ELEC-E4130

Lecture 18: Rectangular Waveguides

Ch. 10

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TE modes

Recall from last time

General Waveguide Equations

$$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right)$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right)$$

$$E_x = \frac{-j}{k_c^2}\left(\omega\mu\frac{\partial H_z}{\partial y} + \beta\frac{\partial E_z}{\partial x}\right)$$

$$E_y = \frac{j}{k_c^2}\left(\omega\mu\frac{\partial H_z}{\partial x} - \beta\frac{\partial E_z}{\partial y}\right)$$

Cutoff wavenumber

$$k_c^2 = k^2 - \beta^2$$

$$k_c^2 = k_x^2 + k_y^2$$

- Once we know the longitudinal components, we know everything else
- k_c, β need to be pre-determined
- We need to determine E_z, H_z, k_x, and k_y
  - Boundary Conditions

More on boundary conditions

Generalized on top of PEC

$$\text{B.C.}\begin{cases} E_{t1} = 0 \\ H_{n1} = 0 \\ i_n \times H_1 = J_S \end{cases}$$

Tangential
Components
(u, v)

Normal
Components
(n)

The curl equation

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$

$$\begin{vmatrix} a_u & a_v & a_n \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial n} \\ H_u & H_v & \cancel{H_n} \end{vmatrix} = j\omega\varepsilon(\mathbf{a_u}\cancel{E_u} + \mathbf{a_v}\cancel{E_v} + \mathbf{a_n}E_n)$$

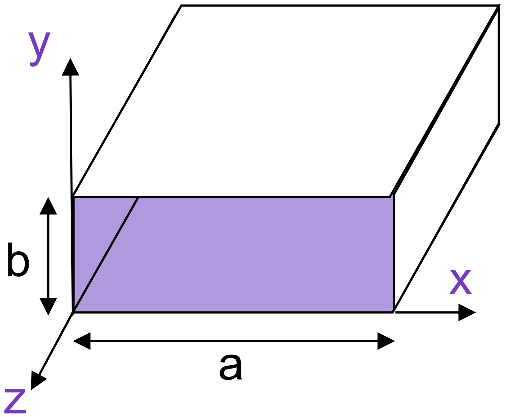
Compare the tangential components (in u, v)

It yields
$$\begin{cases} \frac{\partial}{\partial n}H_v = j\omega\varepsilon E_u = 0 \\ \frac{\partial}{\partial n}H_u = j\omega\varepsilon E_v = 0 \end{cases}$$

Therefore we have on a PEC
$$\begin{cases} \boxed{\frac{\partial H_{t1}}{\partial n} = 0} \\ E_{t1} = 0 \end{cases}$$

(implicit B.C.)

Rectangular Waveguide, TE modes



Which leads to,

$$\frac{d^2 f(x)}{dx^2} g(y) + f(x)\frac{d^2 g(y)}{dy^2} + k_c^2 f(x) g(y) = 0$$

Dividing by f(x)g(x) yields,

$$\underbrace{\frac{1}{f(x)}\frac{d^2 f(x)}{dx^2}}_{-k_x^2} + \underbrace{\frac{1}{g(y)}\frac{d^2 g(y)}{dy^2}}_{-k_y^2} + k_c^2 = 0 \longrightarrow ({k_c}^2 = k_x^2 + k_y^2 = k^2 - \beta^2)$$

The Wave Equation for TE modes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) h_z(x, y) = 0$$

Decoupling to two 1-D wave equations

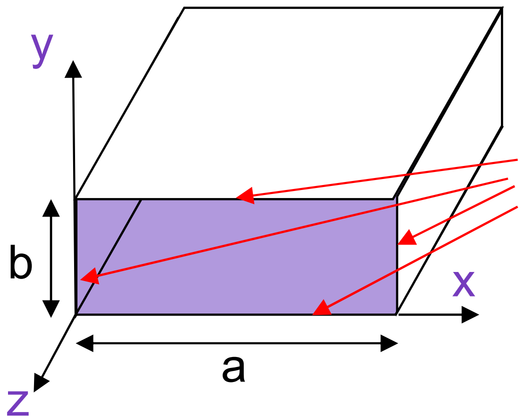
$$\left(\frac{d^2}{dx^2} + k_x^2\right) f(x) = 0$$

$$\left(\frac{d^2}{dy^2} + k_y^2\right) g(y) = 0$$

Separation of variables:

$$h_z(x, y) = f(x) g(y)$$

Rectangular Waveguide, TE modes



$$\frac{\partial H_{t1}}{\partial n} = 0$$

Boundary conditions:

$$\frac{\partial \mathrm{h_z}}{\partial \mathrm{x}} = 0, \ @\ \mathrm{x} = 0, \mathrm{a}$$

$$\frac{\partial \mathrm{h_z}}{\partial \mathrm{y}} = 0, \ @\ \mathrm{y} = 0, \mathrm{b}$$

B.C. at the x = 0 wall, y = 0 floor

$$\left.\frac{\partial \mathrm{h_z(x,y)}}{\partial \mathrm{x}}\right|_{x=0} = 0 \quad \rightarrow \quad \mathrm{A} = 0$$

$$\left.\frac{\partial \mathrm{h_z(x,y)}}{\partial \mathrm{y}}\right|_{y=0} = 0 \quad \rightarrow \quad \mathrm{C} = 0$$

General solutions of electric field:

$$\mathrm{f(x)} = \mathrm{A}\sin(\mathrm{k_x x}) + \mathrm{B}\cos(\mathrm{k_x x})$$

$$\mathrm{g(y)} = \mathrm{C}\sin(\mathrm{k_y y}) + \mathrm{D}\cos(\mathrm{k_y y})$$

Reduced equation

$$\mathrm{h_z(x,y)} = \mathrm{B'}\cos(\mathrm{k_x x})\cos\left(\mathrm{k_y y}\right)$$

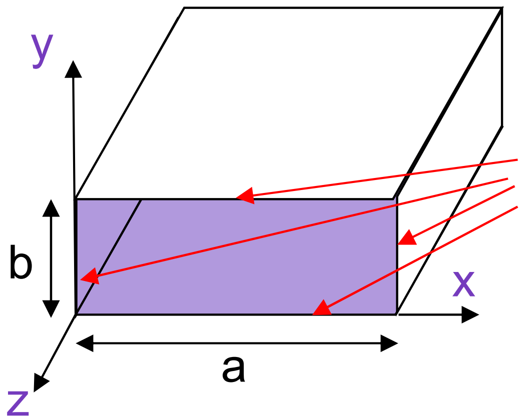
$$B' = BD$$

General solutions of electric field:

$$\mathrm{h_z(x,y)} = \mathrm{f(x)g(y)}$$

$$\mathrm{h_z(x,y)} = (\mathrm{A}\sin(\mathrm{k_x x}) + \mathrm{B}\cos(\mathrm{k_x x}))\left(\mathrm{C}\sin(\mathrm{k_y y}) + \mathrm{D}\cos(\mathrm{k_y y})\right)$$

