ELEC-E4130

Lecture 15: Multilayer (Stratified Media) N layers, oblique incidence

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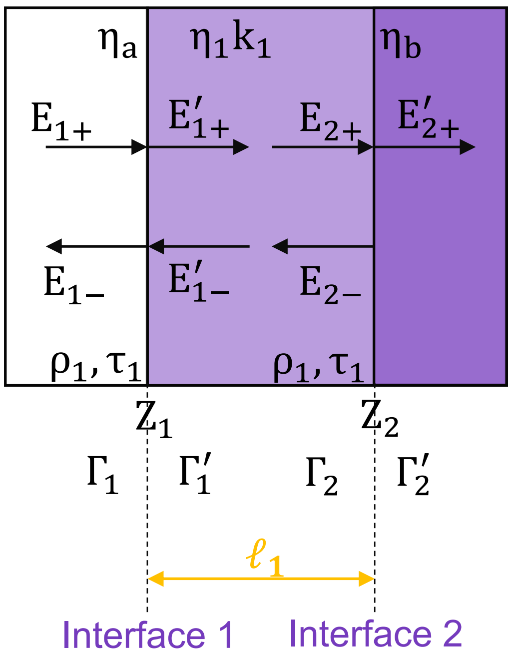
Reflection coefficient of slab



Reflection Coefficient Propagation

$$\Gamma_1 = \frac{\rho_1 + \Gamma'_1}{1 + \rho_1 \Gamma'_1}$$

$$\Gamma'_1 = \Gamma_2 e^{-j2k_1\ell_1}$$

$$\Gamma_1 = \frac{\rho_1 + \Gamma_2 e^{-j2k_1\ell_1}}{1 + \rho_1 \Gamma_2 e^{-j2k_1\ell_1}}$$

$$\Gamma_2 = \rho_2$$

$$\boxed{\Gamma_1 = \frac{\rho_1 + \rho_2 e^{-j2k_1\ell_1}}{1 + \rho_1 \rho_2 e^{-j2k_1\ell_1}}}$$

Matching/propagation Matrices

$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \frac{1}{\tau_1} \cdot \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{jk_1\ell_1} & 0 \\ 0 & e^{-jk_1\ell_1} \end{bmatrix} \cdot \frac{1}{\tau_2} \cdot \begin{bmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} E'_{2+} \\ 0 \end{bmatrix}$$

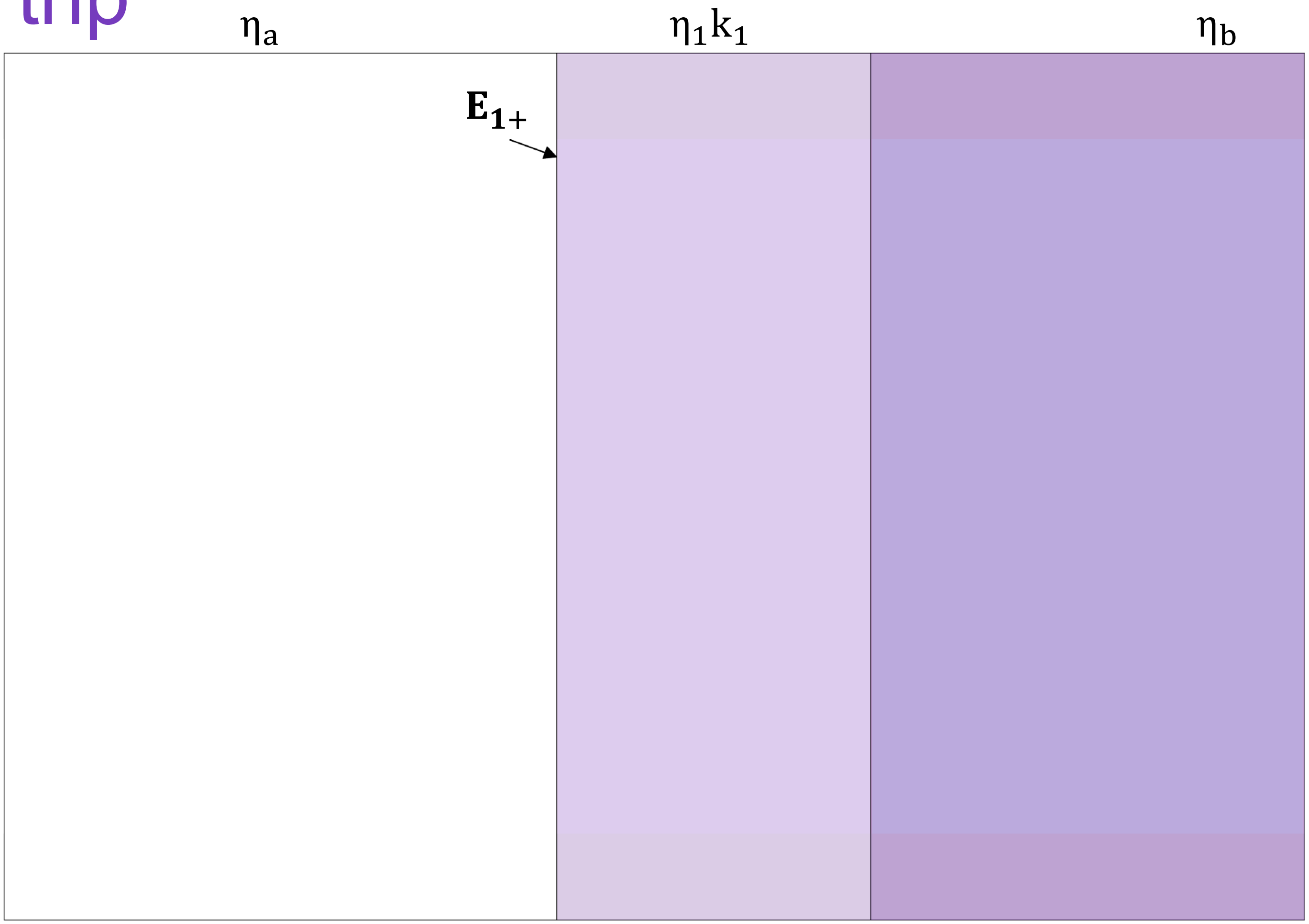
$$E_{1+} = \frac{e^{jk_1\ell_1}}{\tau_1 \tau_2} \left(1 + \rho_1 \rho_2 e^{-j2k_1\ell_1}\right) E'_{2+}$$

$$E_{1-} = \frac{e^{jk_1\ell_1}}{\tau_1 \tau_2} \left(\rho_1 + \rho_2 e^{-j2k_1\ell_1}\right) E'_{2+}$$

