PHYS-E055101 Low Temperature Physics: Nanoelectronics

Cavities and resonators

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This lecture follows mostly Refs. [1, 2]. The input-output theory is done in many other quantum optics textbooks, see for example Ref. [3]. As a prerequisite to what follows and if you want to have a more in-depth understanding of the topic, it would be good to read section D1 from Ref. [1], which deals with transmission line quantization. This allows you to relate the operators $\hat{b}_q$, $\hat{b}_q^\dagger$ discussed below to operators corresponding to voltages and phases on a transmission line. Otherwise the derivation below is quite general, that is, independent on what type of device you have in mind - it applies to cavities with semitransparent mirrors (as used in quantum optics) and to microwaves resonators with capacitive outcouplings.

I. INPUT-OUTPUT THEORY

The formalism refers to a system which is coupled to an environment. In many cases of interest, we trace out completely the degrees of freedom of the environment. The result is a density matrix that evolves according to a master equation. Here we want to keep track of the modes of the environment. In other words, we want to recover some information about the cavity, which otherwise would just spread in the environment. From the point of view of the mode in the cavity, called $\hat{a}(t)$, $\hat{a}^\dagger(t)$ in the following, the effect of the environment will be seen as a decay with some constant κ, but there will be also an effect due to photons being injected by a field operator $b_{\rm in}$ defined by using the modes of the environment.

The basis of the formalism is separating the degrees of freedom into internal cavity modes and external bath modes. The Hamiltonian is then given by

$$\hat{H} = \hat{H}_{\rm sys} + \hat{H}_{\rm bath} + \hat{H}_{\rm int}, \tag{1}$$

where $\hat{H}_{\rm sys}$ is the cavity system Hamiltonian, $\hat{H}_{\rm bath}$ is the bath Hamiltonian,

$$\hat{H}_{\rm bath} = \sum_q \hbar\omega_q \hat{b}_q^\dagger \hat{b}_q, \tag{2}$$

and $\hat{H}_{\rm int}$ is the interaction Hamiltonian between the two. In the rotating wave approximation, i.e. after neglecting the fast oscillating terms that don't conserve the total number of excitations, the interaction can be written as

$$\hat{H}_{\rm int} = -i\hbar \sum_q (f_q \hat{a}^\dagger \hat{b}_q - f_q^* \hat{b}_q^\dagger \hat{a}). \tag{3}$$

Here $\hat{a}$ (for the cavity) and $\hat{b}$ (for the bath) are bosonic operators with commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$ and $[\hat{b}_q, \hat{b}^\dagger_{q'}] = \delta_{q,q'}$.

The time evolution of both the environment and cavity operators is given by the Heisenberg equations of motion. For the bath it reads:

$$\dot{\hat{b}}_q = \frac{i}{\hbar}[\hat{H}, \hat{b}_q] = -i\omega_q \hat{b}_q + f_q^* \hat{a}. \tag{4}$$

The solution for Eq. (4) can be obtained by integrating from some time, t_0, which lies in the far past before any packet launched at the cavity has reached it. The result is

$$\hat{b}_q(t) = e^{-i\omega_q(t-t_0)}\hat{b}_q(t_0) + f_q^* \int_{t_0}^{t} d\tau e^{-i\omega_q(t-\tau)}\hat{a}(\tau). \tag{5}$$

The first term on the right hand side is the evolution of the bath without interaction with the cavity and $\hat{b}(t_0)$ is the cavity operator at t_0. The second term comes from the interaction and it represents the waves radiated by the cavity into the bath.

For the operator of the cavity, the Heisenberg equation of motion reads:

$$\dot{\hat{a}} = \frac{i}{\hbar}[\hat{H}_{\text{sys}}, \hat{a}] - \sum_q f_q \hat{b}_q. \tag{6}$$

With the solution for $\hat{b}_q$, Eq. (5), the second term can be written as

$$\sum_q f_q \hat{b}_q = \sum_q f_q e^{-i\omega_q(t-t_0)}\hat{b}_q(t_0) + \sum_q |f_q|^2 \int_{t_0}^{t} d\tau e^{-i(\omega_q-\omega_c)(t-\tau)} \left[e^{-i\omega_c(t-\tau)}\hat{a}(\tau)\right]. \tag{7}$$

