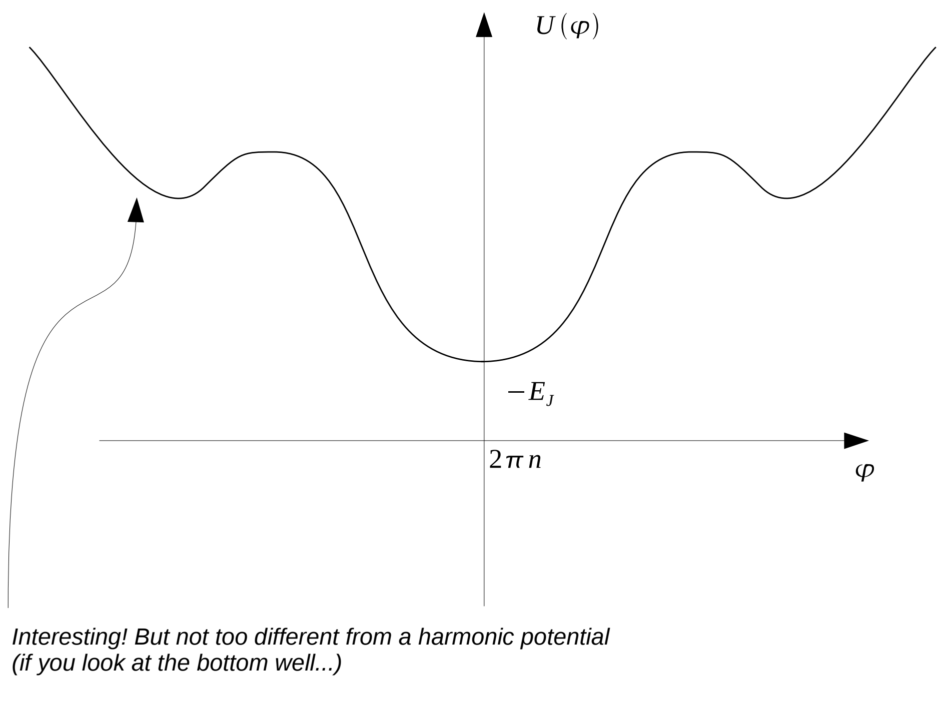
with

$$\hat{H} = -4E_C\frac{\partial^2}{\partial\hat{\varphi}^2} - U\left(\hat{\varphi}\right), \tag{87}$$

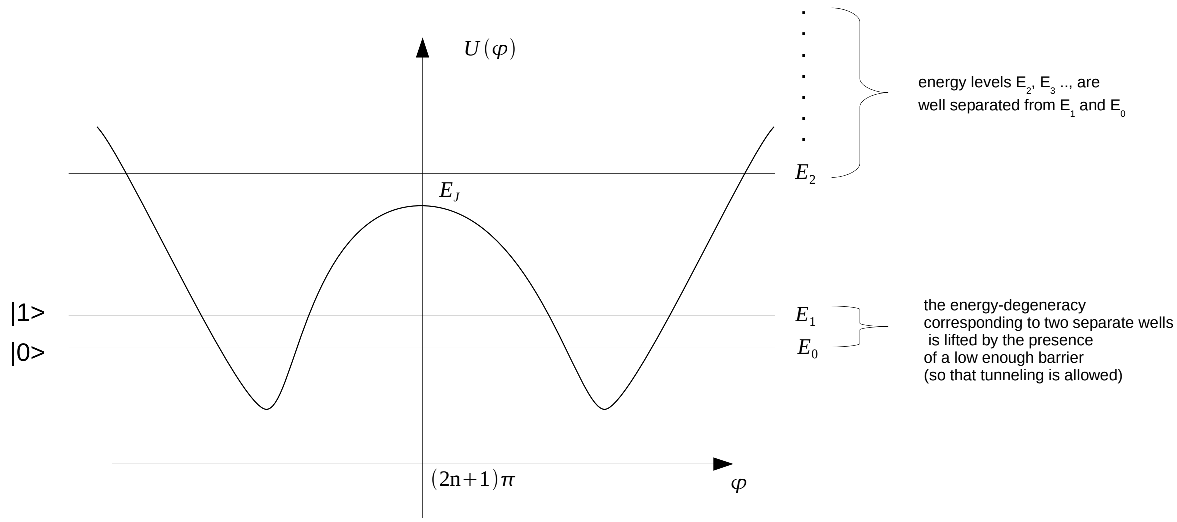
$$U\left(\hat{\varphi}\right) = -E_J\cos\hat{\varphi} + \frac{1}{2L}\left(\frac{\Phi_0}{2\pi}\hat{\varphi} - \Phi_e\right)^2, \tag{88}$$

Let us plot $U\left(\varphi\right)$:

1) if $\Phi_e = n\Phi_0$ it looks like this



2) and if $\Phi_e = \frac{2n+1}{2}\Phi_0$, now it's really cool,



Let us now solve the time-dependent Schrödinger equation

$$\hat{H}\psi_R(\varphi) = E_R\psi_R(\varphi), \tag{89}$$

where

$$\psi_R(\varphi) = \langle\varphi|\, R\rangle, \qquad R \in \{0, 1\}. \tag{90}$$

Note: in reality, for this type of qubit the "computational basis" is $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$, *i.e.* (as if it would have a Hadamard transform applied). The reason is that the states $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ correspond to counterclockwise ↺ and clockwise ↻ currents, which can be measured easier - they produce fluxes with opposite directions, which can be measured by an additional device.