Rectangular Waveguide, TE modes



$$\frac{\partial H_{t1}}{\partial n} = 0$$

Reduced equation

$$h_z(x,y) = B' \cos(k_x x)\cos(k_y y)$$

$$B' = BD$$

Boundary conditions at the x = a wall

$$\left.\frac{\partial h_z(x,y)}{\partial x}\right|_{x=a} = -A' k_x \sin(k_x a)\cos(k_y y) = 0$$

$$\sin(k_x a) = 0 \quad \rightarrow \quad k_x a = m\pi$$

$$k_x = \frac{m\pi}{a} \quad \forall \text{ m integers}$$

Boundary conditions at the y = b ceiling

$$\left.\frac{\partial h_z(x,y)}{\partial y}\right|_{y=b} = -A' k_y \cos(k_x x)\sin(k_y b) = 0$$

$$\sin(k_y b) = 0 \quad \rightarrow \quad k_y b = n\pi$$

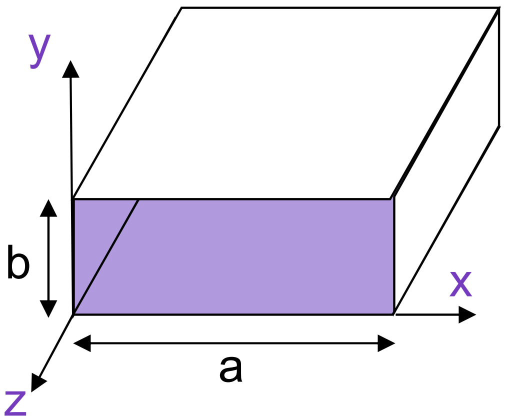
$$k_y = \frac{n\pi}{b} \quad \forall \text{ n integers}$$

Longitudinal component

$$h_z(x,y) = B_{mn} \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$B_{mn} = B'$$

Rectangular Waveguide, TE modes



Longitudinal field

$$h_z(x,y) = B_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_z(x,y,z) = h_z(x,y)e^{-j\beta z}$$

$$\boxed{H_z(x,y,z) = B_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}$$

$\rightarrow$

TE Wave Equations

$$H_x = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial x}$$

$$H_y = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial y}$$

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}$$

- Now that the longitudinal field is full characterized, we can compute all the transverse fields
- Propagation constant β is the same for TE and TM

TE vs TM modes

TM modes

$$E_z(x,y,z) = A_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta z}$$

$$H_x(x,y,z) = \frac{j\omega\varepsilon n\pi}{bk_c^2} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z}$$

$$H_y(x,y,z) = \frac{-j\omega\varepsilon m\pi}{ak_c^2} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z}$$

$$E_x(x,y,z) = \frac{-j\beta m\pi}{ak_c^2} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z}$$

$$E_y(x,y,z) = \frac{-j\beta n\pi}{bk_c^2} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z}$$

$$Z_{TM} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\varepsilon} = \frac{\beta\eta}{k}$$

TE modes

$$H_z(x,y,z) = B_{mn} \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta z}$$

$$E_x(x,y,z) = \frac{j\omega\mu n\pi}{bk_c^2} B_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z}$$

$$E_y(x,y,z) = \frac{-j\omega\mu m\pi}{ak_c^2} B_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z}$$

$$H_x(x,y,z) = \frac{j\beta m\pi}{ak_c^2} B_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z}$$

$$H_y(x,y,z) = \frac{j\beta n\pi}{bk_c^2} B_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z}$$

$$Z_{TE} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta}$$

Cutoff Frequencies

Mode numbers

TM modes

$$E_z(x,y,z) = A_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta z}$$

- If m = 0 **or** n = 0, $E_z = 0$
- If $E_z = 0$ then E_x, E_y, H_x, and $H_y = 0$.
  - No propagation of energy
- If m > 0 **and** n > 0, $E_z \neq 0$
  - YES propagation of enery

TE modes

$$H_z(x,y,z) = B_{mn} \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta z}$$

- If m = 0 **and** n = 0, $H_z = 0$
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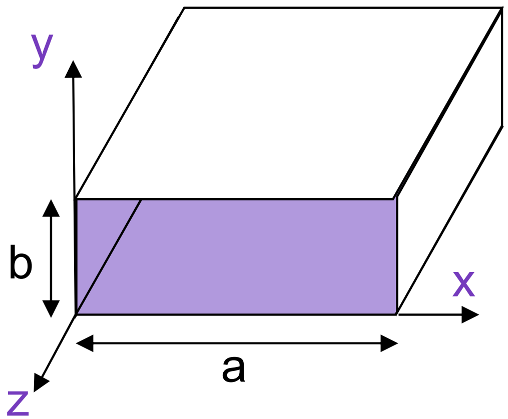
Cutoff Frequency Analysis



$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Two possibilities may happen

1 $\boldsymbol{k > k_c}$, β real, wave propagating $e^{-j\beta z}$

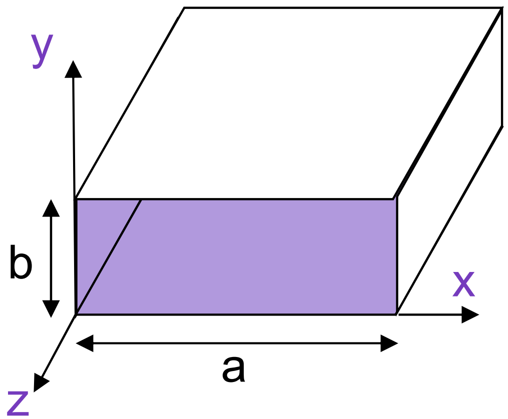
The longitudinal variation of field follows $e^{-j\beta z}$ that is with constant amplitude

2 $\boldsymbol{k < k_c}$, β imaginary, wave decays exponentially

(assuming $\beta = -j\alpha$, $e^{-j\beta z} \Rightarrow e^{-\alpha z}$)

- $k < k_c \rightarrow f < f_c$ propagation constant has a 0 real part meaning NO traveling wave meaning no flow of energy
- Exponentially decaying wave is an evanescent wave

Cutoff Frequency Analysis



Consider TM mode

$$Z_{TM} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\varepsilon} = \frac{\beta\eta}{k} \quad \begin{cases} \text{real} & \text{for } f > f_c \\ \text{imaginary} & \text{for } f < f_c \end{cases}$$

The complex poynting vector:

$$S = E \times H^* = \mathbf{a_z} \frac{|E|^2}{Z_{TM}} \quad \begin{cases} \text{real} & \text{for } f > f_c \\ \text{imaginary} & \text{for } f < f_c \end{cases}$$

- No power will be carried through under the cutoff frequency!