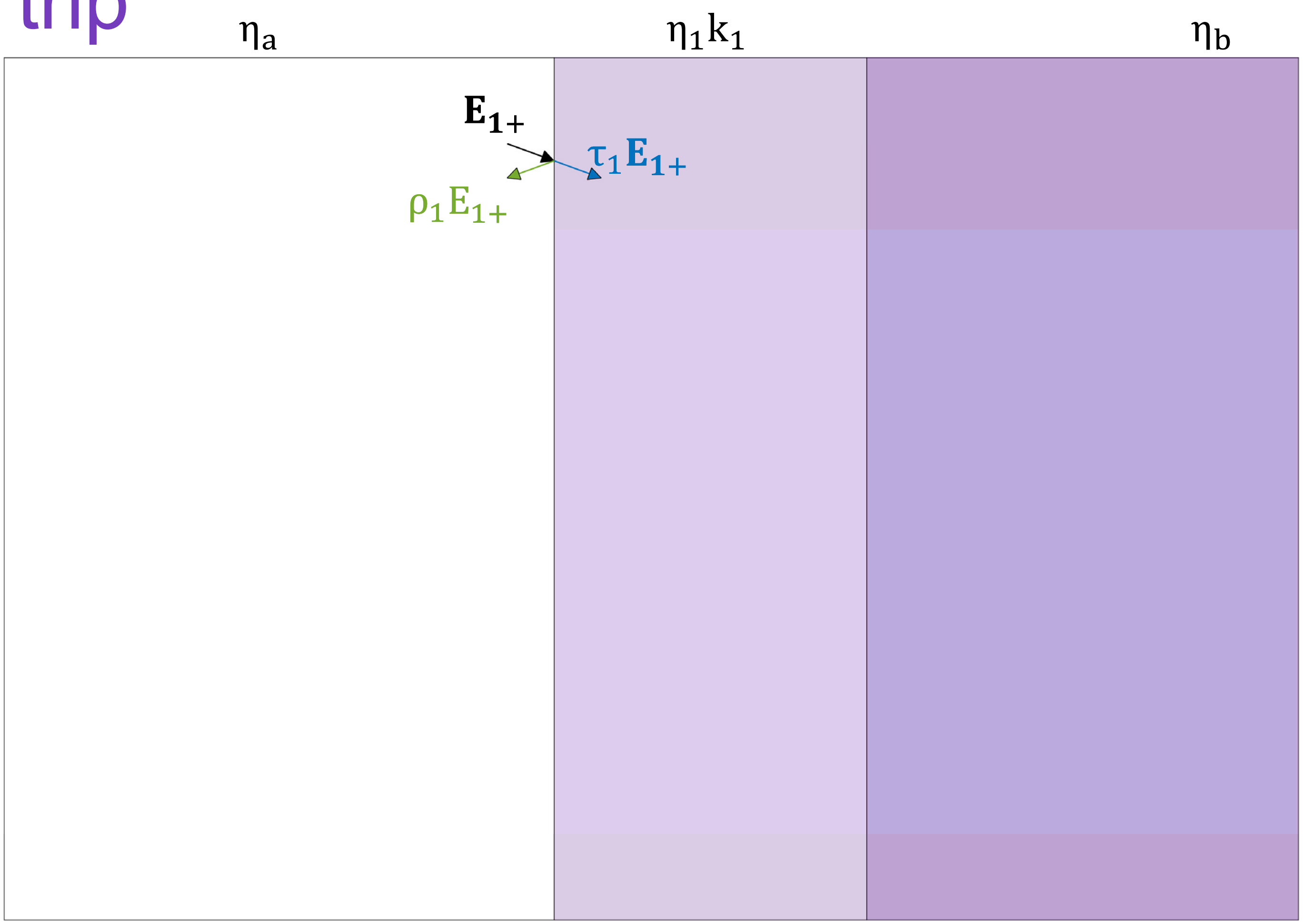
$$\boxed{\Gamma_1 = \frac{E_{1-}}{E_{1+}} = \frac{\rho_1 + \rho_2 e^{-j2k_1\ell_1}}{1 + \rho_1 \rho_2 e^{-j2k_1\ell_1}}}$$

