ELEC-E4130

Lecture 17: Rectangular Waveguides

Ch. 10

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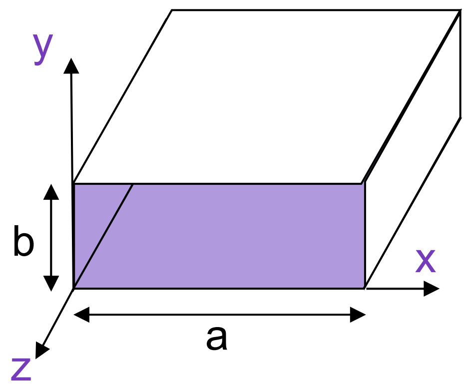
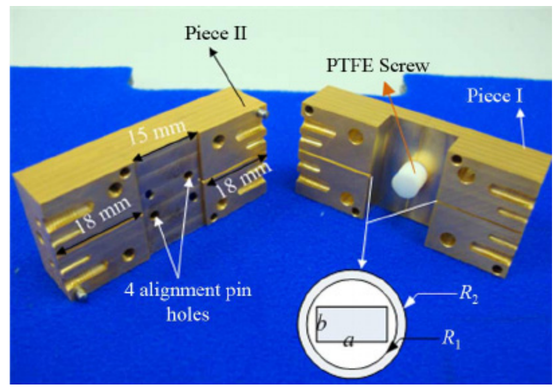
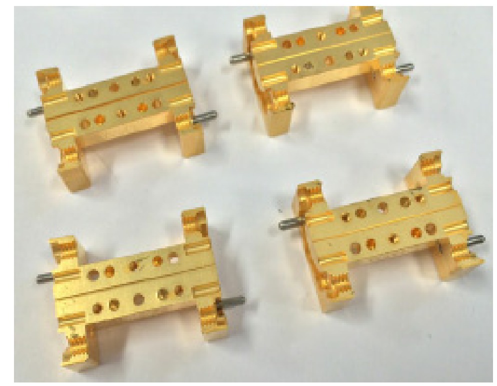
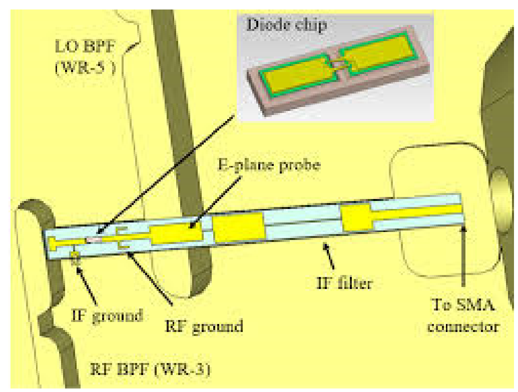
Rectangular waveguides