The average power density:

$$\mathbf{S}_{av} = \frac{1}{2}\mathrm{Re}[\mathbf{E} \times \mathbf{H}^*] = \frac{1}{2}\mathrm{Re}\{\mathbf{S}\} = 0$$

Propagation constant parameters (1/2)

Cutoff wavenumber

$$k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Cutoff frequency

$$f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{c}{2\sqrt{\varepsilon_r}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Cutoff wavelength

$$\lambda_c = \frac{2\pi}{k_c} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

Propagation constant

$$\beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Wave impedance

$$Z_{TE} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta} = \frac{k}{\sqrt{k^2 - k_c^2}}\eta = \frac{1}{\sqrt{1 - (f_c/f)^2}}\frac{\eta_0}{\sqrt{\varepsilon_r}}$$

$$Z_{TM} = \frac{\beta}{\omega\varepsilon} = \frac{\beta\eta}{k} = \frac{\sqrt{k^2 - k_c^2}}{k}\eta = \sqrt{1 - (f_c/f)^2}\frac{\eta_0}{\sqrt{\varepsilon_r}}$$

Wavenumber in medium

$$k = \omega\sqrt{\mu\varepsilon}$$

Propagation constant parameters (2/2)

Phase velocity

$$\frac{1}{u_p} = \frac{\beta}{\omega} = \frac{1}{c}\frac{\beta}{k}$$

$$u_p = f\lambda_g = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{k^2 - k_c^2}} = \frac{c/\sqrt{\varepsilon_r}}{\sqrt{1-(f_c/f)^2}}$$

Group velocity

$$\frac{1}{u_g} = \frac{d\beta}{d\omega} = \frac{1}{c}\frac{d\beta}{dk}$$

$$\frac{d\beta}{dk} = \frac{k}{\sqrt{k^2 - k_c^2}}$$

$$u_g = (c/\sqrt{\varepsilon_r}) \cdot \sqrt{1-(f_c/f)^2}$$

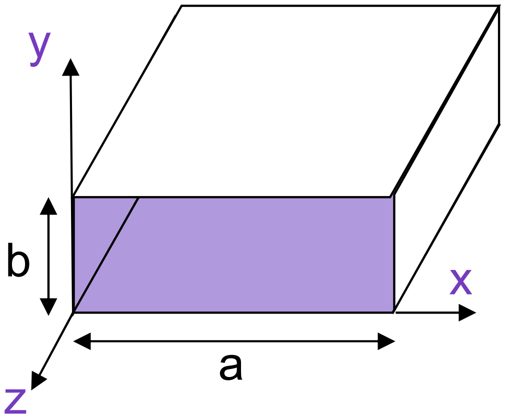
Guide wavelength

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - k_c^2}} = \frac{2\pi}{k\sqrt{1-(f_c/f)^2}} = \frac{\lambda}{\sqrt{1-(f_c/f)^2}}$$

Relation

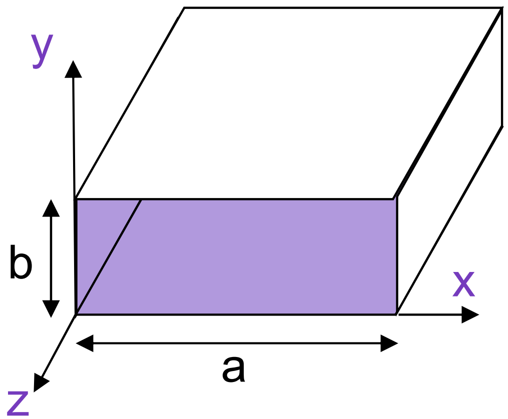
$$u_p u_g = \frac{c^2}{\varepsilon_r}$$

In class exercise: Cutoff Frequencies



- For an air-filled rectangular waveguide with inner dimensions 0.9 in by 0.4in.
- (1) calculate the cutoff frequencies for the TE_{10}, TE_{01}, TE_{20}, TE_{11} and TM_{11} modes.
- (2) Recalculate the cutoff frequencies for the TE_{10}, TE_{01}, TE_{20}, TE_{11} and TM_{11} modes if the same guide is filled with polystyrene (ε_r=2.55).

In class exercise: Cutoff Frequencies



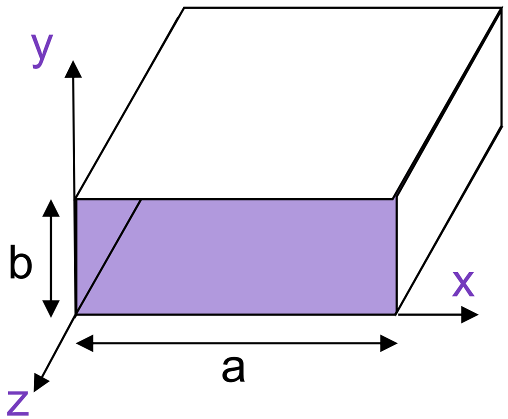
- For an air-filled rectangular waveguide with inner dimensions **0.9 in by 0.45in.**
- (1) calculate the cutoff frequencies for the TE_{10}, TE_{01}, TE_{20}, TE_{11} and TM_{11} modes.
- (2) Recalculate the cutoff frequencies for the TE_{10}, TE_{01}, TE_{20}, TE_{11} and TM_{11} modes if the same guide is filled with polystyrene (ε_r=2.55).

$$\boxed{k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$f_c = \frac{\omega_c}{2\pi}, \quad \omega_c = \frac{k_c c}{\sqrt{\varepsilon_r}}$$

$$\boxed{f_c = \frac{k_c c}{2\pi\sqrt{\varepsilon_r}}}$$

In class exercise: Cutoff Frequencies



- For an air-filled rectangular waveguide with inner dimensions **0.9 in by 0.45in.**
- (1) calculate the cutoff frequencies for the TE_{10}, TE_{01}, TE_{20}, TE_{11} and TM_{11} modes.
- (2) Recalculate the cutoff frequencies for the TE_{10}, TE_{01}, TE_{20}, TE_{11} and TM_{11} modes if the same guide is filled with polystyrene (ε_r=2.55).

$\varepsilon_r = 1.0$

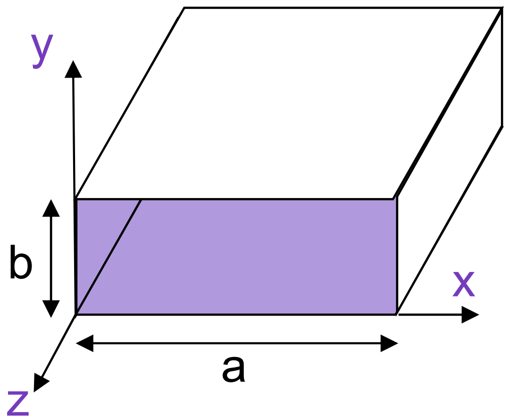
TE_{10}, $$\frac{3\cdot 10^8}{2\pi}\sqrt{\left(\frac{\pi}{0.9\cdot 0.0254}\right)^2} = 6.56 \text{ GHz}$$

TE_{01}, $$\frac{3\cdot 10^8}{2\pi}\sqrt{\left(\frac{\pi}{0.45\cdot 0.0254}\right)^2} = 13.14 \text{ GHz}$$

TE_{20}, $$\frac{3\cdot 10^8}{2\pi}\sqrt{\left(\frac{2\pi}{0.9\cdot 0.0254}\right)^2} = 13.14 \text{ GHz}$$

TE_{11}&TM_{11} $$\frac{3\cdot 10^8}{2\pi}\sqrt{\left(\frac{\pi}{0.9\cdot 0.0254}\right)^2+\left(\frac{\pi}{0.45\cdot 0.0254}\right)^2} = 14.62 \text{ GHz}$$

In class exercise: Cutoff Frequencies



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- (2) Recalculate the cutoff frequencies for the TE_{10}, TE_{01}, TE_{20}, TE_{11} and TM_{11} modes if the same guide is filled with polystyrene (ε_r=2.55).