Single dielectric slab via geometric series

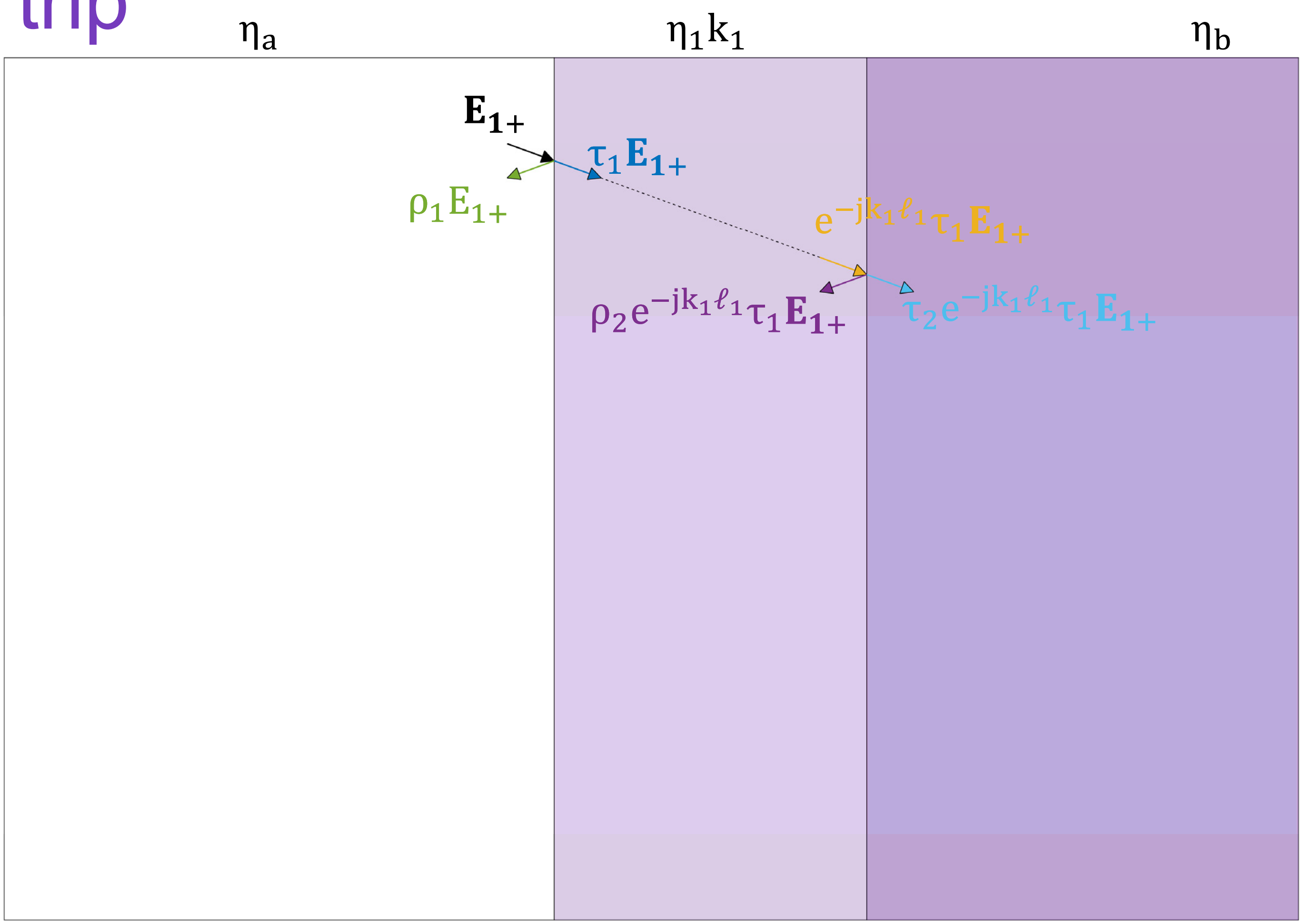
Round trip



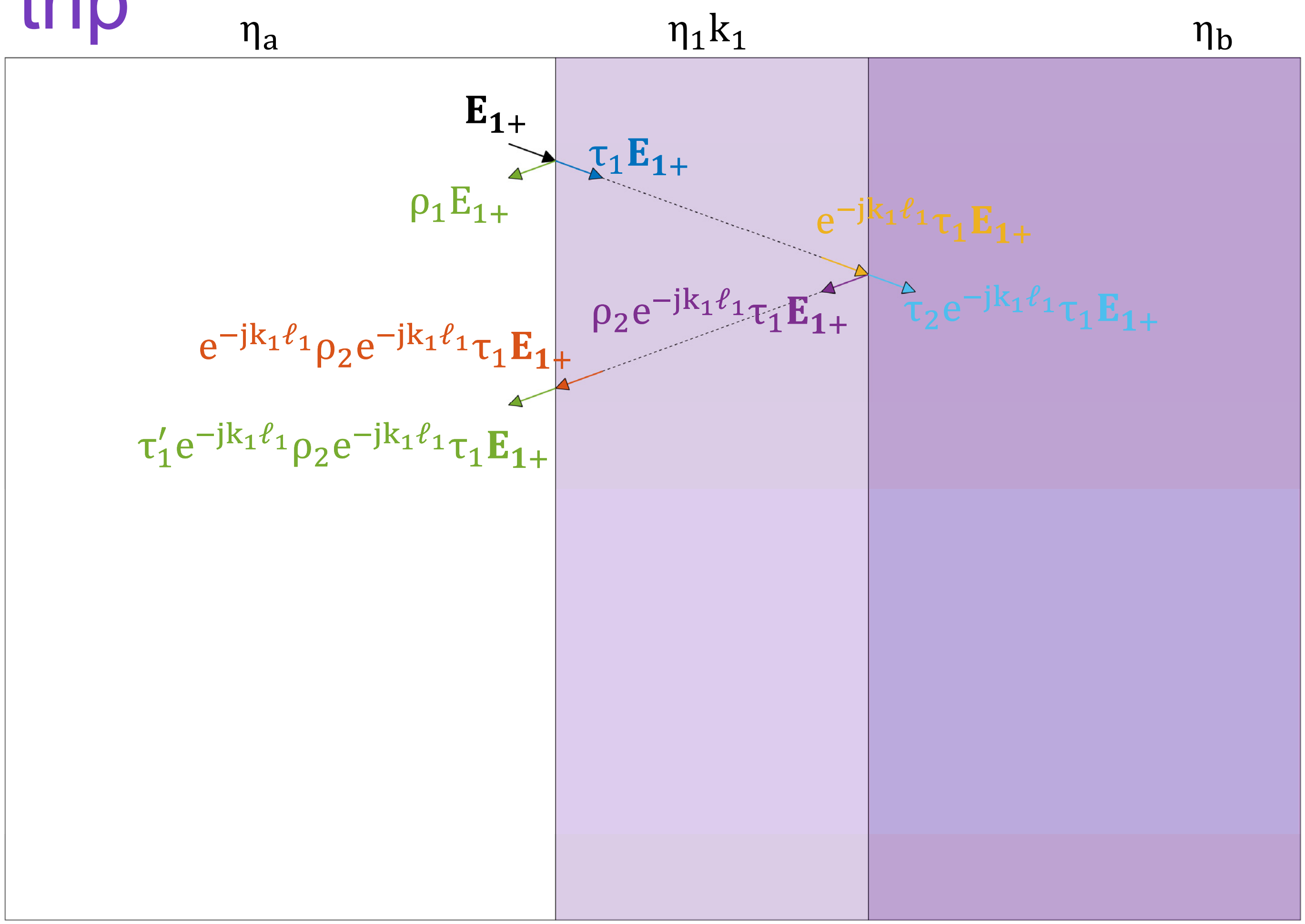
Round trip



Round trip

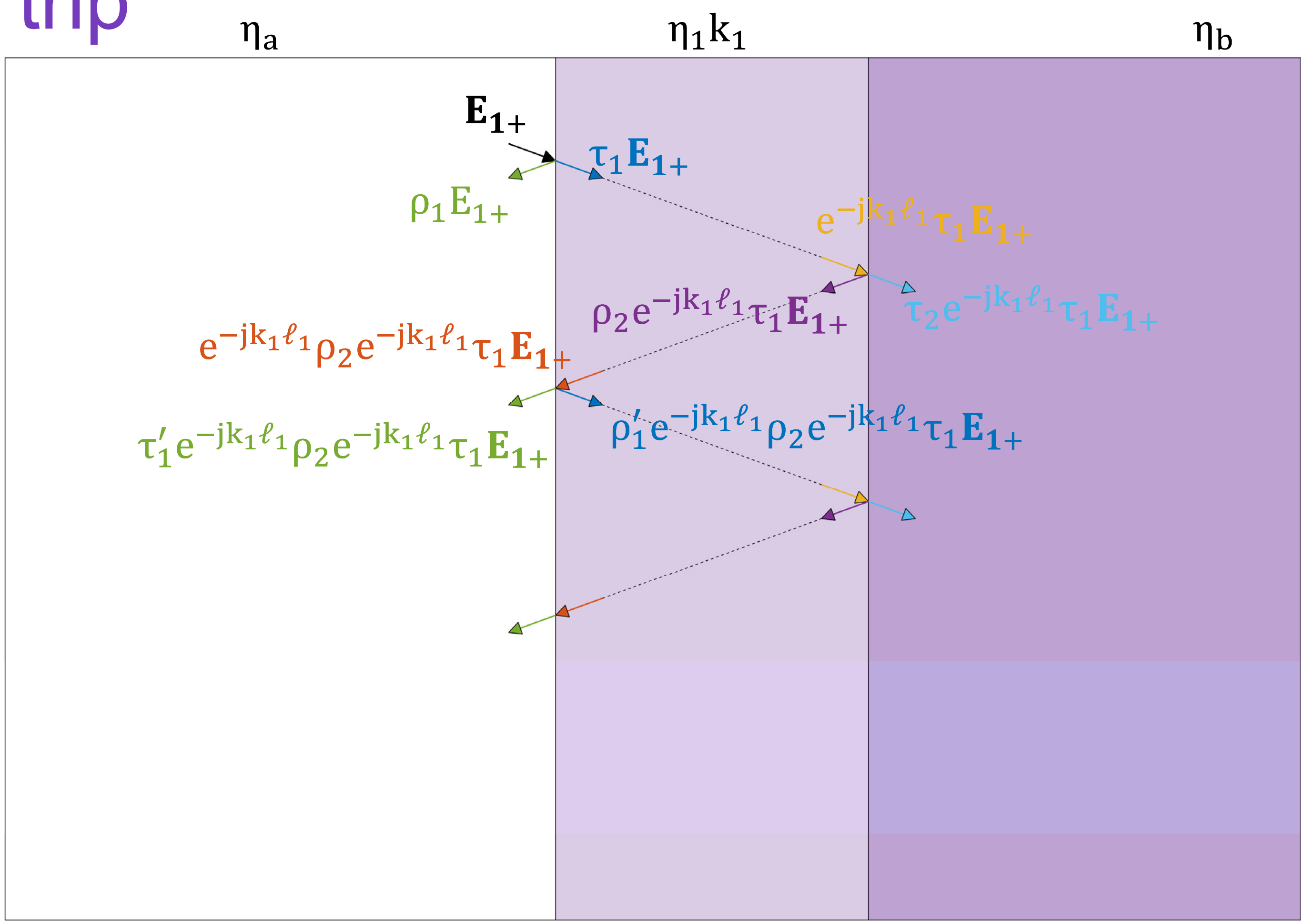

η_a
$\eta_1 k_1$
η_b
$\mathbf{E_{1+}}$
$\tau_1 \mathbf{E_{1+}}$
$\rho_1 E_{1+}$
$e^{-jk_1\ell_1}\tau_1 \mathbf{E_{1+}}$
$\rho_2 e^{-jk_1\ell_1}\tau_1 \mathbf{E_{1+}}$
$\tau_2 e^{-jk_1\ell_1}\tau_1 \mathbf{E_{1+}}$