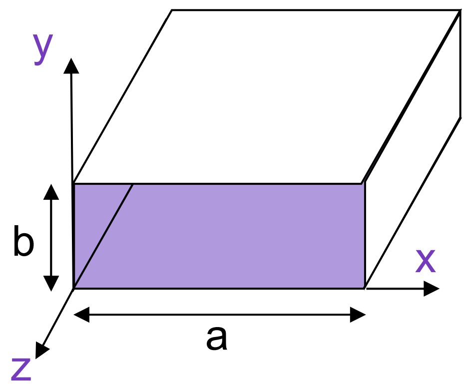


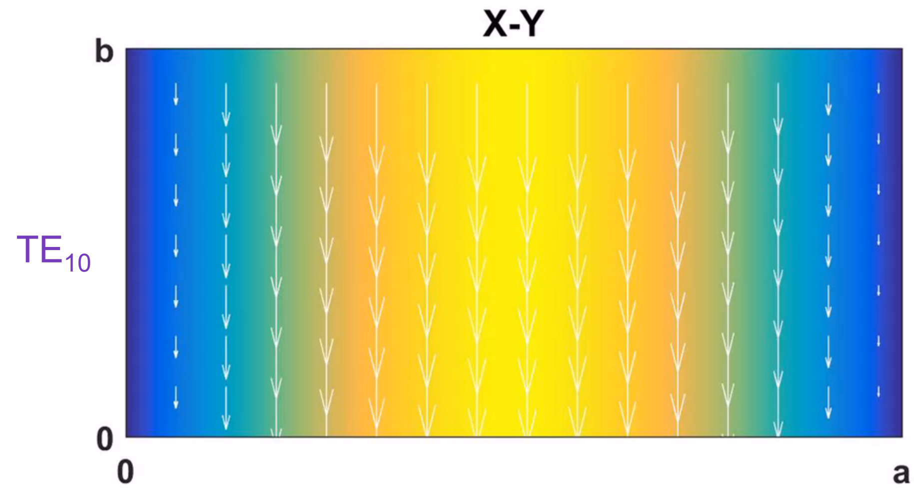


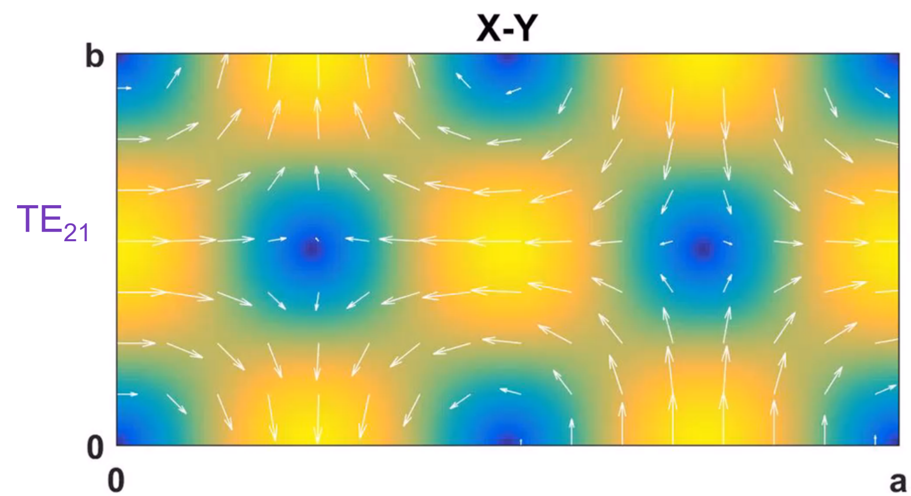


- Metallic pipe with rectangular cross section
- PEC walls
- Homogenous, isotropic dielectric fill
- Low loss, high power handling, easy to fabricate
- Significant dispersion