ε_r = =2.55

TE_{10},
$$\frac{3\cdot 10^8/\sqrt{2.55}}{2\pi}\sqrt{\left(\frac{\pi}{0.9\cdot 0.0254}\right)^2} = 4.37\text{ GHz}$$

TE_{01},
$$\frac{3\cdot 10^8/\sqrt{2.55}}{2\pi}\sqrt{\left(\frac{\pi}{0.45\cdot 0.0254}\right)^2} = 8.74\text{GHz}$$

TE_{20},
$$\frac{3\cdot 10^8/\sqrt{2.55}}{2\pi}\sqrt{\left(\frac{2\pi}{0.9\cdot 0.0254}\right)^2} = 8.74\text{ GHz}$$

TE_{11}&TM_{11}
$$\frac{3\cdot 10^8/\sqrt{2.55}}{2\pi}\sqrt{\left(\frac{\pi}{0.9\cdot 0.0254}\right)^2+\left(\frac{\pi}{0.45\cdot 0.0254}\right)^2} = 9{,}77\text{ GHz}$$

Dominant TE_{10} mode

Lambdas

Wavelength in a vacuum

$$\lambda_0 = \frac{c_0}{f}$$

- Wavelength in a vacuum corresponding to the waveguide operational frequency. Remember that time is fixed.

Wavelength in media

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}}$$

- Wavelength in the waveguide fill medium

Guide wavelength

$$\lambda_g = \frac{2\pi}{\beta}$$

- Peak-to-peak distance for propagating energy
  - **Dependent** on waveguide fill permittivity

Cutoff wavelength

$$\lambda_c = \frac{2\pi}{k_c}$$

- Cutoff wavelength corresponding to cutoff wavenumber
  - **Independent** of waveguide fill permittivity

Guide wavelength vs guide velocity

Propagation constant

$$\beta = \sqrt{k^2 - k_c^2} \quad \rightarrow \quad \beta < k$$

Frequency

$$\frac{u_p}{\lambda_g} = \frac{\frac{\omega}{\beta}}{\frac{2\pi}{\beta}} = \frac{\omega}{2\pi} = f$$

Wavelength

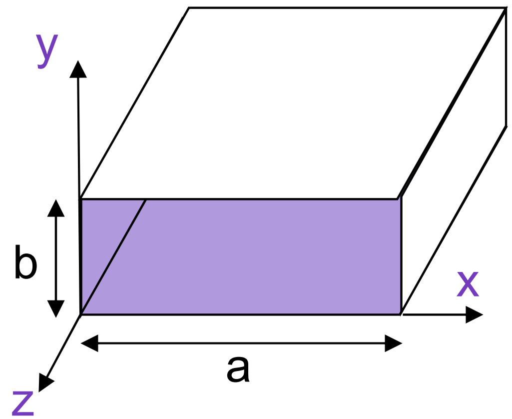
$$\lambda_g = \frac{2\pi}{\beta} > \frac{2\pi}{k} \quad \rightarrow \quad \lambda_g > \lambda$$

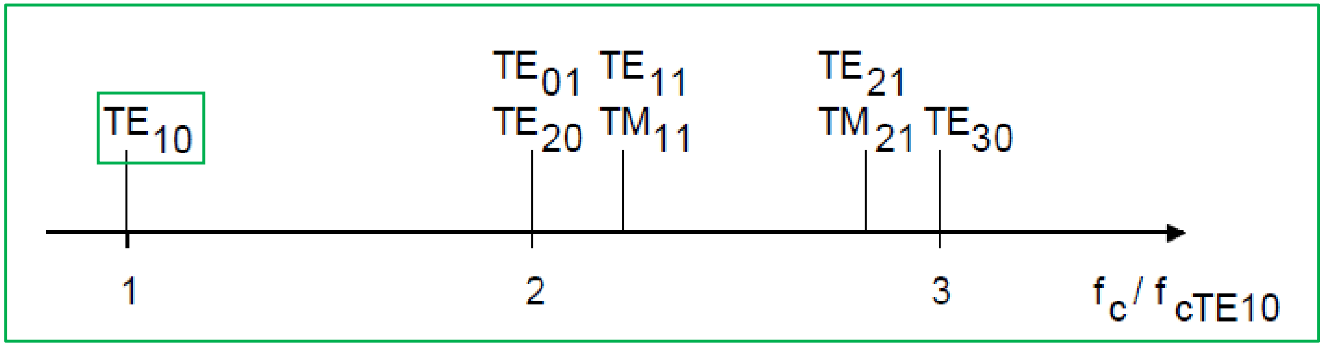
Phase velocity

$$u_p = \frac{\omega}{\beta} > \frac{\omega}{k} \quad \rightarrow \quad u_p > c$$

- The wavelength of a propagating mode inside the waveguide of some frequency f is always larger than the wavelength of a TEM mode of the same frequency
- The phase velocity of a propagating mode inside the waveguide of some frequency f is always larger than the phase velocity of a TEM mode of the same frequency
  - $u_p > c$

Dominant TE_{10} mode





$$\lambda_{c,mn} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

For TE_{10}: $\lambda_c = 2a$

For TE_{01}: $\lambda_c = 2b$

For TE_{11} & TM_{11}: $\lambda_c = \frac{2ab}{\sqrt{a^2 + b^2}}$

Therefore,

$$\lambda_{c,10} > \lambda_{c,01} > \lambda_{c,11}$$

(when a>b)

For rectangular waveguides, when a>b, the longest λ_c and the lowest cutoff frequency exist for TE_{10} mode!!!

The dominant mode!

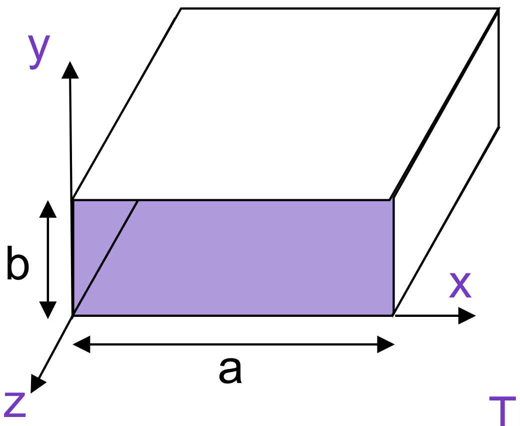
Single mode operation:

$$\lambda_{c,01} < \lambda < \lambda_{c,10} \Rightarrow 2b < \lambda < 2a$$

Only the dominant TE_{10} mode is supported!

Minimum mode dispersion!