Round trip



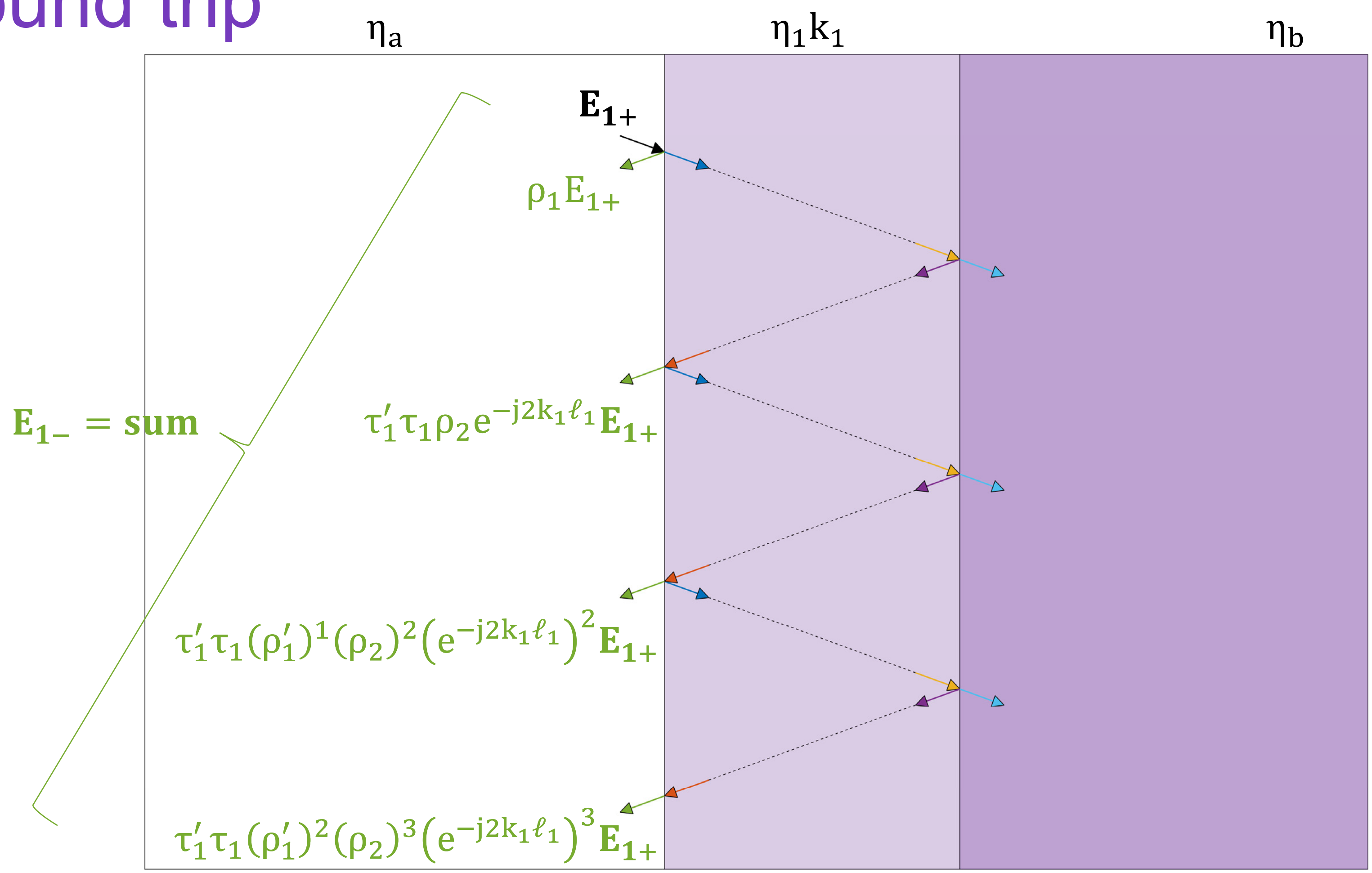
η_a
$\eta_1 k_1$
η_b
$\mathbf{E_{1+}}$
$\tau_1 \mathbf{E_{1+}}$
$\rho_1 E_{1+}$
$e^{-jk_1\ell_1}\tau_1 \mathbf{E_{1+}}$
$\rho_2 e^{-jk_1\ell_1}\tau_1 \mathbf{E_{1+}}$
$\tau_2 e^{-jk_1\ell_1}\tau_1 \mathbf{E_{1+}}$
$e^{-jk_1\ell_1}\rho_2 e^{-jk_1\ell_1}\tau_1 \mathbf{E_{1+}}$
$\tau_1' e^{-jk_1\ell_1}\rho_2 e^{-jk_1\ell_1}\tau_1 \mathbf{E_{1+}}$

Round trip



ηa
η1k1
ηb
E1+
τ1E1+
ρ1E1+
e−jk1ℓ1τ1E1+
ρ2e−jk1ℓ1τ1E1+
τ2e−jk1ℓ1τ1E1+
e−jk1ℓ1ρ2e−jk1ℓ1τ1E1+
τ′1e−jk1ℓ1ρ2e−jk1ℓ1τ1E1+
ρ′1e−jk1ℓ1ρ2e−jk1ℓ1τ1E1+

Round trip



η_a
$\eta_1 k_1$
η_b
$\mathbf{E_{1+}}$
$\rho_1 E_{1+}$
$\mathbf{E_{1-}} = \mathbf{sum}$
$\tau_1' \tau_1 \rho_2 e^{-j2k_1\ell_1} \mathbf{E_{1+}}$
$\tau_1' \tau_1 (\rho_1')^1 (\rho_2)^2 \left(e^{-j2k_1\ell_1}\right)^2 \mathbf{E_{1+}}$
$\tau_1' \tau_1 (\rho_1')^2 (\rho_2)^3 \left(e^{-j2k_1\ell_1}\right)^3 \mathbf{E_{1+}}$

Round trip

$$\Gamma_1 = \frac{E_{1-}}{E_{1+}} = \rho_1 + \sum_{N=1}^{\infty} \tau_1' \tau_1 (\rho_1')^{N-1} (\rho_2)^N \left(e^{-j2k_1\ell_1}\right)^N$$

$$\Gamma_1 = \frac{E_{1-}}{E_{1+}} = \rho_1 + \frac{\tau_1'\tau_1}{\rho_1'} \sum_{N=1}^{\infty} (\rho_1')^N (\rho_2)^N \left(e^{-j2k_1\ell_1}\right)^N$$

$$\Gamma_1 = \frac{E_{1-}}{E_{1+}} = \rho_1 + \frac{\tau_1'\tau_1}{\rho_1'} \left[-1 + \sum_{N=0}^{\infty} \left(\rho_1'\rho_2 e^{-j2k_1\ell_1}\right)^N\right]$$