The term in the brackets on the right is introduced because it is a slow-varying term. If the system part of the Hamiltonian were a simple harmonic oscillator with frequency ω_c, we would get from the Fermi golden rule for the decay rate from state $n = 1$ to $n = 0$:

$$\kappa(\omega_c) = \frac{2\pi}{\hbar} \sum_q |\hbar f_q|^2 \delta(\hbar\omega_c - \hbar\omega_q) \tag{8}$$

$$= 2\pi \sum_q |f_q|^2 \delta(\omega_c - \omega_q). \tag{9}$$

From this, we can observe that

$$\int_{-\infty}^{\infty} \frac{d\nu}{2\pi}\kappa(\omega_c+\nu)e^{-i\nu(t-\tau)} = \int_{-\infty}^{\infty} d\nu \sum_q \delta(\omega_c+\nu-\omega_q)e^{-i\nu(t-\tau)}|f_q|^2 = \sum_q |f_q|^2 e^{-i(\omega_q-\omega_c)(t-\tau)}. \tag{10}$$

Using the Markov approximation, which assumes that $\kappa(\nu) = \kappa$ is constant over the range of relevant frequencies, we get:

$$\sum_q |f_q|^2 e^{-i(\omega_q-\omega_c)(t-\tau)} = \kappa\delta(t-\tau). \tag{11}$$

With this, the second term in Eq. (7) finally becomes

$$\sum_q |f_q|^2 \int_{t_0}^{t} d\tau e^{-i(\omega_q-\omega_c)(t-\tau)} e^{-i\omega_c(t-\tau)}\hat{a}(\tau) = \int_{t_0}^{t} d\tau\kappa\delta(t-\tau)e^{i\omega_c(t-\tau)}\hat{a}(\tau) \tag{12}$$

$$= \frac{\kappa}{2}\hat{a}(t). \tag{13}$$

To obtain the last equality we used

$$\int_{-\infty}^{t} d\tau\delta(t-\tau)f(\tau) = \frac{1}{2}f(t). \tag{14}$$

This identity is not in general valid for any representation of the Dirac function; however, it is clearly valid for the Gaussian representation, which is symmetric with respect to t. In reality there is of course no ideal Dirac distribution - the real waveforms are of typically of some Gaussian form. The equation of motion for the cavity operator finally becomes

$$\dot{\hat{a}} = \frac{i}{\hbar}[\hat{H}_{\mathrm{sys}}, \hat{a}] - \frac{\kappa}{2}\hat{a} - \sum_q f_q e^{-i\omega_q(t-t_0)}\hat{b}_q(t_0). \tag{15}$$

The second is interpreted as the part describing waves radiated by the cavity. The factor of 1/2 is important because it means that the amplitude decays at half of the rate of the energy decay rate κ. Note that this term is linear - within the Markovian approximation the effect of the bath is taken into account here by a simple linear decay term.

The next thing we do is treat $f \equiv \sqrt{|f_q|^2}$ as constant. This allows us to write the decay rate as a function of density of states:

$$\kappa = 2\pi f^2 \rho, \tag{16}$$

where ρ is the density of states which we also treat as constant. With this simplification, replacing $f_q \approx f = \sqrt{\kappa}/\sqrt{2\pi\rho}$, Eq. (15) becomes

$$\dot{\hat{a}} = \frac{i}{\hbar}[\hat{H}_{\mathrm{sys}}, \hat{a}] - \frac{\kappa}{2}\hat{a} - \frac{\sqrt{\kappa}}{\sqrt{2\pi\rho}}\sum_q e^{-i\omega_q(t-t_0)}\hat{b}_q(t_0). \tag{17}$$

Input and output modes

Now we can define the so-called 'input mode'

$$\hat{b}_{\rm in}(t) = \frac{1}{\sqrt{2\pi\rho}} \sum_q e^{-i\omega_q(t-t_0)} \hat{b}_q(t_0). \tag{18}$$

In terms of the input mode, the equation of motion for the cavity operator becomes

$$\dot{\hat{a}} = \frac{i}{\hbar}[\hat{H}_{\rm sys}, \hat{a}] - \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\hat{b}_{\rm in}. \tag{19}$$

Thus the input mode acts as the driving field of the cavity. One can imagine also that at the time $t = t_0 = -\infty$ the field $\hat{b}_{\rm in}$ was launched towards the cavity, and it was evolving under the environment equations of motion until it reached the cavity.

Next, we are going to do the same calculation but instead write the cavity operator in terms of the so-called 'output mode'. Let us first define $t_1 > t$ as some time after the input field has interacted with the cavity. For example if the time t approaches $t_1 = \infty$, then the mode is asymptotically close to what emerges from the cavity after a long enough time (or after a long enough distance from the cavity). From the equation of motion for $\hat{b}_q$, this solution becomes

$$\hat{b}_q(t) = e^{-i\omega_q(t-t_1)} \hat{b}_q(t_1) - f_q^* \int_t^{t_1} d\tau e^{-i\omega_q(t-\tau)} \hat{a}(\tau). \tag{20}$$

So this time we imagine that the initial condition is in the future, after the cavity has interacted with the bath.