Dominant TE_{10} mode



The cutoff wave number is, $k_{c,10} = \dfrac{\pi}{a}$

The phase constant is thus given by,

$$\beta_{10} = \sqrt{k^2 - k_{c,10}^2} = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2} = k\sqrt{1 - \left(\frac{\lambda}{2a}\right)}$$

The cutoff wavelength is, $\lambda_{c,10} = \dfrac{2\pi}{k_c} = \dfrac{2\pi}{\pi/a} = 2a$

(the waveguide has to be at least greater than half of the wavelength in order for the wave to propagate)

The cutoff frequency is thus: $f_{c,10} = \dfrac{c/\sqrt{\varepsilon_r}}{\lambda_{c,10}} = \dfrac{c/\sqrt{\varepsilon_r}}{2a} = \dfrac{1}{2a\sqrt{\mu\varepsilon}}$

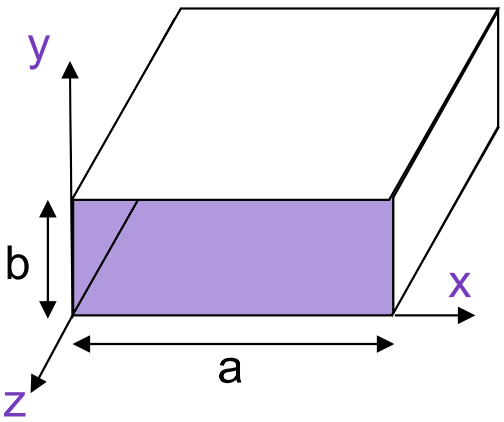
The guide wavelength:

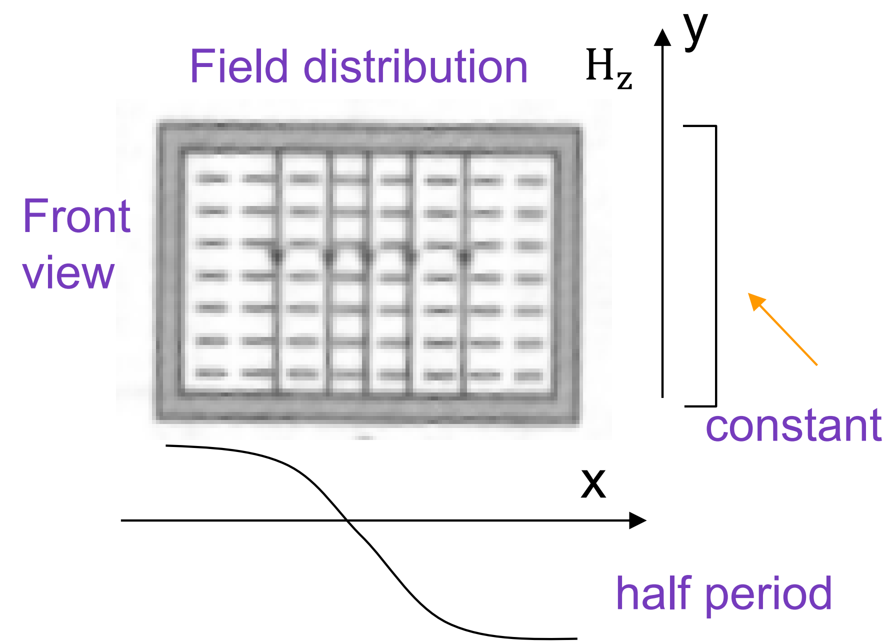
$$\lambda_g = \frac{2\pi}{\beta_{10}} = \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}},$$

The phase velocity:

$$v_p = \frac{\omega}{\beta_{10}} = \frac{c/\sqrt{\varepsilon_r}}{\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}} = \frac{c/\sqrt{\varepsilon_r}}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}},$$

Dominant TE_{10} mode





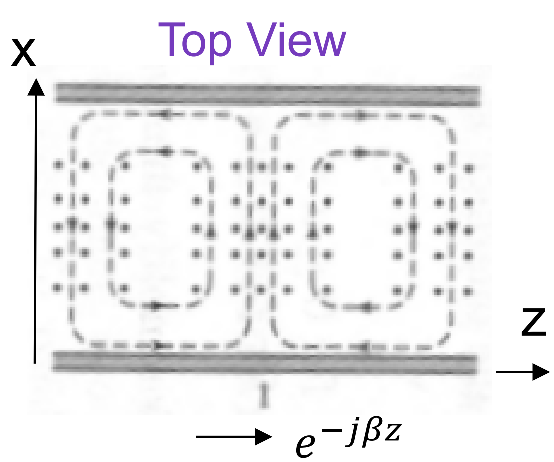
TE_{10} fields

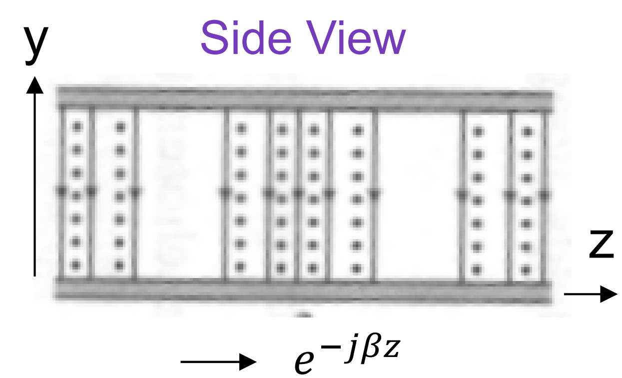
$$E_y = \frac{-j\omega\mu a}{\pi} B_{10} \sin\frac{\pi x}{a} e^{-j\beta z}$$

$$H_x = \frac{j\beta a}{\pi} B_{10} \sin\frac{\pi x}{a} e^{-j\beta z}$$

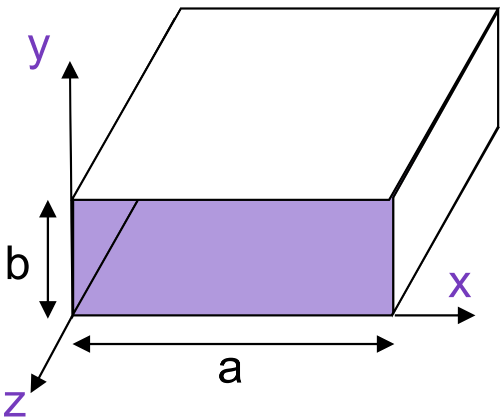
$$H_z = B_{10} \cos\frac{\pi x}{a} e^{-j\beta z}$$