Rectangular waveguides







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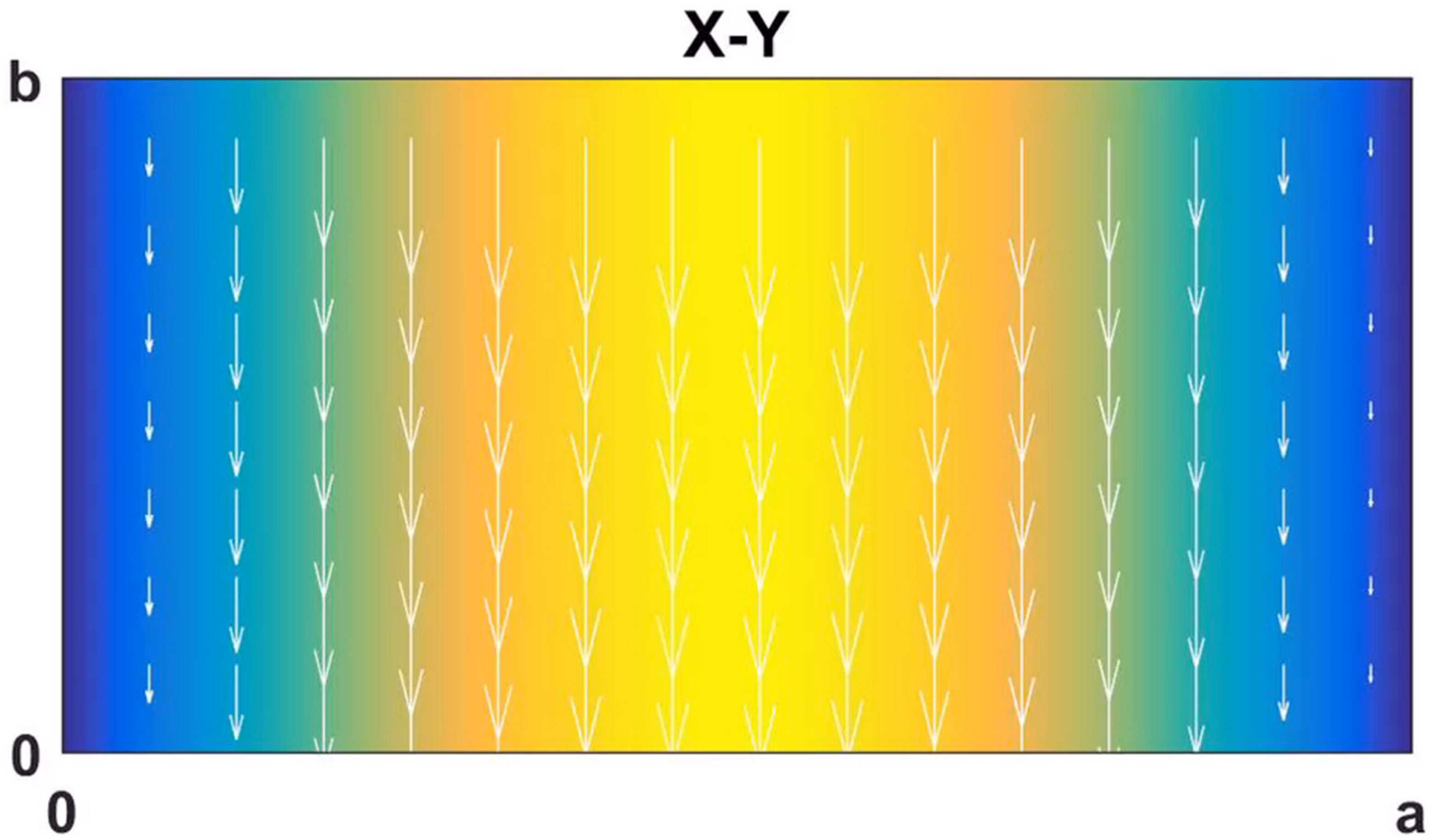
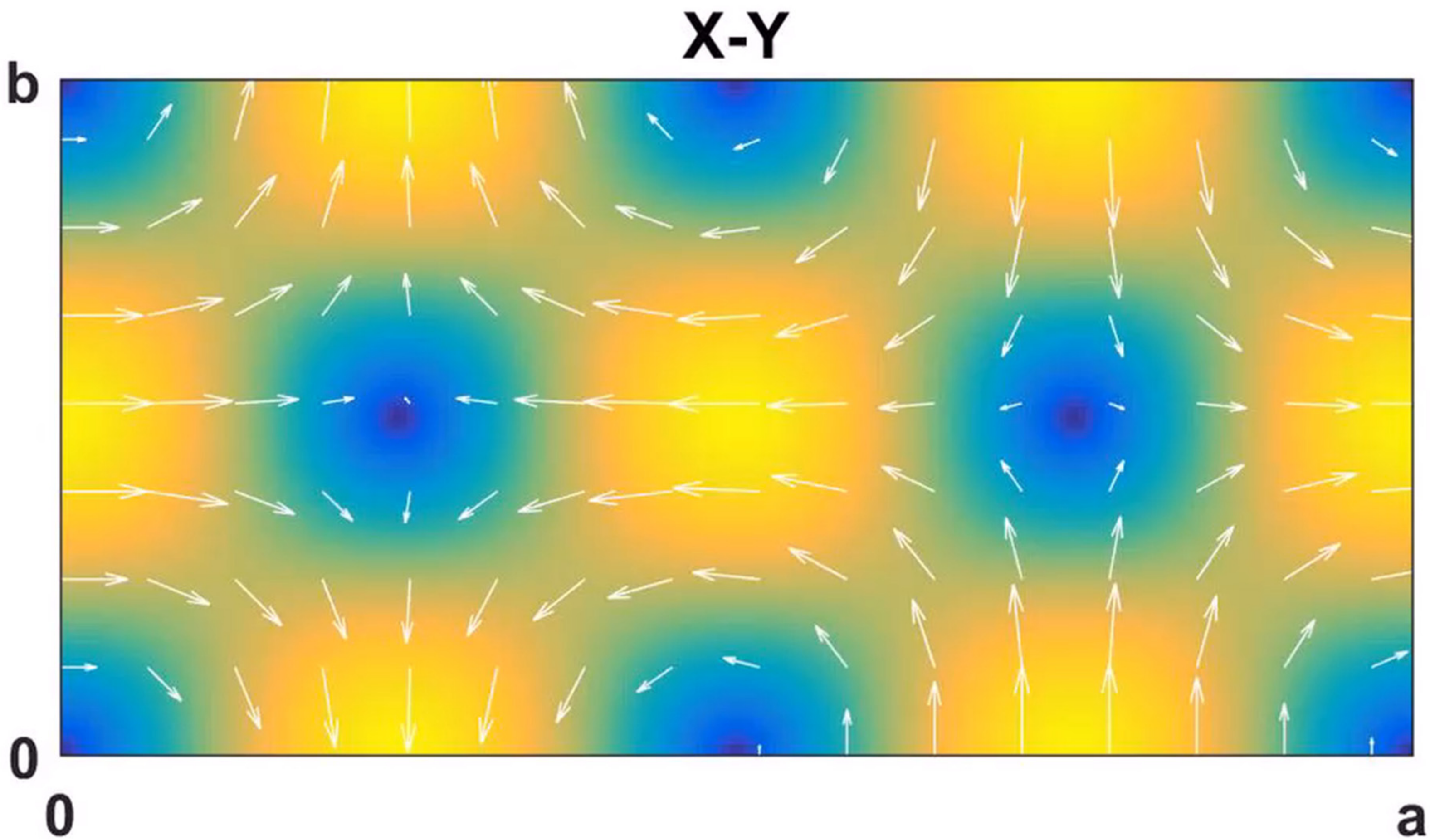
Rectangular waveguides



