Aalto University
School of Science

Lecture 9:
Why donuts are so good …

Menu of today

- Why donuts are so good: Hairy ball theorem
- Drift *orbits*
- Neoclassical transport
- Noether's theorem
- Tokamak ABC
- More on 'anomalous' transport:
  - TF ripple,
  - microinstabilities
- Drifts due to non-stationary fields
- Plasma heating by adiabatic compression

Depressing news from last week

Previously, in IPPFSA …

Analytical work led to findings that are ***4 orders of magnitude*** from reality:

- We derived expression for the diffusion coefficient D_{cl} and let the continuity equation give us the *confinement time*

We already identified a few possible causes for increased *transport* – not necessarily *diffusive* in nature.

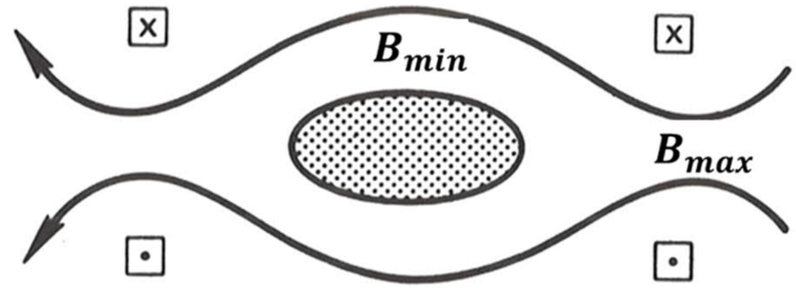
But so far we ignored the effect of *geometry!*

- Recall lecture#2: in linear geometry (cylinder) we have a loss cone

$$\frac{v_{\parallel,0}^2}{v_0^2} \equiv \xi_0^2 > 1 - B_{min}/B_{max}$$

These are called *end losses* and they certainly add to diffusive losses.

Loss cone and collisions



Features of the end losses:

- Losses via loss cone are along field lines, and, thus, almost immediate
  - ➔ loss cone is ’empty’ in no time

Recall the effect of Coulomb collisions:

- Even small-angle collisions change the direction of a particle
  - ➔ a particle with $\xi_0^2 < 1 - B_{min}/B_{max}$ can be kicked to the loss cone

Here even like-particle collisions can feed the loss cone

➔ Eventually entire plasma depleted at rate determined by ν_{coll}

Note: loss cone independent of particle species (*q,m)*

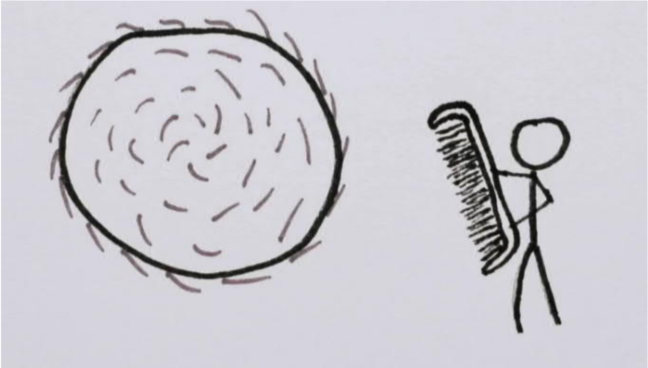
How to eliminate end losses?

This is equal to asking: ***”how to confine magnetic field lines”***?

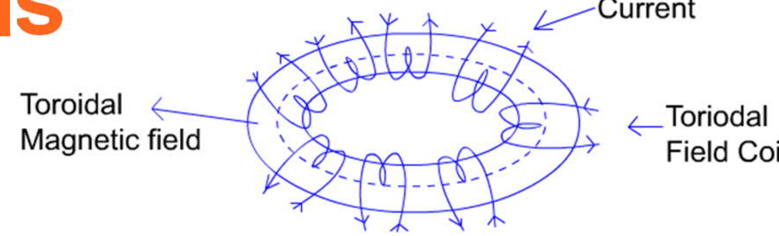
The answer is given by a mathematical theorem in algebraic topology, also known as

The Hairy Ball theorem

https://www.youtube.com/watch?v=B4UGZEjG02s



Magnetic field lines in a torus



When the axial field is turned into a toroidal one, it becomes non-uniform.