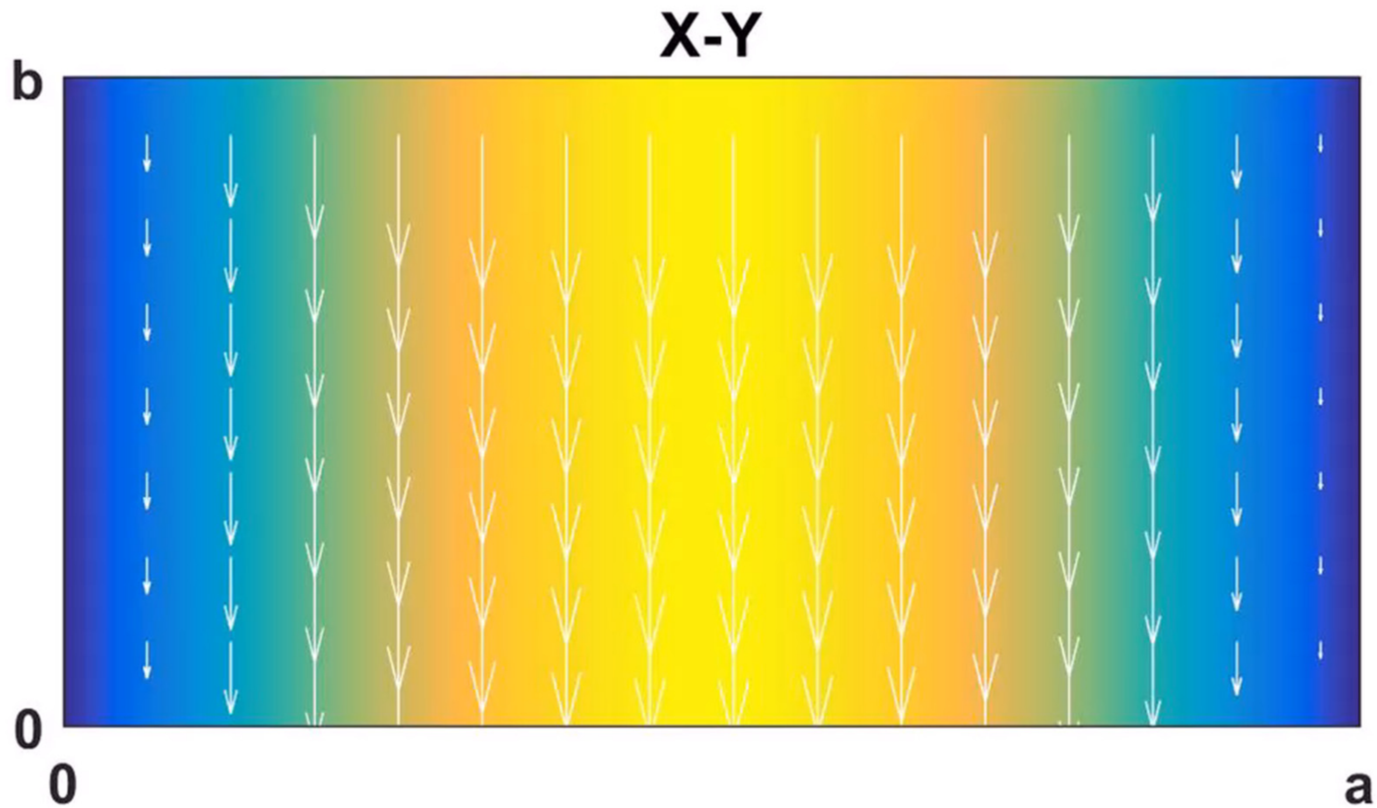
$$E_z = E_x = H_y = 0$$





Dominant TE_{10} mode

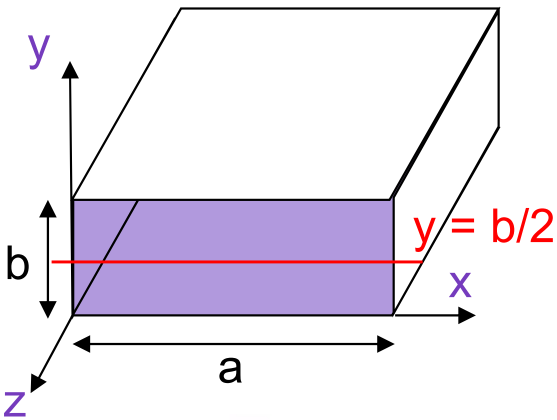




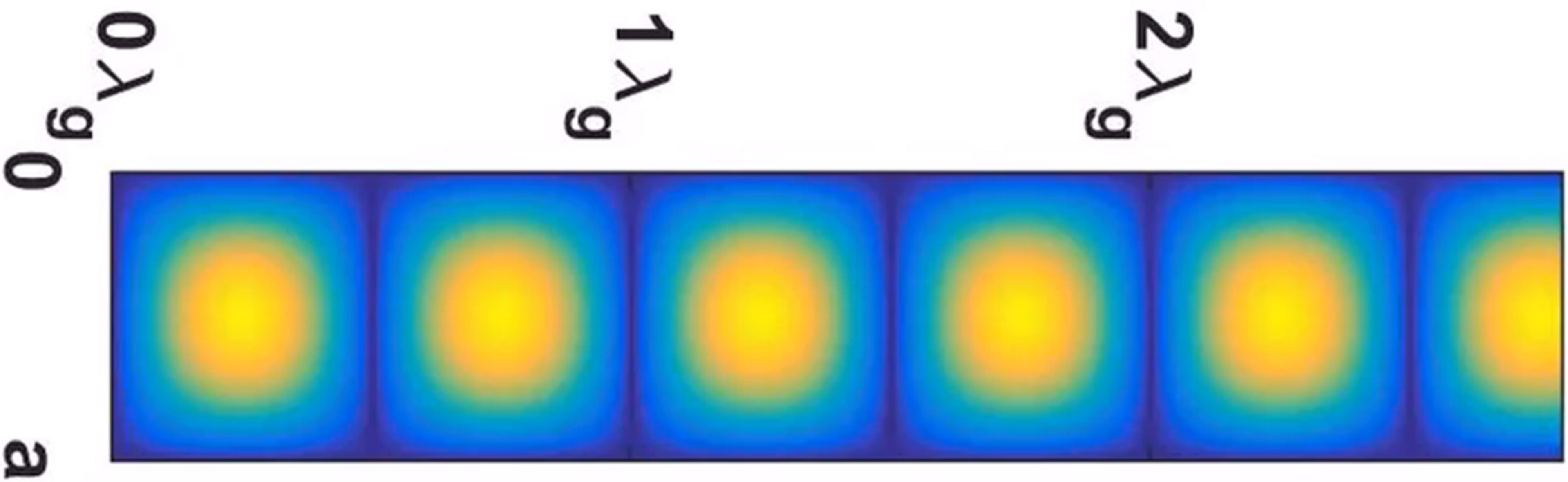
- E field modulus and vector field components
- Z = 0
- Electric field in the along horizontal axis is ½ period which corresponds to n = 1

- Electric field in the vertical axis is constant which corresponds to m = 0 (not sinusoid)

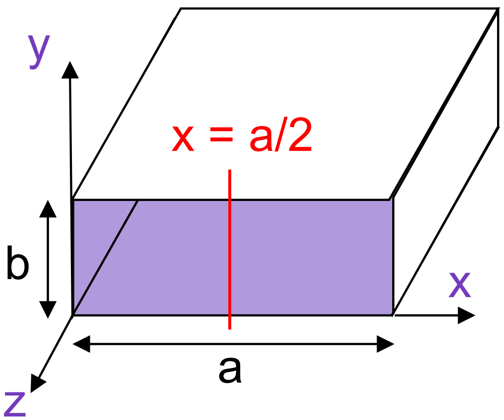
Dominant TE_{10} mode



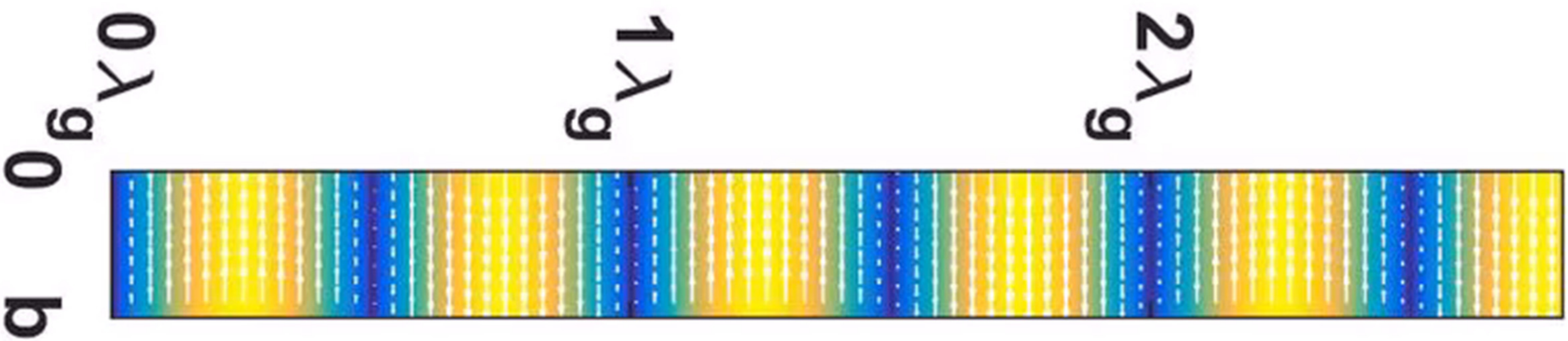
- E field modulus and vector field components
- Cross sectional view at y = b/2
- Oscillating Vector components not visible as the are aligned orthogonal to the page