Geometric Series

$$\sum_{N=0}^{\infty} (x)^N = \frac{1}{x} \qquad \text{when } |x| < 1$$

$$\Gamma_1 = \frac{E_{1-}}{E_{1+}} = \rho_1 + \frac{\tau_1'\tau_1}{\rho_1'} \left[-1 + \frac{1}{1 - \rho_1'\rho_2 e^{-j2k_1\ell_1}}\right]$$

Round trip

$$\Gamma_1 = \rho_1 + \frac{\tau_1'\tau_1}{\rho_1'}\left[-1 + \frac{1}{1 - \rho_1'\rho_2 e^{-j2k_1\ell_1}}\right]$$

$$\Gamma_1 = \rho_1 + \frac{\tau_1'\tau_1}{\rho_1'}\left[\frac{-1 + \rho_1'\rho_2 e^{-j2k_1\ell_1} + 1}{1 - \rho_1'\rho_2 e^{-j2k_1\ell_1}}\right]$$

Matching Matrices from Lecture 13

$$\Gamma_1 = \rho_1 + \frac{\tau_1'\tau_1}{\rho_1'}\left[\frac{\rho_1'\rho_2 e^{-j2k_1\ell_1}}{1 - \rho_1'\rho_2 e^{-j2k_1\ell_1}}\right]$$

$$\tau_1'\tau_1 = 1 - \rho_1^2$$

$$\rho_1 = -\rho_1'$$

$$\Gamma_1 = \rho_1 + (1 - \rho_1^2)\left[\frac{\rho_2 e^{-j2k_1\ell_1}}{1 + \rho_1\rho_2 e^{-j2k_1\ell_1}}\right]$$

Round trip

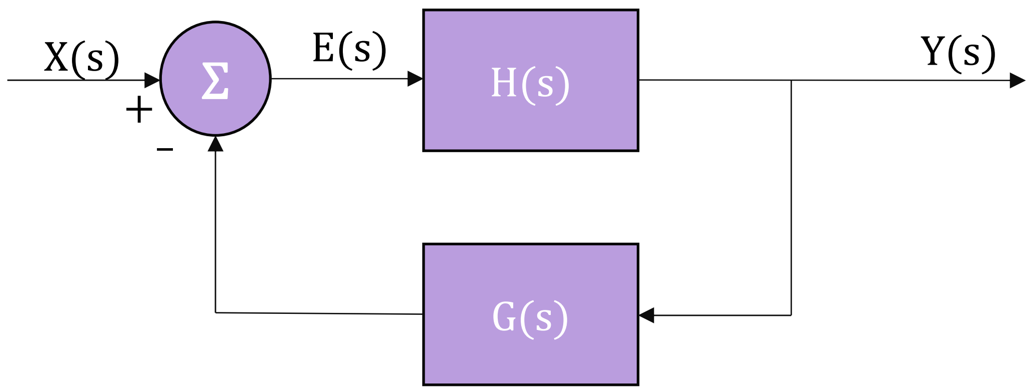
$$\Gamma_1 = \rho_1 + (1 - \rho_1^2)\left[\frac{\rho_2 e^{-j2k_1\ell_1}}{1 - \rho_1\rho_2 e^{-j2k_1\ell_1}}\right]$$

$$\Gamma_1 = \frac{\rho_1\left(1 + \rho_1\rho_2 e^{-j2k_1\ell_1}\right) + (1 - \rho_1^2)\rho_2 e^{-j2k_1\ell_1}}{1 + \rho_1\rho_2 e^{-j2k_1\ell_1}}$$

$$\boxed{\Gamma_1 = \frac{\rho_1 + \rho_2 e^{-j2k_1\ell_1}}{1 + \rho_1\rho_2 e^{-j2k_1\ell_1}}}$$

- Reflection from a single slab can be computed from the infinite sum of plane waves bouncing around the slab
- Amplitude of waves decays geometrically
- Interference at surface dictated by
  - $k_1\ell_1$
  - Phase contributions of ρ_1, ρ_2

Comparisons to feedback control theory



$$E(s) = X(s) - G(s)Y(s)$$
$$Y(s) = H(s)E(s)$$
$$Y(s) = H(s)\big(X(s) - G(s)Y(s)\big)$$

$$\frac{Y(s)}{X(s)} = \frac{H(s)}{1 + G(s)H(s)}$$

From slide 11

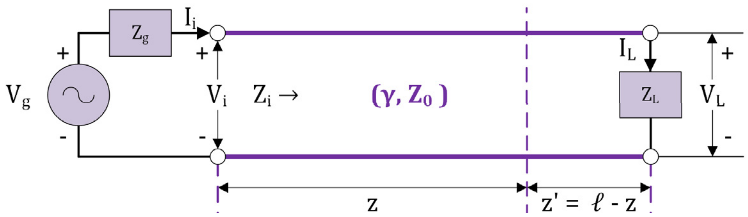
$$\Gamma_1 = \rho_1 + (1 - \rho_1^2)\left[\frac{\rho_2 e^{-j2k_1\ell_1}}{1 + \rho_1\rho_2 e^{-j2k_1\ell_1}}\right]$$

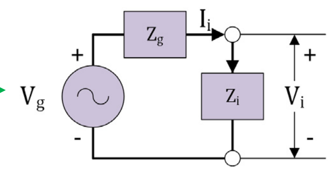
Feedback term

$$H(s) \sim \rho_2 e^{-j2k_1\ell_1}$$
$$G(s) \sim \rho_1$$

- Can you design the layers to act like a filter? YES!
- Can you use filter/linear system theory to design the layers? YES!

Comparisons to transmission line theory





$$Z_i = Z_0 \frac{Z_L + Z_0 \tanh(\gamma\ell)}{Z_0 + Z_L \tanh(\gamma\ell)}$$

Reflection Coefficient of dielectric slab

$$\Gamma_1 = \frac{\rho_1 + \rho_2 e^{-j2k_1\ell_1}}{1 + \rho_1\rho_2 e^{-j2k_1\ell_1}}$$

Voltage and current phasors, arb. Line

$$V(z') = V_g \frac{Z_0}{Z_0 + Z_g} e^{\gamma(z'-\ell)} \frac{1 + \Gamma e^{-2\gamma z'}}{1 - \Gamma_g \Gamma e^{-2\gamma\ell}}$$