In the first approximation (geometrical considerations): $B = B_0 \frac{R_0}{R}$

(Previous "*Food for thought*"): $v_{\nabla B} = \pm \frac{1}{2} v_\perp r_L \frac{\boldsymbol{B} \times \nabla B}{B^2}$ (HW: for 1/R field)

➔ Electrons move upward, ions downward (or vice versa)

➔ A vertical electric field is established!

➔ ExB drift makes both species march horizontally out of the cage …

The need for *helical* field lines

The physics approach to field line problem:

- Having fixed the geometry as torus, the gradient drift is unavoidable
- The charge separation *can* be prevented:

Short-circuit the system!

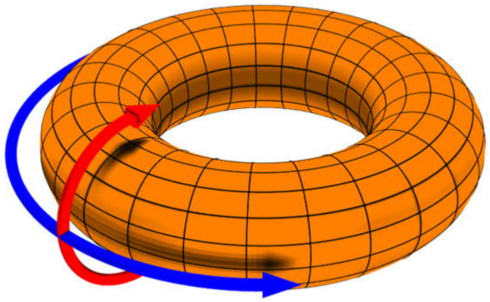
The electrons need a route to the ions ➔ a vertical B-field?

No, that is not natural to toroidal geometry

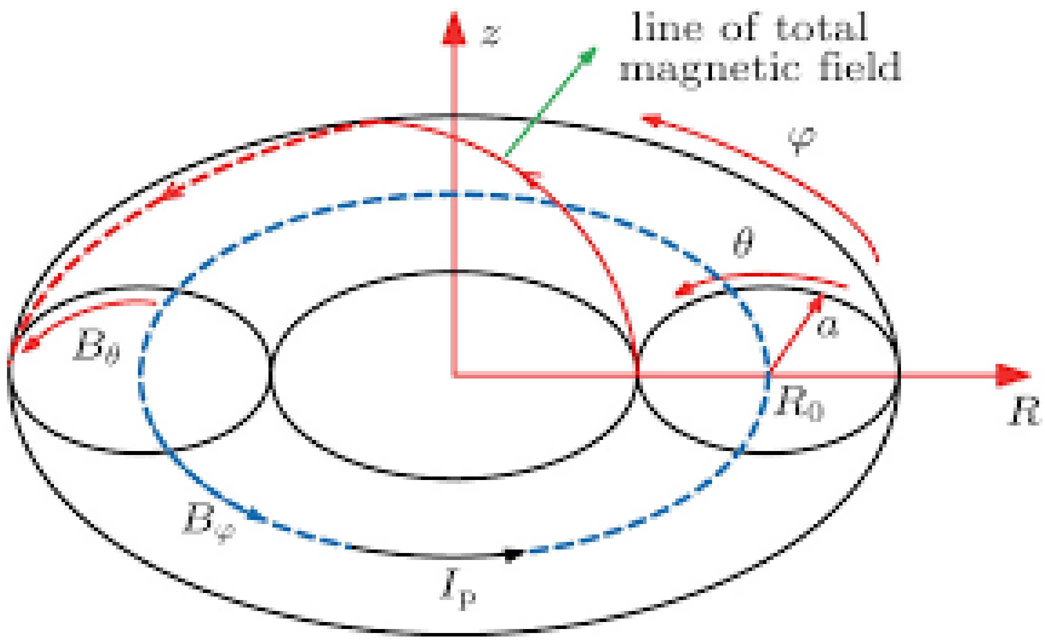
➔ introduce a

poloidal magnetic field

to supplement the *toroidal field*



Toroidal magnetic cage



Toroidal angle: φ
Poloidal angle: θ
Toroidal field: B_φ
Poloidal field: B_θ
Major radius: R
Minor radius: r (a)
Magnetic axis at $R = R_0$

There is no such thing as free lunch

Things look perfect now – or do they?