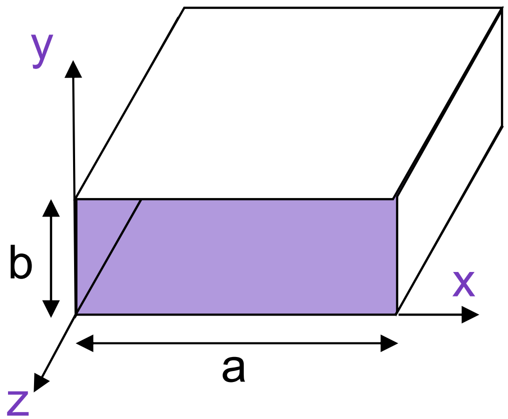
Dominant TE_{10} mode



- E field modulus and vector field components
- Cross sectional view at x = a/2

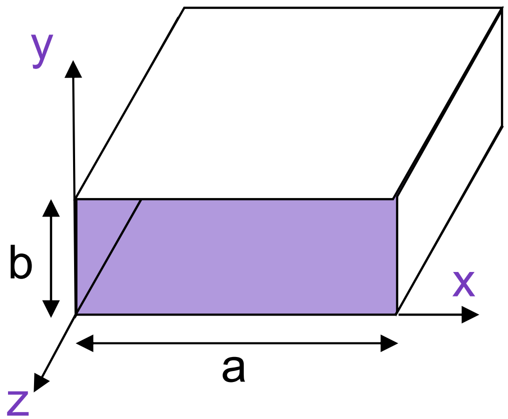


In class exercise: Cutoff Frequencies



- For an air-filled rectangular waveguide with inner dimensions 0.9 in by 0.4in. Assume f = 9 GHz
- (1) calculate Z_{TM} and Z_{TE} for each of the TE_{10}, TE_{01}, TE_{20}, TE_{11} and TM_{11} modes
- (2) what is λ_g, u_p and u_g for TE_{10} mode

In class exercise: Cutoff Frequencies



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$$Z_{TE} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta} = \frac{k}{\sqrt{k^2 - k_c^2}}\eta = \frac{1}{\sqrt{1-(f_c/f)^2}}\frac{\eta_0}{\sqrt{\varepsilon_r}}$$

$$Z_{TM} = \frac{\beta}{\omega\varepsilon} = \frac{\beta\eta}{k} = \frac{\sqrt{k^2 - k_c^2}}{k}\eta = \sqrt{1-(f_c/f)^2}\frac{\eta_0}{\sqrt{\varepsilon_r}}$$

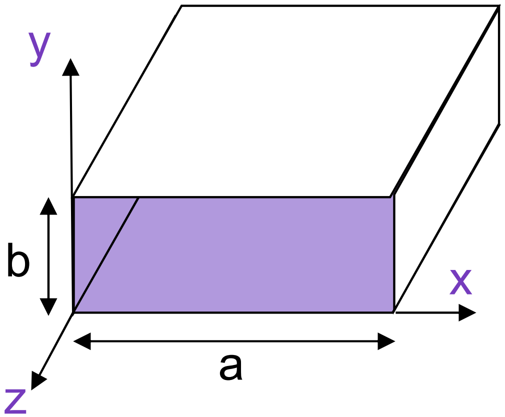
$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - k_c^2}} = \frac{2\pi}{k\sqrt{1-(f_c/f)^2}} = \frac{\lambda}{\sqrt{1-(f_c/f)^2}}$$

$$f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{c}{2\sqrt{\varepsilon_r}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$v_p = f\lambda_g = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{k^2 - k_c^2}} = \frac{c/\sqrt{\varepsilon_r}}{\sqrt{1-(f_c/f)^2}}$$

$$v_g = \frac{\left(c/\sqrt{\varepsilon_r}\right)^2}{v_p} = \left(c/\sqrt{\varepsilon_r}\right)\cdot\sqrt{1-(f_c/f)^2}$$

In class exercise: Cutoff Frequencies



- For an air-filled rectangular waveguide with inner dimensions 0.9 in by 0.4in. Assume f = 9 GHz
- (1) calculate Z_{TM} and Z_{TE} for each of the TE_{10}, TE_{01}, TE_{20}, TE_{11} and TM_{11} modes
- (2) what is λ_g, u_p and u_g for TE_{10} mode

Part (1)

$\mathrm{f_c(TE10)} = 6.55\ \mathrm{GHz}$ $\quad$ $\mathrm{Z(TE10)} = 550.38\ \Omega$

$\mathrm{f_c(TE01)} = 14.75\ \mathrm{GHz}$ $\quad$ $\mathrm{Z(TE01)} = 0.00 - \mathrm{j}355.70\ \Omega$

$\mathrm{f_c(TE20)} = 13.11\ \mathrm{GHz}$ $\quad$ $\mathrm{Z(TE20)} = 0.00 - \mathrm{j}355.70\ \Omega$

$\mathrm{f_c(TE11)} = 16.15\ \mathrm{GHz}$ $\quad$ $\mathrm{Z(TE11)} = 0.00 - \mathrm{j}292.92\ \Omega$

$\mathrm{f_c(TM11)} = 16.15\ \mathrm{GHz}$ $\quad$ $\mathrm{Z(TM11)} = 0.00 - \mathrm{j}292.92\Omega$

Part (2)

$$\lambda_0 = 33.3\ mm$$

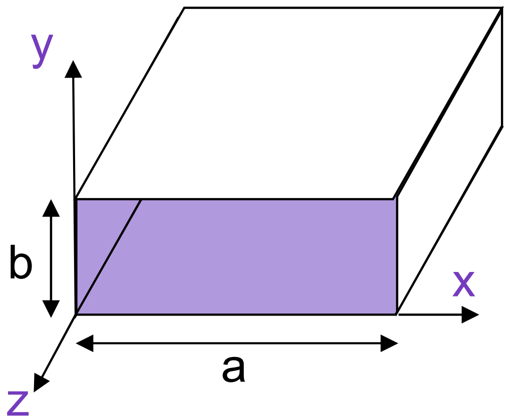
$$\lambda_g = 48.6\ mm$$

$$\mathrm{u_p} = 4.38 \times 10^8\ m/s$$

$$\mathrm{u_g} = 2.05 \times 10^8\ m/s$$

Conclusions

Real WG



Some standard waveguide specs.