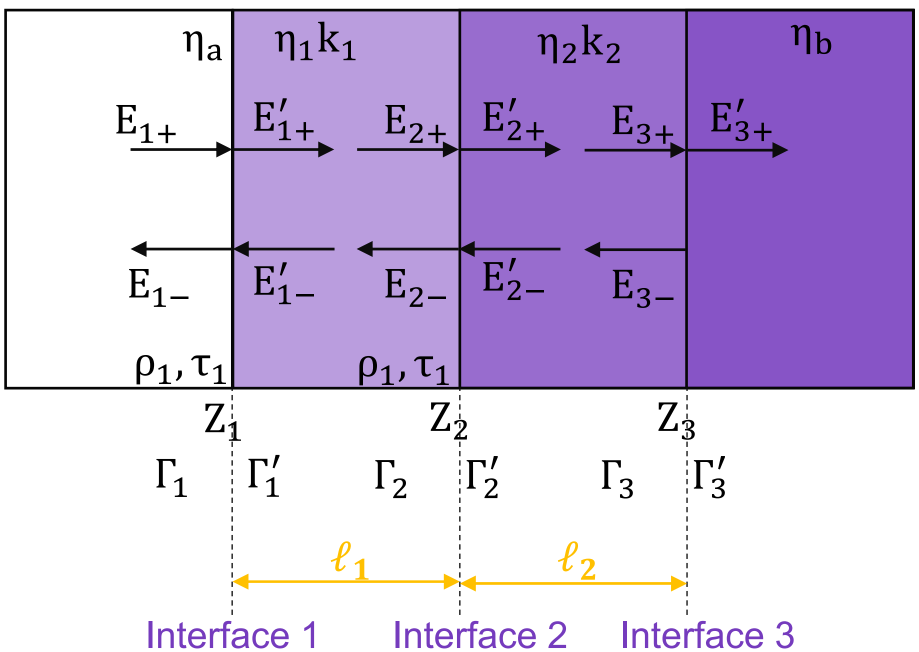
$$I(z') = V_g \frac{1}{Z_0 + Z_g} e^{\gamma(z'-\ell)} \frac{1 + \Gamma e^{-2\gamma z'}}{1 - \Gamma_g \Gamma e^{-2\gamma\ell}}$$

- Physical thickness and propagation constant dependent

Normal Incidence, M layers

2…M layers

Lets let $E'_{2-} \neq 0$



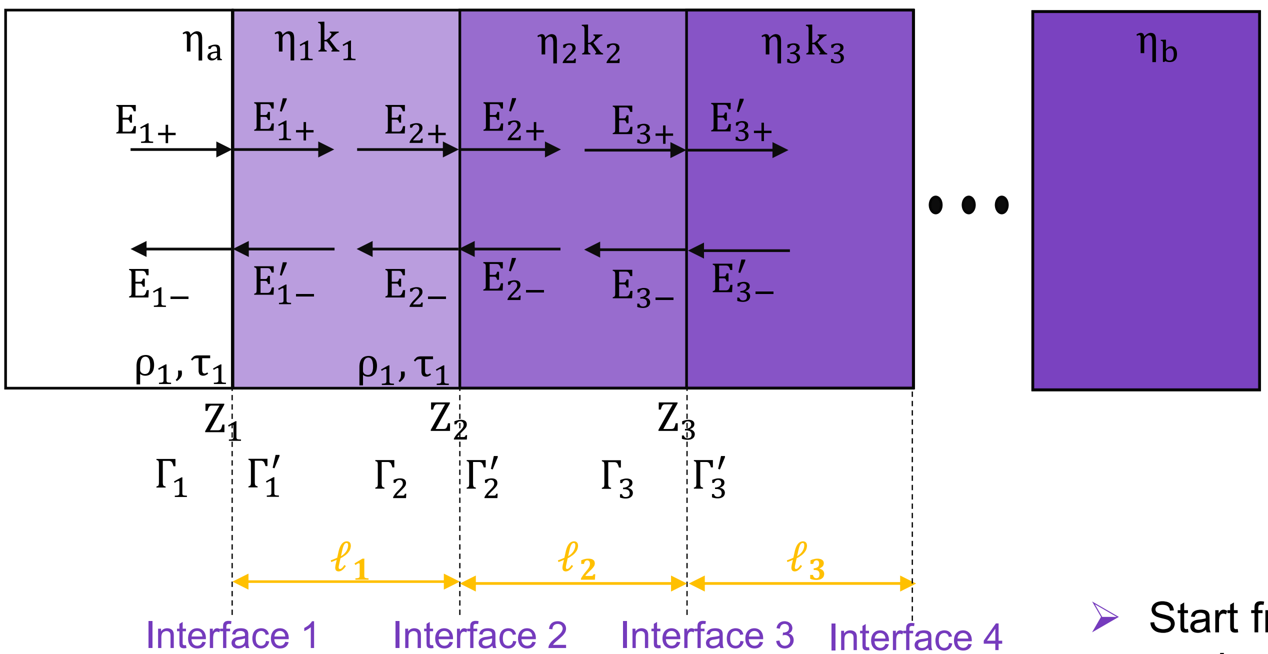
$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \frac{1}{\tau_1} \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} e^{jk_1\ell_1} & 0 \\ 0 & e^{-jk_1\ell_1} \end{bmatrix} \frac{1}{\tau_2} \begin{bmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{bmatrix} \begin{bmatrix} E'_{2+} \\ E'_{2-} \end{bmatrix}$$

$$E_{1+} = \frac{e^{jk_1\ell_1}}{\tau_1\tau_2}\left(1 + \rho_1\rho_2 e^{-j2k_1\ell_1}\right)E'_{2+}$$

$$E_{1-} = \frac{e^{jk_1\ell_1}}{\tau_1\tau_2}\left(\rho_1 + \rho_2 e^{-j2k_1\ell_1}\right)E'_{2+}$$

$$\Gamma_1 = \frac{E_{1-}}{E_{1+}} = \frac{\rho_1 + \frac{E_{2-}}{E_{2+}} e^{-j2k_1\ell_1}}{1 + \rho_1 \frac{E_{2-}}{E_{2+}} e^{-j2k_1\ell_1}} = \frac{\rho_1 + \Gamma_2 e^{-j2k_1\ell_1}}{1 + \rho_1 \Gamma_2 e^{-j2k_1\ell_1}}$$

2…M layers



M slabs

M+1 interfaces

m = 1…M+1

$$\Gamma_m = \frac{\rho_m + \Gamma_{m+1} e^{-j2k_m \ell_m}}{1 + \rho_m \Gamma_{m+1} e^{-j2k_m \ell_m}}$$

$$\Gamma_1 = \text{goal coefficient}$$

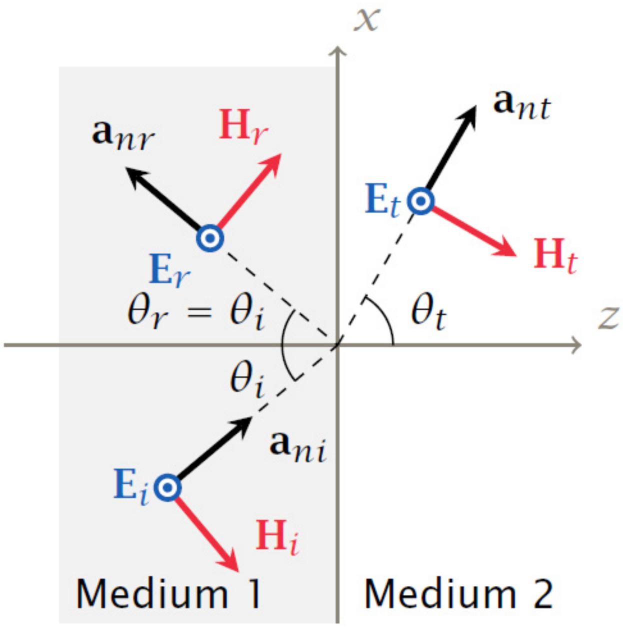
$$\Gamma_{M+1} = \rho_{m+1}$$

- Start from the second half space and work your way back to thee first
- Second half space only has a forward propagating wave thus ending the recursion