There is a price to pay for the more complicated geometry …

Life in toroidal geometry

Single particle in *non-uniform* B-field
Part III: curved B

In a tokamak, the field lines have *curvature*, R_c

Geometry makes math complicated ➔ let's take a more intuitive approach:

A particle on a curved path experiences centrifugal force $\boldsymbol{F}_{cf} = \frac{mv_\parallel^2}{R_c}\boldsymbol{r} = mv_\parallel^2 \frac{\boldsymbol{R_c}}{R_c^2}$

➔ *GC* drift called the ***curvature drift***: $v_R = \frac{1}{q}\frac{\boldsymbol{F}_{cf}\times\boldsymbol{B}}{B^2} = \frac{mv_\parallel^2}{qB^2}\frac{\boldsymbol{R_c}\times\boldsymbol{B}}{R_c^2}$

But this is not all, folks …

When modifying our magnetic field on pen-and-paper, we have to make sure our new field is *physical* !

Curvature drift with a physical B field

A curved field cannot be uniform ➔ accompanying gradient drift.

Ampere' law in vacuum: $\nabla \times \boldsymbol{B} = 0$

Cylindrical coordinates (natural choice): $\nabla \times \boldsymbol{B} = (\nabla \times \boldsymbol{B})_z = \frac{1}{R}\frac{\partial}{\partial R}\left(RB_\varphi\right) = 0$

➔ $B_\varphi \propto \frac{1}{R}$ ➔ $B \propto \frac{1}{R_c}$ ➔ $\frac{\boldsymbol{\nabla} B}{B} = -\frac{\boldsymbol{R}_c}{R_c^2}$ accompanying $\boldsymbol{v}_{\nabla B} = \frac{1}{2}\frac{m}{q}v_\perp^2 \frac{\boldsymbol{R}_c \times \boldsymbol{B}}{R_c^2 B^2}$

➔ Total drift in curved B field: $\boldsymbol{v}_{\boldsymbol{R}} + \boldsymbol{v}_{\nabla B} = \frac{m}{q}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)\frac{\boldsymbol{R}_c \times \boldsymbol{B}}{R_c^2 B^2}$

Drifts add. Bad news for toroidal confinement = bending field lines …

Direction of magnetic drifts in a torus

The poloidal field is much smaller than the toroidal field, $B_{pol} \sim \frac{1}{10} B_{tor}$

➔ the direction of the drifts is dominated by B_{tor}

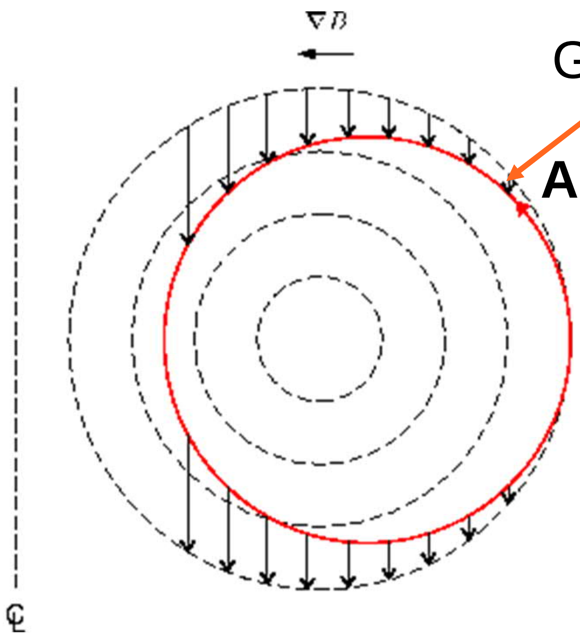
Gradient drift: $v_{\nabla B} \propto \boldsymbol{B} \times \nabla B \propto (\pm\hat{\phi}) \times (-\hat{R})$ ➔ gradient drift ≈ vertical !