Rectangular waveguides



Recap of separability in plane wave solutions

Plane wave review

Arb vector field in $\mathbb{R}^3$

$$\mathbf{E}(\mathrm{x,y,z}) = \mathbf{a_x}\mathrm{E_x}(\mathrm{x,y,z}) + \mathbf{a_y}\mathrm{E_y}(\mathrm{x,y,z}) + \mathbf{a_z}\mathrm{E_z}(\mathrm{x,y,z})$$

Shorthand notation

$$\left.\begin{aligned} \mathrm{E_x}(\mathrm{x,y,z}) &= \mathrm{E_x} \\ \mathrm{E_y}(\mathrm{x,y,z}) &= \mathrm{E_y} \\ \mathrm{E_z}(\mathrm{x,y,z}) &= \mathrm{E_z} \end{aligned}\right\} \rightarrow \mathbf{E} = \mathbf{a_x}\mathrm{E_x} + \mathbf{a_y}\mathrm{E_y} + \mathbf{a_z}\mathrm{E_z}$$

Vector wave equation

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0$$

$$\nabla^2\left(\mathbf{a_x}\mathrm{E_x} + \mathbf{a_y}\mathrm{E_y} + \mathbf{a_z}\mathrm{E_z}\right) + k^2\left(\mathbf{a_x}\mathrm{E_x} + \mathbf{a_y}\mathrm{E_y} + \mathbf{a_z}\mathrm{E_z}\right) = 0$$

Scalar wave equation

$$\nabla^2\mathrm{E_x} + \mathrm{k}^2\mathrm{E_x} = 0$$
$$\nabla^2\mathrm{E_y} + \mathrm{k}^2\mathrm{E_y} = 0$$
$$\nabla^2\mathrm{E_z} + \mathrm{k}^2\mathrm{E_z} = 0$$

- Consider the arbitrary electric vector field in cartesian coordinates
- Each unit vector is associated with a scalar function of the spatial variables x,y,z
  - $E_x(x,y,z)$: $\mathbb{R}^3 \rightarrow \mathbb{R}$
- Linear, homogenous, isotropic medium therefore no cross-coupling between field components
- Vector wave equation can be solved as three separate scalar wave equations

Plane wave review

Scalar wave equation

$$\nabla^2 E_x + k^2 E_x = 0$$

Separable: product or three $\mathbb{R}^1$ equations

$$E_x(x, y, z) = f(x)g(y)h(z)$$

Rewrite scalar wave equation with separated functions

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x)g(y)h(z) + k^2 f(x)g(y)h(z) = 0$$

Cancel out terms whose derivatives are 0

$$g(y)h(z)\frac{\partial^2 f(x)}{\partial x^2} + f(x)h(z)\frac{\partial^2 g(y)}{\partial y^2} + f(x)g(y)\frac{\partial^2 h(z)}{\partial z^2} + k^2 f(x)g(y)h(z) = 0$$

Collect like terms

$$\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2} + \frac{1}{g(y)}\frac{\partial^2 g(y)}{\partial y^2} + \frac{1}{h(z)}\frac{\partial^2 h(z)}{\partial z^2} + k^2 = 0$$

- As a demonstration, lets focus on the x component Ex
- E_x(x,y,z) is separable therefore we can write E_x(x,y,z) as the product of three 1D functions
- Dividing by these functions enables collection of like terms
- Each function is a function of a single, independent variable that can vary independently from the other variables therefore each term must equal a constant

Plane wave review

Solve for each spatial variable separately by rewriting k

$$\left.\begin{aligned}\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2} &= -k_x^2\\ \frac{1}{g(y)}\frac{\partial^2 g(y)}{\partial y^2} &= -k_y^2\\ \frac{1}{h(z)}\frac{\partial^2 h(z)}{\partial z^2} &= -k_z^2\end{aligned}\right\} \rightarrow k^2 = k_x^2 + k_y^2 + k_z^2$$

1D, 2nd order ODE

$$\left.\begin{aligned}\left(\frac{\partial^2}{\partial x^2} + k_x^2\right) f(x) &= 0\\ \left(\frac{\partial^2}{\partial y^2} + k_y^2\right) g(y) &= 0\\ \left(\frac{\partial^2}{\partial z^2} + k_z^2\right) h(z) &= 0\end{aligned}\right\} \rightarrow \begin{array}{c}\text{threee separte}\\ \text{2nd order, ODEs}\end{array}$$

- Define constant k^2 as a the sum of three constants associated with each differential term
- Problem is now couched as three 2nd order, 1D, hopmogenousODEs.