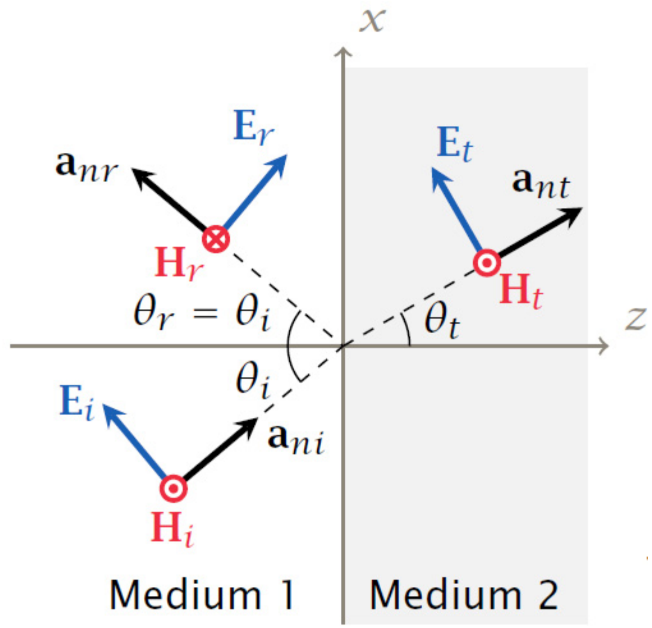
Oblique incidence (Dielectrics)

Review of oblique angle and polarization

Perpendicular (TE)



Parallel (TM)



Wave impedance

$$\eta_T = \begin{cases} \eta\cos(\theta) & \parallel, \mathrm{TM} \\ \dfrac{\eta}{\cos(\theta)} & \perp, \mathrm{TE} \end{cases}$$

Refractive index

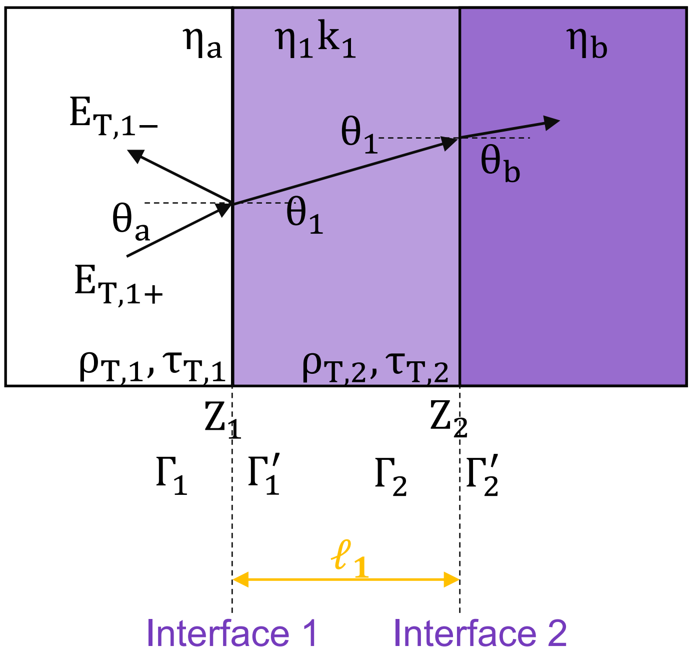
$$n_T = \begin{cases} \dfrac{n}{\cos(\theta)} & \parallel, \mathrm{TM} \\ n\cos(\theta) & \perp, \mathrm{TE} \end{cases}$$

Snell's law

$$\sin(\theta_i) = \sin(\theta_t)$$

- Obtained by continuity of transverse component
- Results matching coefficients indicated by the ***T*** subscript

Oblique Incidence: Transverse components



$$\eta_T = \begin{cases} \eta\cos(\theta) & \parallel, \mathrm{TM} \\ \dfrac{\eta}{\cos(\theta)} & \perp, \mathrm{TE} \end{cases}$$

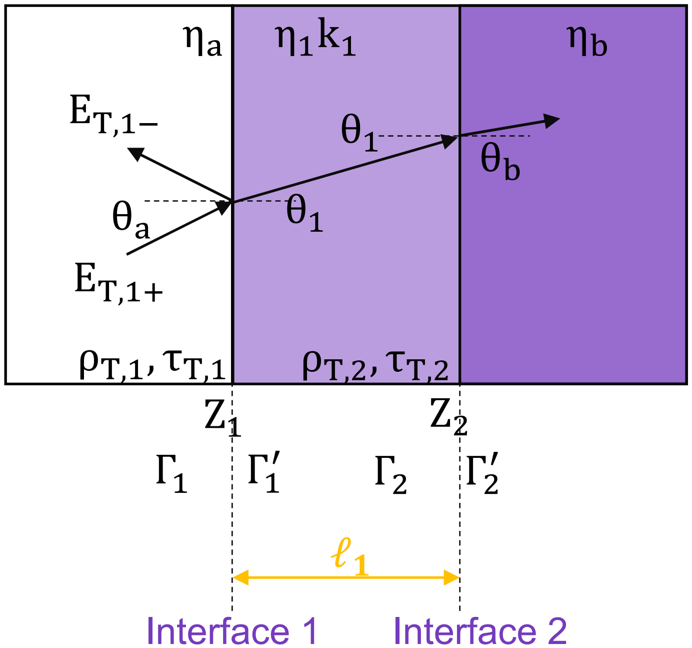
$$\eta_T' = \begin{cases} \eta'\cos(\theta') & \parallel, \mathrm{TM} \\ \dfrac{\eta'}{\cos(\theta')} & \perp, \mathrm{TE} \end{cases}$$

$$n_T = \begin{cases} \dfrac{n}{\cos(\theta)} & \parallel, \mathrm{TM} \\ n\cos(\theta) & \perp, \mathrm{TE} \end{cases}$$

$$n_T' = \begin{cases} \dfrac{n'}{\cos(\theta')} & \parallel, \mathrm{TM} \\ n'\cos(\theta') & \perp, \mathrm{TE} \end{cases}$$

Medium → Medium' → Medium 1 → Medium 2

Oblique Incidence: Transverse components



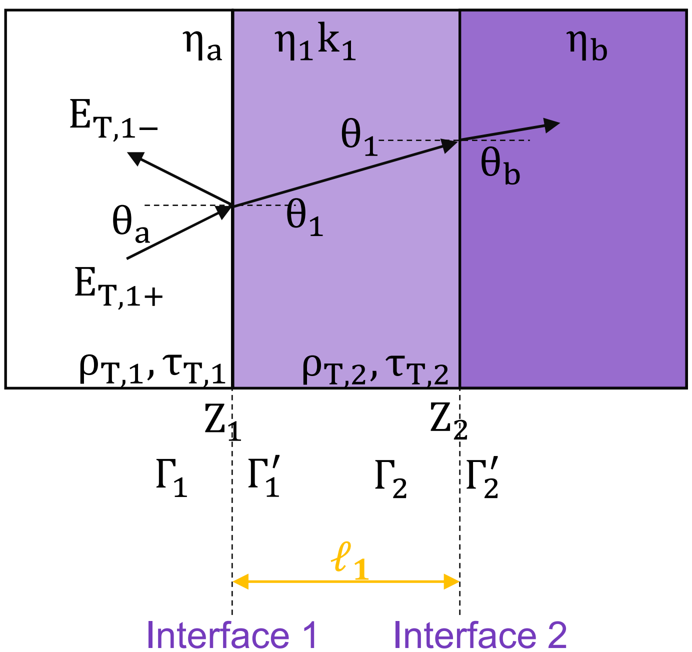
Fresnel's Coefficients

$$\rho_{T,1} = \frac{\eta_{T,1} - \eta_{T,a}}{\eta_1 + \eta_a} \qquad \tau_{T,1} = 1 + \rho_{T,1}$$

$$\rho_{T,2} = \frac{\eta_{T,b} - \eta_{T,1}}{\eta_{T,b} + \eta_{T,1}} \qquad \tau_{T,2} = 1 + \rho_{T,2}$$