Define now output mode as

$$\hat{b}_{\rm out}(t) = \frac{1}{\sqrt{2\pi\rho}} \sum_q e^{-i\omega_q(t-t_1)} \hat{b}_q(t_1). \tag{21}$$

After doing the calculations similarly as before, we get for the cavity operator in terms of the output mode

$$\dot{\hat{a}} = \frac{i}{\hbar}[\hat{H}_{\rm sys}, \hat{a}] + \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\hat{b}_{\rm out}. \tag{22}$$

Now we have two equations of motion for $\hat{a}$. One in terms of the input mode, Eq. (19), and one in terms of the output mode, Eq. (22). Substracting the two from each other yields a relationship between the input and the ouput modes:

$$\hat{b}_{\rm out} = \hat{b}_{\rm in} + \sqrt{\kappa}\hat{a}. \tag{23}$$

In Fig. I we present a schematic of the input and output modes.

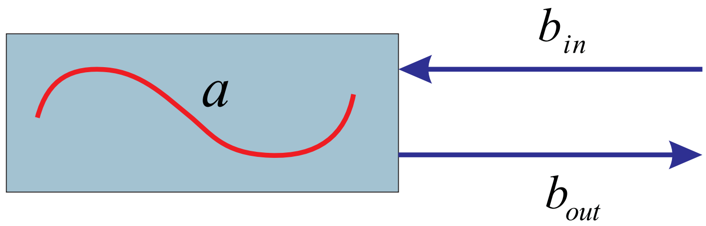

FIG. 1. Schematic of the input and output modes in a one-sided cavity.

II. APPLICATIONS: CAVITIES AND RESONATORS

Let us now take our system to be a simple non-interacting cavity with

$$\hat{H}_{\rm sys} = \hbar\omega_c \hat{a}^\dagger \hat{a}. \tag{24}$$

A. One-sided cavity

The equation of motion in terms of the input mode now reads

$$\dot{\hat{a}}(t) = -i\omega_c \hat{a}(t) - \frac{\kappa}{2}\hat{a}(t) - \sqrt{\kappa}\hat{b}_{\rm in}(t). \tag{25}$$

In the Fourier space, this equation becomes a simple algebraic equation

$$-i\omega\hat{a}[\omega] = -i\omega_c\hat{a}[\omega] - \frac{\kappa}{2}\hat{a}[\omega] - \sqrt{\kappa}\hat{b}_{\rm in}[\omega], \tag{26}$$

where the Fourier transform is defined as

$$\hat{a}[\omega] = \int_{-\infty}^{\infty} dt e^{i\omega t}\hat{a}(t), \quad \hat{a}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\hat{a}[\omega], \tag{27}$$

and correspondingly for the inverse transformation. The solution is given by

$$\hat{a}[\omega] = -\frac{\sqrt{\kappa}}{i(\omega_c - \omega) + \kappa/2}\hat{b}_{\rm in}[\omega] \equiv -\sqrt{\kappa}\chi_c(\omega - \omega_c)\hat{b}_{\rm in}[\omega]. \tag{28}$$

Here we have defined the cavity susceptibility as

$$\chi_c(\omega - \omega_c) = \frac{1}{\kappa/2 - i(\omega - \omega_c)}. \tag{29}$$

The output field will represent what is effectively seen by an observer situated somewhere far enough from the cavity. Using $\hat{b}_{\rm out} = \hat{b}_{\rm in} + \sqrt{\kappa}\hat{a}$, we get

$$\hat{b}_{\rm out}[\omega] = \frac{\omega - \omega_c - i\kappa/2}{\omega - \omega_c + i\kappa/2}\hat{b}_{\rm in}[\omega]. \tag{30}$$

Now we can consider two limits:

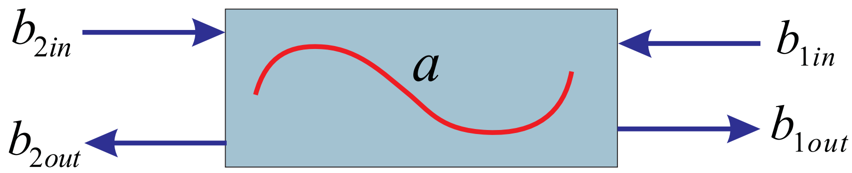

FIG. 2. Schematic of the input and output modes in a double-sided cavity.