Plane wave review

Traveling wave solution

$$f_1(x) = A_1 e^{-jk_x x} + B_1 e^{jk_x x}$$

$$0's \text{ of } f_1(x) \rightarrow \begin{Bmatrix} k_x \rightarrow -j\infty \\ k_x \rightarrow +j\infty \end{Bmatrix} \rightarrow \begin{matrix} k_x\text{ , x real} \\ \text{Not realizable} \end{matrix}$$

Rectangular waveguides

Standing wave solution

$$f_2(x) = C_1 \cos(k_x x) + D_1 \sin(k_x x)$$

$$0's \text{ of } f_2(x) \rightarrow \begin{Bmatrix} \cos(k_x x) & k_x x = \pm\left(n + \frac{1}{2}\right)\pi & \forall\, n\ int \\ \sin(k_x x) & k_x x = \pm(n)\pi & \forall\, n\ int \end{Bmatrix}$$

- Rectangular waveguides follow similar principles
- Solutions to differential equations are separable
- Standing wave solutions exist due to boundary conditions with finite distance between boundaries
- PEC boundary conditions provide straight forward solutions to 2nd order ODEs.

Transverse Electric (TE) and Transverse Magnetic (TM) Solutions

Waveguide solutions

General z-direction solutions can be written as:

transverse variation

$$\mathbf{E}(x,y,z) = [\mathbf{e_t}(x,y) + \hat{z}e_z(x,y)]e^{-j\beta z}$$

longitudinal variation

$$\mathbf{H}(x,y,z) = [\mathbf{h_t}(x,y) + \hat{z}h_z(x,y)]e^{-j\beta z}$$

transverse component

longitudinal component

TEM waves:	$E_z = H_z = 0$
TE waves:	$E_z = 0, H_z \neq 0$
TM waves:	$E_z \neq 0, H_z = 0$
Other hybrid waves	

- Define z as the direction of propagation
- Consider the axial component separate from the transverse component

General Waveguide solutions

Unknown phase constant yet at this point!

$$\beta \rightarrow k_z$$

Definition of general waveguide solution

$$\frac{\partial \mathbf{E}}{\partial z} = -j\beta\mathbf{E} \qquad \& \qquad \frac{\partial \mathbf{H}}{\partial z} = -j\beta\mathbf{H}$$

- z dependent variation described entirely by propagation constant β
- Take curl and equate vector components

The curl equation

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \quad \rightarrow \quad \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = j\omega\mu\left(\mathbf{a_x}H_x + \mathbf{a_y}H_y + \mathbf{a_z}H_z\right)$$

$\frac{\partial}{\partial z} \rightarrow -j\beta$

General Waveguide solutions

The curl equation No.1

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{E}$$

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x$$

$$-\frac{\partial E_z}{\partial x} - j\beta E_x = -j\omega\mu H_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

The curl equation No.2

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\varepsilon E_x$$

$$-\frac{\partial H_z}{\partial x} - j\beta H_x = j\omega\varepsilon E_y$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z$$

Solve for E_y, H_x in terms of E_z, H_z

Solve for E_x, H_y in terms of E_z, H_z

Relationship between Trans. and Long.

General Waveguide Equations

$$H_x = \frac{j}{k_c^2}(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x})$$

$$H_y = \frac{-j}{k_c^2}(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y})$$

$$E_x = \frac{-j}{k_c^2}(\omega\mu\frac{\partial H_z}{\partial y} + \beta\frac{\partial E_z}{\partial x})$$

$$E_y = \frac{j}{k_c^2}(\omega\mu\frac{\partial H_z}{\partial x} - \beta\frac{\partial E_z}{\partial y})$$

Cutoff wavenumber

$$k_c^2 = k^2 - \beta^2$$

$$k_c^2 = k_x^2 + k_y^2$$

- Once we know the longitudinal components, we know everything else
- k_c, β need to be pre-determined
- We need to determine Ez, Hz, k_x, and k_y
  - Boundary Conditions