How do we produce single-qubit gates in this system?

Consider a small change $\delta\Phi_e$ around the value $\frac{2n+1}{2}\Phi_0$. Then $H_{\text{total}} = H + \delta H$, where δH is due to the corresponding charge in $U(\varphi)$, namely

$$U(\varphi) \to -E_J\cos\varphi + \frac{1}{2L}\left(\frac{\Phi_0}{2\pi}\varphi - \frac{2n+1}{2}\Phi_0\right)^2 + \frac{1}{L}\left(\frac{\Phi_0}{2\pi}\varphi - \frac{2n+1}{2}\Phi_0\right)\delta\Phi_e + \\ + \frac{1}{2L}(\delta\Phi_e)^2 \tag{91}$$

$$\delta H = \frac{1}{L}\left(\frac{\Phi_0}{2\pi}\varphi - \frac{2n+1}{2}\Phi_0\right)\delta\Phi_e \tag{92}$$

Here we neglect $\frac{1}{2L}\left(\delta\Phi_e\right)^2$ because $\delta\Phi_e \ll \Phi_0$.

Let's express H_{total} in the two-state basis $|0\rangle$,$|1\rangle$. We have:

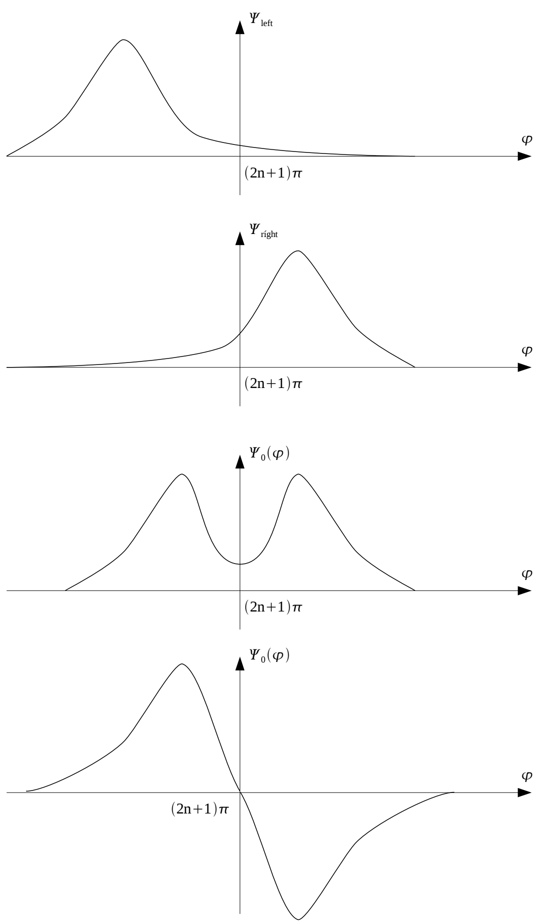
$$\langle 0|\, H_{\text{total}}\, |0\rangle = \langle 0|\, H\, |0\rangle + \langle 0|\, \delta H\, |0\rangle = E_0 + \delta\Phi_e \, \langle 0|\, (\varphi - (2n+1)\,\pi)\, |0\rangle \, \frac{\Phi_0}{2\pi L} \simeq E_0, \quad (93)$$

$$\langle 1|\, H_{\text{total}}\, |1\rangle = \langle 1|\, H\, |1\rangle + \langle 1|\, \delta H\, |1\rangle = E_1 + \delta\Phi_e \, \langle 1|\, (\varphi - (2n+1)\,\pi)\, |1\rangle \, \frac{\Phi_0}{2\pi L} \simeq E_1, \quad (94)$$

$$\langle 1|\, H_{\text{total}}\, |0\rangle = \langle 1|\, H\, |0\rangle + \langle 0|\, \delta H\, |1\rangle = \delta\Phi_e \, \langle 0|\, (\varphi - (2n+1)\,\pi)\, |1\rangle \, \frac{\Phi_0}{2\pi L} = \Sigma\delta\Phi_e, \quad (95)$$

$$\langle 0|\, H_{\text{total}}\, |1\rangle = \langle 0|\, H\, |1\rangle + \langle 1|\, \delta H\, |0\rangle = \Sigma\delta\Phi_e. \quad (96)$$

Why has the matrix element $\langle 0|\, (\varphi - (2n+1)\,\pi)\, |0\rangle$ vanished and $\langle 0|\, (\varphi - (2n+1)\,\pi)\, |1\rangle$ has not? In principle, you can solve numerically the Scrödinger equation and convince yourself that this is the case.