- At resonance, $\omega = \omega_c$, we get that $\chi_c = 2/\kappa$ and

$$\hat{b}_{\text{out}}[\omega] = \frac{\sqrt{\kappa}}{2}\hat{a}[\omega] = -\hat{b}_{\text{in}}[\omega]. \tag{31}$$

  Also, we get that the power radiated to the cavity is the same as power radiated out of the cavity,

$$P = \hbar\omega\langle\hat{b}^{\dagger}_{\text{out}}(t)\hat{b}_{\text{out}}(t)\rangle = \hbar\omega\langle\hat{b}^{\dagger}_{\text{in}}(t)\hat{b}_{\text{in}}(t)\rangle = \frac{\kappa}{4}\hbar\omega\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle. \tag{32}$$

  Note that this is not the same as the naive guess, $\hbar\omega\kappa\langle\hat{a}^{\dagger}\hat{a}\rangle$, due to interference between the part of the incoming wave which is reflected from the cavity and the field radiated by the cavity.

- Far off-resonance limit. In this case we have

$$\hat{b}_{\text{out}}[\omega] = \hat{b}_{\text{in}}[\omega]. \tag{33}$$

  Everything is reflected back from the cavity.

B. Double-sided cavity

We now consider the case of a double-sided cavity, see Fig. II B for a schematic. The equation of motion in terms of the input mode reads

$$\dot{\hat{a}}(t) = -i\omega_c\hat{a}(t) - \left(\frac{\kappa_1}{2} + \frac{\kappa_2}{2}\right)\hat{a}(t) - \sqrt{\kappa_1}\hat{b}_{1\text{in}}(t) - \sqrt{\kappa_2}\hat{b}_{2\text{in}}(t). \tag{34}$$

In the Fourier space, this equation becomes a simple algebraic equation

$$-i\omega\hat{a}[\omega] = -i\omega_c\hat{a}[\omega] - \left(\frac{\kappa_1}{2} + \frac{\kappa_2}{2}\right)\hat{a}[\omega] - \sqrt{\kappa_1}\hat{b}_{1\text{in}}(\omega) - \sqrt{\kappa_1}\hat{b}_{2\text{in}}[\omega], \tag{35}$$

and therefore

$$\hat{a}[\omega] = -\frac{\sqrt{\kappa_1}\hat{b}_{1\text{in}}[\omega] + \sqrt{\kappa_2}\hat{b}_{2\text{in}}[\omega]}{i(\omega_c - \omega) + \left(\frac{\kappa_1}{2} + \frac{\kappa_2}{2}\right)}. \tag{36}$$

Next, we write the relation between the input and the output field at each mirror,

$$\hat{b}_{1\text{out}} = \hat{b}_{1\text{in}} + \sqrt{\kappa_1}\hat{a}, \tag{37}$$

$$\hat{b}_{2\text{out}} = \hat{b}_{2\text{in}} + \sqrt{\kappa_2}\hat{a}. \tag{38}$$

After a little algebra, we find

$$b_{1\text{out}}[\omega] = \frac{\left[\frac{\kappa_1-\kappa_2}{2} + i(\omega-\omega_c)\right]\hat{b}_{1\text{in}}[\omega] + \sqrt{\kappa_1\kappa_2}\hat{b}_{2\text{in}}[\omega]}{i(\omega-\omega_c) - \left(\frac{\kappa_1}{2}+\frac{\kappa_2}{2}\right)}, \tag{39}$$

$$b_{2\text{out}}[\omega] = \frac{\left[\frac{\kappa_2-\kappa_1}{2} + i(\omega-\omega_c)\right]\hat{b}_{2\text{in}}[\omega] + \sqrt{\kappa_1\kappa_2}\hat{b}_{1\text{in}}[\omega]}{i(\omega-\omega_c) - \left(\frac{\kappa_1}{2}+\frac{\kappa_2}{2}\right)}. \tag{40}$$

Symmetric cavity

Consider for example the case of equally transmitted mirrors, $\kappa_1 = \kappa_2 = \kappa$. In this case Eq. (39,40) becomes

$$\hat{b}_{1\text{out}}[\omega] = \frac{i(\omega-\omega_c)\hat{b}_{1\text{in}}[\omega] + \kappa\hat{b}_{2\text{in}}[\omega]}{i(\omega-\omega_c) - \kappa}. \tag{41}$$

This expression can be analized in two limits:

- Near the resonance

$$\hat{b}_{1\text{out}}[\omega] \approx \frac{\kappa}{i(\omega-\omega_c) - \kappa}\hat{b}_{2\text{in}}[\omega], \tag{42}$$

  which is a through-pass Lorenzian filter.