TM waves: $H_z = 0$, $E_z \neq 0$

General Waveguide Equations

$$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right)$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right)$$

$$E_x = \frac{-j}{k_c^2}\left(\omega\mu\frac{\partial H_z}{\partial y} + \beta\frac{\partial E_z}{\partial x}\right)$$

$$E_y = \frac{j}{k_c^2}\left(\omega\mu\frac{\partial H_z}{\partial x} - \beta\frac{\partial E_z}{\partial y}\right)$$

TM Wave Equations

$$H_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial y}$$

$$H_y = \frac{-j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial x}$$

$$E_x = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial x}$$

$$E_y = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial y}$$

Wave Impedance:

$$k = \omega\sqrt{\mu\varepsilon}$$

$$Z_{TM} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\varepsilon} = \frac{\beta\eta}{k}$$

- Once we know E_z we know everything else
- Notice that the waveguide wave impedance does not equal the free space impedance
- The presence of a traveling, axial directed component significantly changes the wave impedance

Wave equation for E_z

Cutoff wave number

$$k_c \neq 0, \quad \Rightarrow \quad \beta^2 = k^2 - k_c^2$$

- One only needs to solve a wave equation that is only defined in the cross-section!!

Original 3D wave equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) E_z = 0$$

$$\frac{\partial^2}{\partial z^2} \rightarrow -\beta^2$$

Definition for the longitudinal component

$$E_z(x, y, z) = e_z(x, y)e^{-j\beta z}$$

2-D Wave equation or Helmholtz equation !!

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) e_z = 0$$

In class Exercise: TE waves: $H_z \neq 0$, $E_z = 0$

General Wave Equations

$$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon\cancel{\frac{\partial E_z}{\partial y}} - \beta\frac{\partial H_z}{\partial x}\right)$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\varepsilon\cancel{\frac{\partial E_z}{\partial x}} + \beta\frac{\partial H_z}{\partial y}\right)$$

$$E_x = \frac{-j}{k_c^2}\left(\omega\mu\frac{\partial H_z}{\partial y} + \beta\cancel{\frac{\partial E_z}{\partial x}}\right)$$

$$E_y = \frac{j}{k_c^2}\left(\omega\mu\frac{\partial H_z}{\partial x} - \beta\cancel{\frac{\partial E_z}{\partial y}}\right)$$

TM Wave Equations

$$H_x = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial x}$$

$$H_y = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial y}$$

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}$$

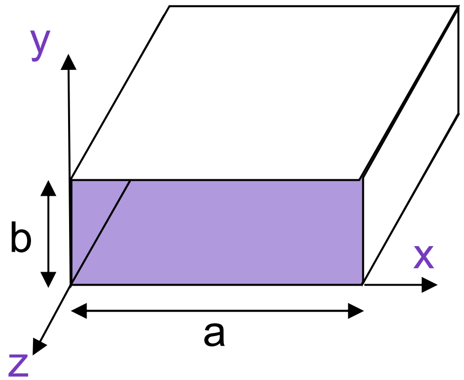
Wave Impedance:

$$Z_{TE} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta}$$

- $Z_{TE} \neq Z_{TM}$
- This is true independent of waveguide geometry

Transverse Magnetic (TM) Modes in a Rectangular Waveguide

Rectangular Waveguide, TM modes



The Wave Equation for TM modes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) e_z(x,y) = 0$$

Separation of variables:

$$e_z(x,y) = f(x)g(y)$$

Which leads to,

$$\frac{d^2 f(x)}{dx^2} g(y) + f(x)\frac{d^2 g(y)}{dy^2} + k_c^2 f(x)g(y) = 0$$

Dividing by f(x)g(x) yields,

$$\underbrace{\frac{1}{f(x)}\frac{d^2 f(x)}{dx^2}}_{-k_x^2} + \underbrace{\frac{1}{g(y)}\frac{d^2 g(y)}{dy^2}}_{-k_y^2} + k_c^2 = 0$$

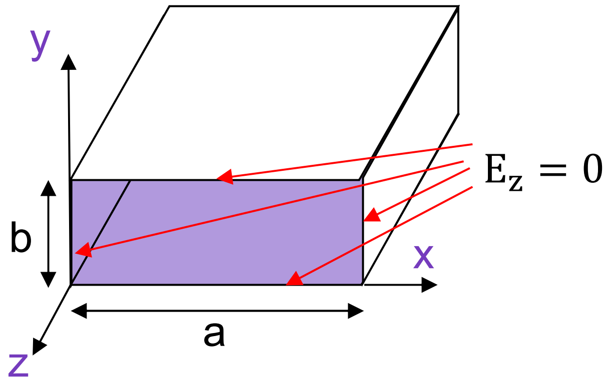
$$\rightarrow ({k_c}^2 = k_x^2 + k_y^2 = k^2 - \beta^2)$$