But here is a simple argument $\psi_0\,(\varphi)$ and $\psi_1\,(\varphi)$ are symmetric and anti-symmetric combinations of wavefunctions localized in the left well (call it $\psi_{\text{left}}\,(\varphi)$) and right well (call it

$\psi_{\text{right}}(\varphi)$). As mentioned earlier, these corresponds to counterclockwise ↺ and clockwise ↻ currents in the loop.

$$\psi_{\text{left}}(\varphi) = \langle\varphi|\,\text{left}\rangle\,, \tag{97}$$

$$\psi_{\text{right}}(\varphi) = \langle\varphi|\,\text{right}\rangle\,. \tag{98}$$

So,

$$|0\rangle \simeq \frac{|\text{left}\rangle + |\text{right}\rangle}{\sqrt{2}}, \tag{99}$$

$$|1\rangle \simeq \frac{|left\rangle - |right\rangle}{\sqrt{2}}. \tag{100}$$

In the following we neglect the overlap under the barrier

$$\langle\text{left}|\ldots|\text{right}\rangle \simeq 0, \tag{101}$$

so

$$\langle 0|\left(\hat{\varphi} - (2n+1)\,\pi\right)|0\rangle \simeq \frac{1}{2}\langle\text{left}|\left(\varphi - (2n+1)\,\pi\right)|\text{left}\rangle + \frac{1}{2}\langle\text{right}|\left(\varphi - (2n+1)\,\pi\right)|\text{right}\rangle \simeq 0 \tag{102}$$

Remember here that $(\varphi - (2n+1)\,\pi)$ is negative where $\psi_{\text{left}}(\varphi) \neq 0$ and positive $\psi_{\text{right}}(\varphi) \neq 0$. The sum is zero simply because ψ_{left} and ψ_{right} are symmetric with respect to the axis $\varphi = (2n+1)\,\pi$.

$$\langle 0|\left(\varphi - (2n+1)\,\pi\right)|1\rangle \simeq -\frac{1}{2}\langle\text{right}|\left(\varphi - (2n+1)\,\pi\right)|\text{right}\rangle + \tag{103}$$

$$\frac{1}{2}\langle\text{left}|\left(\varphi - (2n+1)\,\pi\right)|\text{left}\rangle \tag{104}$$

$$= \langle\text{right}|\left(\varphi - (2n+1)\,\pi\right)|\text{right}\rangle \neq 0. \tag{105}$$

Similarly,

$$\langle 1|\left(\varphi - (2n+1)\,\pi\right)|1\rangle \simeq 0. \tag{106}$$

So, we have shown that, in the basis $|0\rangle$, $|1\rangle$:

$$\hat{H}_{\text{total}} = E_0\,|0\rangle\langle 0| + E_1\,|1\rangle\langle 1| + \Sigma\delta\Phi_e\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right) =$$

$$= \frac{E_0 + E_1}{2}\hat{I} + \frac{E_1 - E_0}{2}\hat{\sigma}_z + \Sigma\delta\Phi_e\hat{\sigma}_x. \tag{107}$$

But $\frac{E_0+E_1}{2}\hat{I}$ is constant, so

$$\boxed{\hat{H}_{\text{total}} = \frac{E_1 - E_0}{2}\hat{\sigma}_z + \Sigma\delta\Phi_e\hat{\sigma}_x} \tag{108}$$

Now we can make rotations around x, by taking $\delta\Phi_e = \frac{\hbar\Omega}{\Sigma}\cos\omega t$, $E_1 - E_0 = \hbar\nu$.

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[1] Tero T. Heikkilä, The physics of nanoelectronics, Oxford University Press (2013).

[2] A. M. Zagoskin, Quantum Engineering: Theory and Design of Quantum Coherent Structures, Cambridge University Press (2011).

[3] M. H. Devoret, A. Wallraff, J. M. Martinis, "Superconducting Qubits: A Short Review", arXiv:cond-mat/0411174.

[4] John M. Martinis, Kevin Osborne, "Superconducting Qubits and the Physics of Josephson Junctions", arXiv:cond-mat/0402415

[5] G. Wendin, V.S. Shumeiko, "Superconducting Quantum Circuits, Qubits and Computing", arXiv:cond-mat/0508729.

[6] Yuriy Makhlin, Gerd Schön, Alexander Shnirman, "Quantum-state engineering with Josephson-junction devices", Rev. Mod. Phys. 73, 357–400 (2001).