Curvature drift: $\boldsymbol{v}_c \propto \boldsymbol{R}_c \times \boldsymbol{B} \propto \hat{R} \times \left(\pm\hat{\phi}\right) = (\pm\hat{\phi}) \times (-\hat{R})$

Thus it is further verified: gradient and curvature drifts add.

What do they do to our charged particles?

Drift orbits in toroidal geometry



Simon Pinches, PhD thesis

In this course the magnetic field strength can be approximated as

$$B = \frac{B_0 R_0}{R}$$

Taking into account how the particles move along the field lines, this can be expressed a little more informatively:

$$B = \frac{B_0 R_0}{R_0 + r\cos\theta}$$

With large *aspect ratio* $A = R_0/a \gg 1$

$$B = \frac{B_0}{1 + \epsilon\cos\theta} \approx B_0(1 - \epsilon\cos\theta)\,, \epsilon = 1/A$$

Trapping in toroidal geometry

1/R nature of the magnetic geometry ➔ field lines are denser at the inner part of the donuts

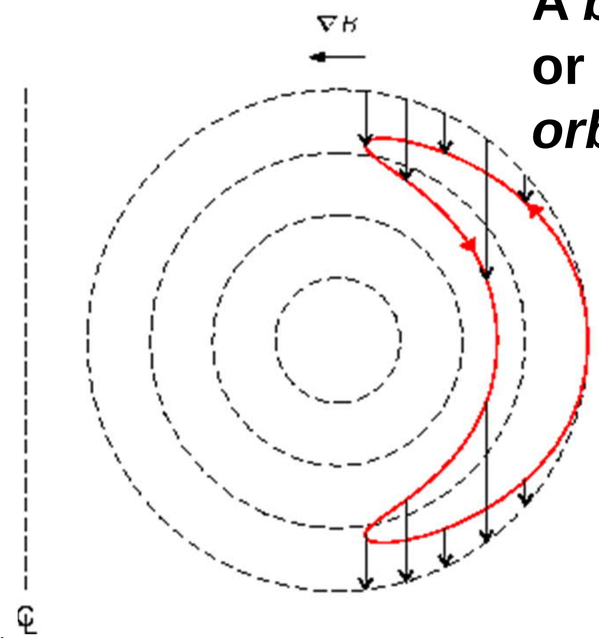
➔ there is a *gradient* in magnetic field towards the symmetry axis of the torus

➔ reflection can occur! ➔ HW

- Find the mirror point
- Fraction of banana particles

Width of this banana orbit:

poloidal Larmor radius $\Delta_b = mv_{\parallel}/qB_{pol} \sim 10 r_L$



A *banana* or *trapped orbit*

Simon Pinches, PhD thesis

Drift orbits and transport

Remember the intuitive basis for diffusion via random walk:

$$D \propto (step\ length)^2 / \tau_{coll}$$

For classical diffusion: step length = Larmor radius

Now collisions can throw a particle from one banana orbit to another

➔ step length = *poloidal* Larmor radius

➔ so-called *neoclassical diffusion*, which is about $\frac{B_{tor}^2}{B_{pol}^2}$ ~100 times larger !

But we are still about a factor of 100 away from the observed transport …

More on ‘anomalous’ transport

More geometrical effects

”Previously on IPPFSA: … *misalignments of field coils can lead to stray field lines prematurely ending up at the wall”*

In toroidal geometry an additional loss channel, related to the field line topology, can appear:

Think: *how can we be sure that the drift orbit closes upon itself in the poloidal plane???*

Intuitive conclusion: *only if the drift in the upper and lower hemisphere are identical…*

But it is dangerous to blindly trust intuition. Luckily we have ….