Decoupling to two 1-D wave equations

$$\left(\frac{d^2}{dx^2} + k_x^2\right) f(x) = 0$$

$$\left(\frac{d^2}{dy^2} + k_y^2\right) g(y) = 0$$

Rectangular Waveguide, TM modes



Boundary conditions at the x,y = 0

$$e_z(x = 0, y) = 0 \quad \rightarrow \quad B = 0$$ → Left wall

$$e_z(x, y = 0) = 0 \quad \rightarrow \quad D = 0$$ → Bottom Floor

General solutions of electric field:

$$f(x) = A\sin(k_x x) + B\cos(k_x x)$$

$$g(y) = C\sin(k_y y) + D\cos(k_y y)$$

Reduced equation

$$e_z(x, y) = A'\sin(k_x x)\sin(k_y y)$$

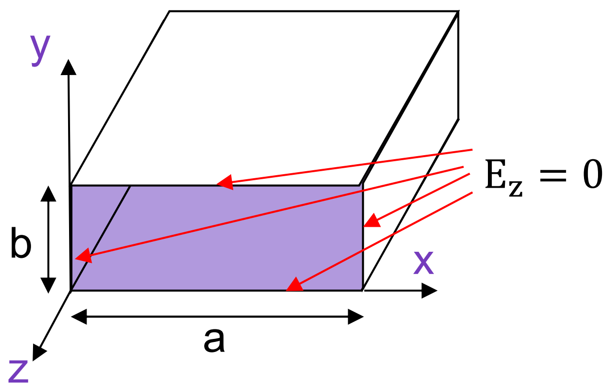
→ $$A' = AC$$

General solutions of electric field:

$$e_z(x, y) = f(x)g(y)$$

$$e_z(x, y) = (A\sin(k_x x) + B\cos(k_x x))(C\sin(k_y y) + D\cos(k_y y))$$

Rectangular Waveguide, TM modes



Reduced equation

$$e_z(x,y) = A' \sin(k_x x)\sin(k_y y)$$

$$A' = AC$$

Boundary conditions at the x = a wall

$$e_z(x=a,y) = A' \sin(k_x a)\sin(k_y y) = 0$$

$$\sin(k_x a) \quad \rightarrow \quad k_x a = m\pi$$

$$k_x = \frac{m\pi}{a} \quad \forall \text{ m integers}$$

Boundary conditions at the y = b ceiling

$$e_z(x,y=b) = A' \sin(k_x x)\sin(k_y b) = 0$$

$$\sin(k_y b) \quad \rightarrow \quad k_y b = n\pi$$

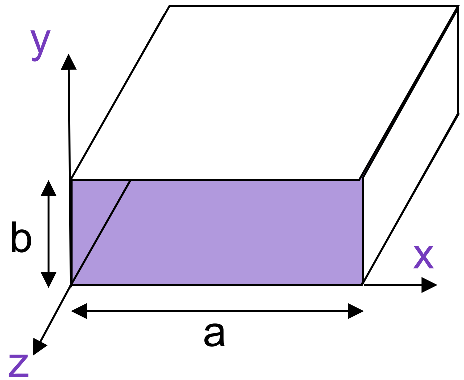
$$k_y = \frac{n\pi}{b} \quad \forall \text{ n integers}$$

Boundary conditions at the x = a wall

$$e_z(x,y) = A_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$A_{mn} = A'$$

Rectangular Waveguide, TM modes



Longitudinal field

$$e_z(x,y) = A_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$E_z(x,y,z) = e_z(x,y)e^{-j\beta z}$$

$$E_z(x,y,z) = A_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

TM Wave Equations

$$H_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial y}$$

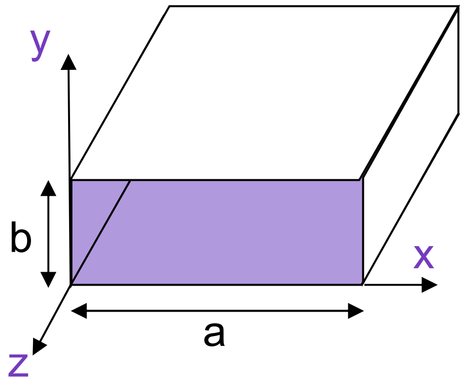
$$H_y = \frac{-j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial x}$$

$$E_x = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial x}$$

$$E_y = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial y}$$

- Now that the longitudinal field is full characterized, we can compute all the transverse fields