- Same form as those computed with normal incidence

Oblique Incidence: Transverse components



Transverse field matching

Matching the z-component of the k-vector

$$k \rightarrow k_z$$

$$k_z = k\cos(\theta)$$

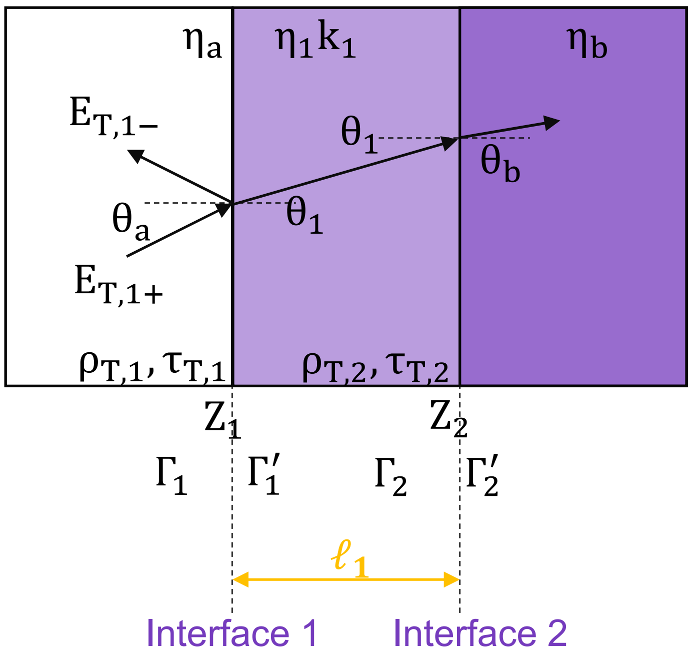
Propagation matrix

$$\begin{bmatrix} e^{jk_1\ell_1} & 0 \\ 0 & e^{-jk_1\ell_1} \end{bmatrix} \rightarrow \begin{bmatrix} e^{jk_{z,1}\ell_1} & 0 \\ 0 & e^{-jk_{z,1}\ell_1} \end{bmatrix}$$

$$\begin{bmatrix} e^{jk_{z,1}\ell_1} & 0 \\ 0 & e^{-jk_{z,1}\ell_1} \end{bmatrix} \rightarrow \begin{bmatrix} e^{jk_1\ell_1\cos(\theta)} & 0 \\ 0 & e^{-jk_1\ell_1\cos(\theta)} \end{bmatrix}$$

The transverse fields across the slap are related by $\cos(\theta)$

Oblique Incidence: Transverse components



$$\begin{bmatrix} E_{T,1+} \\ E_{T,1-} \end{bmatrix} = \frac{1}{\tau_1} \cdot \begin{bmatrix} 1 & \rho_{T,1} \\ \rho_{T,1} & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{jk_1 \ell_1 \cos(\theta)} & 0 \\ 0 & e^{-jk_1 \ell_1 \cos(\theta)} \end{bmatrix}$$

$$\cdot \frac{1}{\tau_2} \cdot \begin{bmatrix} 1 & \rho_{T,2} \\ \rho_{T,2} & 1 \end{bmatrix} \cdot \begin{bmatrix} E'_{T,2+} \\ E'_{T,2-} \end{bmatrix}$$

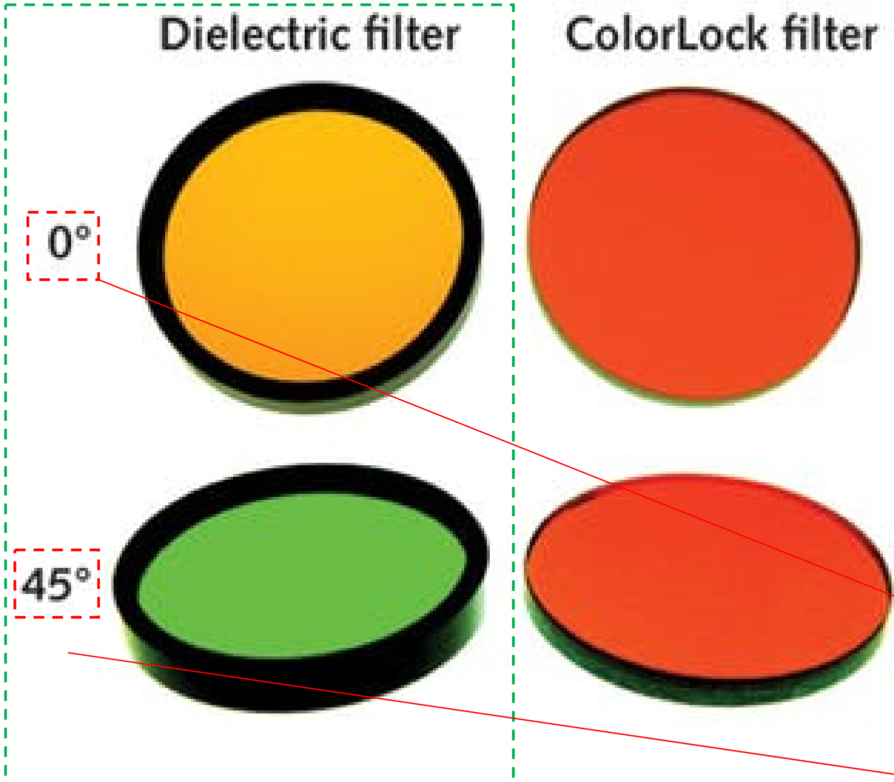
$$\Gamma_{T,m} = \frac{\rho_{T,m} + \Gamma_{T,m+1} e^{-j2\delta_m}}{1 + \rho_{T,m} \Gamma_{T,m+1} e^{-j2\delta_m}}$$

$$\delta_m = k_m \ell_m \cos(\theta)$$

$$\Gamma_1 = \text{goal coefficient}$$

$$\Gamma_{T,M+1} = \rho_{T,m+1}$$

Angle dependent behavior



- Transmission and reflection are defined by steady state constructive and destructive interference
- Interference controlled by optical path length within the dielectric
- Optical path is increased going from normal to obliques incidence angle

$$\Gamma_{T,m} = \frac{\rho_{T,m} + \Gamma_{T,m+1} e^{-j2\delta_m}}{1 + \rho_{T,m}\Gamma_{T,m+1} e^{-j2\delta_m}}$$

$$\delta_m = k_m \ell_m \cos(\theta)$$

Conclusions and next time

- This time we
  - Solve for the reflection via geometric series (ray tracing)
  - Extrapolate to multiple layers
  - Explore oblique incidence angle
- Stack of dielectric layers acts like a filter by modulating optical path lengths
  - Delayed and attenuated copies of the incident field
- This Thursday we'll do some numerical examples