- Far from resonance $|\omega - \omega_c| \gg \kappa$ the input field is completely reflected

$$\hat{b}_{1\text{out}}[\omega] = \hat{b}_{1\text{in}}[\omega]. \tag{43}$$

A simple model for attenuators

The two-sided cavity is the simplest model for attenuation. Consider Eq. (39). Then let us imagine that $b_{2\text{in}}$ is a signal that enters the cavity. Suppose we do not input any signal $b_{1\text{in}}$ (except the vacuum, which enters in inevitably).

$$b_{1\text{out}}[\omega] = \frac{\sqrt{\kappa_1\kappa_2}}{i(\omega-\omega_c) - \left(\frac{\kappa_1}{2}+\frac{\kappa_2}{2}\right)}\hat{b}_{2\text{in}}[\omega]. \tag{44}$$

We notice that $|b_{1\text{out}}| < |b_{2\text{in}}|$, and this is due to the fact that

$$\left|\frac{\sqrt{\kappa_1\kappa_2}}{i(\omega-\omega_c) - \left(\frac{\kappa_1}{2}+\frac{\kappa_2}{2}\right)}\right| < 1. \tag{45}$$

III. CONVENTION USED FOR TIME-FOURIER TRANSFORMS OF OPERATORS

It is important to spell out explicitly the conventions we use for the Fourier transform, especially for the writing of the adjoint of the Fourier-transformed operator $\hat{a}(t)$ or $\hat{b}(t)$. The question is: which one is the canonically conjugate operator of $\hat{a}[\omega]$, is it $(\hat{a}[\omega])^\dagger$ or $\hat{a}^\dagger[\omega]$. We follow here the convention introduced for the first time in Ref. [1]. Before this paper, in most textbooks and papers the conventions varied or were not very explicitly spelled out.

The Fourier transform is

$$\hat{a}[\omega] = \int_{-\infty}^{\infty} dt e^{i\omega t} \hat{a}(t), \tag{46}$$

with inverse

$$\hat{a}[t] = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \hat{a}[\omega], \tag{47}$$

and the commutation relations are

$$\left[\hat{a}(t), \hat{a}^\dagger(t')\right] = \delta(t - t'). \tag{48}$$

Taking the adjoint of Eq. (47) results in

$$\hat{a}^\dagger[\omega] = (\hat{a}[-\omega])^\dagger = \int_{-\infty}^{\infty} dt e^{i\omega t} \hat{a}^\dagger(t). \tag{49}$$

Thus in this convention the Fourier transform acts in the same way (with the same exponent $e^{i\omega t}$) no matter on what operator - either $\hat{a}$ or $\hat{a}^\dagger$,

$$(\hat{a}[\omega])^\dagger = \int_{-\infty}^{\infty} dt e^{-i\omega t} \hat{a}^\dagger(t), \tag{50}$$

and

$$\left[\hat{a}[\omega], (\hat{a}[\omega'])^\dagger\right] = 2\pi\delta(\omega - \omega'), \tag{51}$$

and similarly for the $\hat{b}$ operators.

For example, if the modes are at thermal equilibrium, with photon (bosonic) statistics, we can write

$$\left\langle \left(\hat{b}_{\text{in}}[\omega]\right)^\dagger \hat{b}_{\text{in}}[\omega'] \right\rangle = 2\pi\delta(\omega - \omega') n_{\text{B}}(\hbar\omega), \tag{52}$$

$$\left\langle \hat{b}_{\text{in}}[\omega'] \left(\hat{b}_{\text{in}}[\omega]\right)^\dagger \right\rangle = 2\pi\delta(\omega - \omega')[1 + n_{\text{B}}(\hbar\omega)]. \tag{53}$$

Finally, it is worth reminding the following representation of the Dirac delta function,

$$2\pi\delta(t-t') = \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')}. \tag{54}$$

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[1] A. A. Clerk, M. H. Devoret, S. M. Girvin, F. Marquardt, R. J. Schoelkopf, Rev. Mod. Phys. **82**, 1155 (2010), arXiv:0810.4729.

[2] D. F. Walls and G.J. Milburn, Quantum Optics, Springer-Verlag, Berlin (1994).

[3] C. W. Gardiner, Quantum Noise, Springer-Verlag, Berlin, (1991).