Rectangular Waveguide, TM modes



Longitudinal field

$$E_z(x,y,z) = A_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta z}$$

Transverse Fields

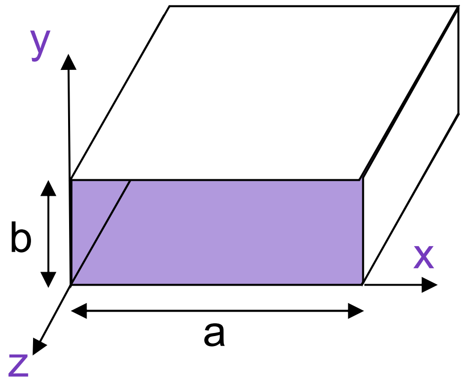
$$H_x(x,y,z) = \frac{j\omega\varepsilon n\pi}{bk_c^2} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z}$$

$$H_y(x,y,z) = \frac{-j\omega\varepsilon m\pi}{ak_c^2} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z}$$

$$E_x(x,y,z) = \frac{-j\beta m\pi}{ak_c^2} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z}$$

$$E_y(x,y,z) = \frac{-j\beta n\pi}{bk_c^2} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z}$$

Rectangular Waveguide, TM modes



Boundary conditions also determine that,

$$\begin{cases} k_x = \dfrac{m\pi}{a}, m = 1,2,3\ldots \\ k_y = \dfrac{n\pi}{b}, n = 1,2,3\ldots \end{cases} \Rightarrow k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

Cutoff wave number

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

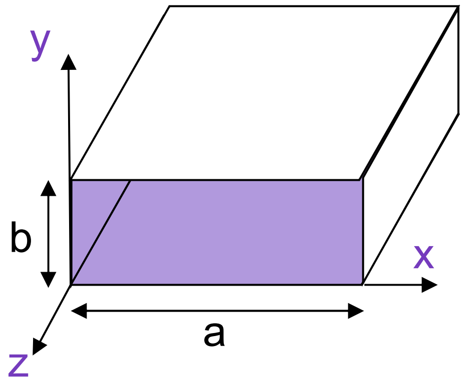
Propagation constant

$$\Rightarrow \quad \beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

k: $\omega\sqrt{\mu\varepsilon}$

$\left(\frac{m\pi}{a}\right)^2$, $\left(\frac{n\pi}{b}\right)^2$: dependent on the geometry of the waveguide

Cutoff Frequency



Free space wave number

$$k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda} = 2\pi f\sqrt{\mu\varepsilon} \rightarrow$$

Frequency

$$f = \frac{k}{2\pi\sqrt{\mu\varepsilon}}$$

Cutoff point is defined when

$$k = k_c \rightarrow f = f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}}$$

Cutoff angular frequency:

$$\omega_c = k_c \cdot c/\sqrt{\varepsilon_r},$$

Put it together

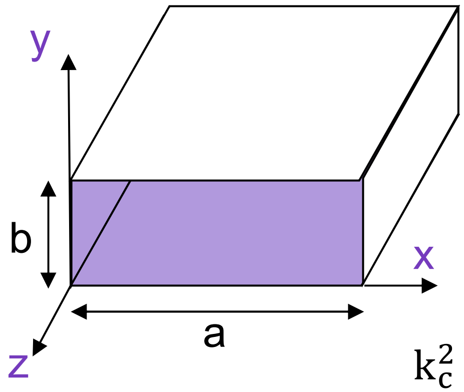
$$f_c = \frac{\omega_c}{2\pi} = \frac{c \cdot k_c}{2\pi\sqrt{\varepsilon_r}} = \frac{c}{2\pi\sqrt{\varepsilon_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$f_c = \frac{c}{2\sqrt{\varepsilon_r}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

- Cutoff frequency a function of waveguide geometry and waveguide fill
- As we'll see next time, below this frequency the mode will not propagate

Conclusions

Propagation



$$\beta \neq k$$

$$\left.\begin{aligned} k_c^2 &= k^2 - \beta^2 \\ k_c^2 &= k_x^2 + k_y^2 \end{aligned}\right\} \rightarrow \beta^2 = k^2 - k_x^2 - k_y^2 \rightarrow \beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

- The one conductor geometry supports TE/TM operation
- The longitudinal phase variation of a TE/TM is not equal to the free space (plane wave) TEM phase variation
  - β → rectangular waveguide longitudinal phase variation
  - k → free space longitudinal phase variation
- β is a strong function of frequency and geometry

Next time

- We'll derive the expressions for the TE modes
- We'll focus on lower order modes and compare and contrast TE and TM.