Noether's theorem (https://www.youtube.com/watch?v=ahf0zCaqrwM)

Much of the physics you learned already in school is based on the work of a *mathematician*, Emmy Noether:

Any symmetry is accompanied by a conservation law

- Translational symmetry ➔ conservation of *linear momentum*
- Rotational symmetry ➔ conservation of *angular momentum*
- Symmetry in time ➔ conservation of *energy*

If our toroidal plasma is *axisymmetric*, the conserved quantity, called *toroidal canonical momentum*, is conserved and guarantees closing of the drift orbits!

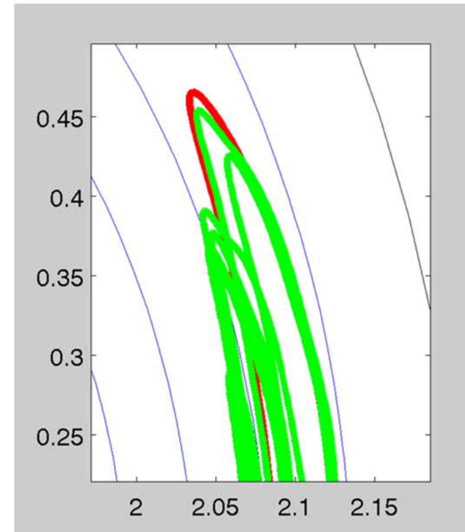
But…there is no such thing as an axisymmetric tokamak…

- The toroidal field is produced by $N \ll \infty$ external coils
- The extent of these coils is finite

➔ $R_{mirror} = R_{mirror}(\phi)$

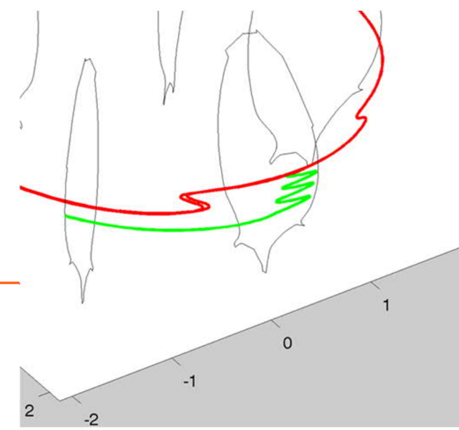
➔ the banana orbit does not close upon itself

➔ banana orbit starts wandering off …



Notes:

- this effect is limited to the very edge of the toroidal plasma
  ➔ not responsible for global degradation of a plasma
- very small ξ_0 ➔ can even get trapped between two coils



But what about non-uniform *E* field??

Requires straight-forward but complicated (= time consuming) math

➔ skip here. Will give the so-called *finite-Larmor-radius effects*:

$$v_E = \left(1 + \frac{1}{4} r_L^2 \nabla^2\right) \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}$$

Note: the *ExB* drift is no longer independent of species due to r_L^2 !

➔ if a density clump appears… ➔ charge separation ➔ $\boldsymbol{E}$

A feedback mechanism ➔ a *drift instability* ➔transport

Note: important for large $k^2 r_L^2$, i.e., small wavelengths ➔ *microinstability*

What about time-varying fields???

Time-varying E field …

Plasma is not a passive media but responds to changes (recall plasma oscillations) ➔ take uniform but sinusoidally varying field: $\boldsymbol{E} = E_0 e^{-i\omega t}\boldsymbol{x}$

➔ $\frac{dE_x}{dt} = i\omega E_x$ ➔ $\frac{d^2 v_x}{dt^2} = -\Omega_c^2\left(v_x \mp \frac{i\omega}{\Omega_c}\frac{E_x}{B}\right)$; ($\boldsymbol{B} = B_0\boldsymbol{z}$)