Guide	Size (inch)	Rec. (GHz)	f_c (GHz)	Band	(GHz)
WR650	6.500×3.250	1.12 - 1.70	0.91	L	(1.0 - 2.0)
WR284	2.840×1.340	2.60 - 3.95	2.08	S	(2.0 - 4.0)
WR187	1.872×0.872	3.95 - 5.85	3.15	C	(4.0 - 8.0)
WR90	0.900×0.400	8.20 - 12.40	6.56	X	(8.0 - 12.0)
WR62	0.622×0.311	12.40 - 18.00	9.49	Ku	(12.0 - 18.0)
WR42	0.420×0.170	18.00 - 26.50	14.05	K	(18.0 - 27.0)
WR28	0.280×0.140	26.50 - 40.00	21.08	Ka	(27.0 - 40.0)

a b

$f_c(TE_{10})$

$<f_c(TE_{20})$

- All the above waveguides are a ~ 2b
- They are all intended to operate in the TE_{10} (lowest order mode)
- Recommended operating range provides buffer above the TE10 cutoff frequency and below the TE_{20} and TE_{01} cutoff frequencies
- a ~ 2b → $f_c(TE_{20}) \sim f_c(TE_{01})$

Real WG

Virginia Diodes Inc. Waveguide Band Designations

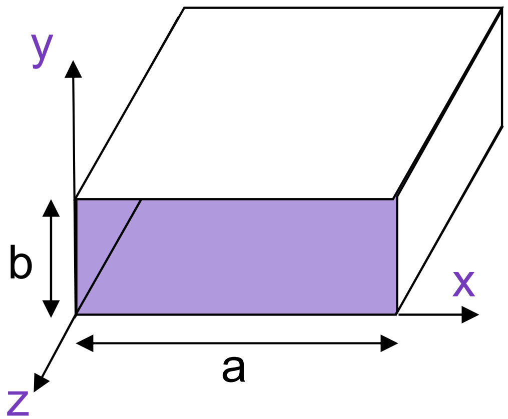
Last Modified : 6/29/2010

Virginia Diodes, Inc.

Internal Band Designation	EIA Band Designation	Internal Dimensions (mils)	Internal Dimensions (mm)	Frequency Range (GHz)	TE(10) Cutoff (GHz)	WG Loss Low - High[1] (dB/mm)	Flange Designation	Description	Letter Desig.
WR- 51.0	WR- 51	510 x 255	12.954 x 6.477	15.0 - 22.0	11.6	0.0005 - 0.0004			
WR- 42.0	WR- 42	420 x 170	10.668 x 4.318	17.5 - 26.5	14.0	0.0008 - 0.0006			K
WR- 34.0	WR- 34	340 x 170	8.636 x 4.318	22.0 - 33.0	17.4	0.001 - 0.0007			
WR- 28.0	WR- 28	280 x 140	7.112 x 3.556	26.5 - 40.0	21.1	0.0013 - 0.0009	UG-599/U	Square, Four hole fixing	Ka
WR- 22.4	WR- 22	224 x 112	5.690 x 2.845	33.0 - 50.5	26.3	0.0019 - 0.0013	UG-383/U	Circular, Four hole fixing/doweled	Q
WR- 18.8	WR- 19	188 x 94	4.775 x 2.388	40.0 - 60.0	31.4	0.0023 - 0.0016	UG-383/UM	Circular, Four hole fixing/doweled	U
WR- 14.8	WR- 15	148 x 74	3.759 x 1.880	50.5 - 75.0	39.9	0.0034 - 0.0024	UG-385/U	Circular, Four hole fixing/doweled	V
WR- 12.2	WR- 12	122 x 61	3.099 x 1.549	60.0 - 90.0	48.4	0.0047 - 0.0032	UG-387/U	Circular, Four hole fixing/doweled	E
WR- 10.0	WR- 10	100 x 50	2.540 x 1.270	75.0 - 110.0	59.0	0.0061 - 0.0043	UG-387/UM	Circular, Four hole fixing/doweled	W
WR- 8.0	WR- 8	80 x 40	2.032 x 1.016	90.0 - 140.0	73.8	0.0092 - 0.0059	UG-387/UM	Circular, Four hole fixing/doweled	F
WR- 6.5	WR- 6	65 x 32.5	1.651 x 0.826	110.0 - 170.0	90.8	0.0128 - 0.0081	UG-387/UM	Circular, Four hole fixing/doweled	D
WR- 5.1	WR- 5	51 x 25.5	1.295 x 0.648	140.0 - 220.0	116	0.0185 - 0.0117	UG-387/UM	Circular, Four hole fixing/doweled	G
WR- 4.3	WR- 4	43 x 21.5	1.092 x 0.546	170.0 - 260.0	137	0.0227 - 0.0151	UG-387/UM	Circular, Four hole fixing/doweled	
WR- 3.4	WR- 3	34 x 17	0.864 x 0.432	220.0 - 330.0	174	0.0308 - 0.0214	UG-387/UM	Circular, Four hole fixing/doweled	
WR- 2.8	n/a	28 x 14	0.711 x 0.356	260.0 - 400.0	211	0.0436 - 0.0287	UG-387/UM	Circular, Four hole fixing/doweled	
WR- 2.2	n/a	22 x 11	0.559 x 0.279	330.0 - 500.0	268	0.063 - 0.041	UG-387/UM	Circular, Four hole fixing/doweled	
WR- 1.9	n/a	19 x 9.5	0.483 x 0.241	400.0 - 600.0	311	0.072 - 0.051	UG-387/UM	Circular, Four hole fixing/doweled	
WR- 1.5	n/a	15 x 7.5	0.381 x 0.191	500.0 - 750.0	393	0.105 - 0.073	UG-387/UM	Circular, Four hole fixing/doweled	
WR- 1.2	n/a	12 x 6	0.305 x 0.152	600.0 - 900.0	492	0.159 - 0.104	UG-387/UM	Circular, Four hole fixing/doweled	
WR- 1.0	n/a	10 x 5	0.254 x 0.127	750.0 - 1100.0	590	0.192 - 0.135	n/a		
WR- 0.8	n/a	8 x 4	0.203 x 0.102	900.0 - 1400.0	738	0.292 - 0.188	n/a		
WR- 0.65	n/a	6.5 x 3.25	0.165 x 0.083	1100.0 - 1700.0	908	0.406 - 0.258	n/a		
WR- 0.51	n/a	5.1 x 2.55	0.130 x 0.065	1400.0 - 2200.0	1157	0.586 - 0.369	n/a		

1) The waveguide loss is calculated assuming the conductivity of Gold, and a surface roughness factor of 1.5.
The two values listed represent the loss at the low end and high end of the frequency range.

Propagation



$$\beta \neq k$$

- The one conductor geometry supports TE/TM operation
- The longitudinal phase variation of a TE/TM is not equal to the free space (plane wave) TEM phase variation
  - $\beta \rightarrow$ rectangular waveguide longitudinal phase variation
  - $k \rightarrow$ free space longitudinal phase variation
- β is a strong function of frequency and geometry

Conclusions and Next time

- TE and TM modes derived in the same manner
- TE mode uses implicit boundary conditions
- If the frequency of operation is below the so called cutoff frequency, propagation down the waveguide is not supported
  - No power will flow
  - “Cut-on” frequency, the waveguide is a high pass filter
- Analysis of cutoff frequencies for and mode numbers indicates the TE10 mode has the lowest cutoff frequency
- TE_{10} mode is the dominant mode

- Next time we’ll go more in depth on group and phase velocity and compute the surface current densities on the walls in support of waveguide propagation loss calculations