Previous results ➔ define $v_E \equiv -\frac{E_x}{B}$ and $v_p \equiv \pm\frac{i\omega}{\Omega_c}\frac{E_x}{B}$

$$\frac{d^2 v_x}{dt^2} = -\Omega_c^2(v_x - v_p)$$

$$\frac{d^2 v_y}{dt^2} = -\Omega_c^2(v_y - v_E)$$

… and the *polarization drift*

From previous experience, let's make *trial solutions*

$v_x = v_\perp e^{-i\Omega_c t} + v_p$ $\qquad$ $\frac{d^2 v_x}{dt^2} = -\Omega_c^2\left(v_x - v_p\right) - \omega^2 v_p$

$v_y = \pm i v_\perp e^{-i\Omega_c t} + v_E$ $\qquad$ $\frac{d^2 v_y}{dt^2} = -\Omega_c^2\left(v_y - v_E\right) - \omega^2 v_E$

This is NOT the same as the original set of equations… ☹

However, IF $\omega^2 \ll \Omega_c^2$, then we can neglect the last term.

➔ For *sufficiently* slowly varying E-field the trial solutions are OK

➔ *Polarization drift* : $\boldsymbol{v}_p \equiv \pm \frac{1}{\Omega_c B}\frac{d\boldsymbol{E}}{dt}$

Physics of polarization drift etc

1. The polarization drift is *parallel* to the electric field!
2. Polarization drift depends on charge ➔ *polarization current !*

$$\boldsymbol{j}_p = \frac{\rho}{B^2}\frac{d\boldsymbol{E}}{dt}$$, where ρ is the mass density of the plasma

3. Polarization effect is similar to that in any dielectric where $\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}$
   - In a plasma, the dipoles are formed by ions and electrons separated by the distance r_L
   - However, since ions and electrons are mobile and try to keep quasineutrality, a stationary electric field cannot sustain polarization current

Time-varying magnetic field …

We already know what happens in time-varying B field – at least if the variation is sufficiently slow: $\mu \equiv \frac{1}{2}\frac{mv_\perp^2}{B} = constant$

➔ Increasing B will increase $v_\perp$! … But how much?

Time-varying magnetic field is associated with an electric field:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

EoM ➔ $\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = q\boldsymbol{E}\cdot\boldsymbol{v}_\perp = q\boldsymbol{E}\cdot\frac{d\boldsymbol{l}}{dt}$, where $\boldsymbol{v}_\perp = \frac{d\boldsymbol{l}}{dt}$.

Slowly varying field ➔ change over one gyro orbit: $\delta\left(\frac{1}{2}mv_\perp^2\right) = \oint q\boldsymbol{E}\cdot d\boldsymbol{l}$

… and adiabatic compression = heating !

Use Stokes' theorem

$$\delta\left(\frac{1}{2}mv_{\perp}^{2}\right) = q\int(\boldsymbol{\nabla}\times\boldsymbol{E})\cdot d\boldsymbol{S} = -q\int\frac{\partial\boldsymbol{B}}{\partial t}\cdot d\boldsymbol{S}$$

Plasma is diamagnetic ➔ $\boldsymbol{B}\cdot\boldsymbol{dS}$ < 0 for ions and > 0 for electrons

➔ $\delta\left(\frac{1}{2}mv_{\perp}^{2}\right) = \pm q\frac{\partial\boldsymbol{B}}{\partial t}\pi r_L^2 = \mu\frac{2\pi\frac{\partial\boldsymbol{B}}{\partial t}}{\Omega_c} = \mu\frac{1}{f_c}\frac{\partial\boldsymbol{B}}{\partial t}$; $\delta B_{orbit} = \frac{1}{f_c}\frac{\partial\boldsymbol{B}}{\partial t}$

➔ $\delta\left(\frac{1}{2}mv_{\perp}^{2}\right) = \mu\delta B_{orbit}$ & additional result: $\mu = constant$ (… again)

So plasma can be heated by increasing B slowly: $v_{\perp}^{2}$ increases but r_L shrinks ➔ by *compressing* it.