ELEC E5440 Statistical Signal Processing. Homework set #2
due December 23, 2021

1. MS Estimator for sensor array

Consider the array signal model

$$\mathbf{x} = \mathbf{A}(\boldsymbol{\theta})\mathbf{s} + \mathbf{v}, \tag{1}$$

where:
$\mathbf{x}$ is the $M \times 1$ received signal vector,
$\mathbf{A}(\boldsymbol{\theta})$ is an $M \times K$ array steering matrix $K < M$,
$\mathbf{s}$ is a $K \times 1$ transmitted signal vector and
$\boldsymbol{\theta} = [\theta_1, \dots, \theta_K]^T$ is a known vector of DOAs that determines the matrix $\mathbf{A}(\boldsymbol{\theta})$.

Let $\mathbf{v}$ be complex Gaussian noise with mean ($E[\mathbf{v}] = \mathbf{0}$) and covariance matrix $E[\mathbf{v}\mathbf{v}^H] = \sigma^2\mathbf{I}_M$. The noise is uncorrelated with the signal. Assign a prior density $\mathbf{s} \sim \mathcal{CN}(0, \mathbf{P})$ for the transmitted signal (complex zero mean and covariance $E[\mathbf{s}\mathbf{s}^H] = \mathbf{P}$). Show that the mean square estimate of $\mathbf{s}$ based on $\mathbf{x}$ is:

$$\hat{\mathbf{s}}_{MS} = \mathbf{P}\mathbf{A}^H(\mathbf{A}\mathbf{P}\mathbf{A}^H + \sigma^2\mathbf{I}_M)^{-1}\mathbf{x}, \tag{2}$$

Bonus quiz: What is the MAP estimator in this case? Why?

Hint: In your derivation you may use the following result:

Result

Assuming that

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{CN} \begin{bmatrix} \boldsymbol{\mu}_\mathbf{x} \\ \boldsymbol{\mu}_\mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{C}_{\mathbf{xx}} & \mathbf{C}_{\mathbf{xy}} \\ \mathbf{C}_{\mathbf{yx}} & \mathbf{C}_{\mathbf{yy}} \end{bmatrix} \tag{3}$$

where $\boldsymbol{\mu}_\mathbf{x} = E[\mathbf{x}]$, $\boldsymbol{\mu}_\mathbf{y} = E[\mathbf{y}]$,
$\mathbf{C}_{\mathbf{xx}} = E[(\mathbf{x} - \boldsymbol{\mu}_\mathbf{x})(\mathbf{x} - \boldsymbol{\mu}_\mathbf{x})^H]$, $\mathbf{C}_{\mathbf{yy}} = E[(\mathbf{y} - \boldsymbol{\mu}_\mathbf{y})(\mathbf{y} - \boldsymbol{\mu}_\mathbf{y})^H]$, $\mathbf{C}_{\mathbf{yx}} = E[(\mathbf{y} - \boldsymbol{\mu}_\mathbf{y})(\mathbf{x} - \boldsymbol{\mu}_\mathbf{x})^H]$,
then the posterior density $p(\mathbf{y}|\mathbf{x})$ is also Gaussian with mean:

$$E[\mathbf{y}|\mathbf{x}] = \boldsymbol{\mu}_\mathbf{y} + \mathbf{C}_{\mathbf{yx}}\mathbf{C}_{\mathbf{xx}}^{-1}(\mathbf{x} - \boldsymbol{\mu}_\mathbf{x}) \tag{4}$$

2. Experiment on the Kalman filter and state variable model

You are tracking a target moving along a straight line. The state vector consists of *the position* and *the velocity* of the target. The sampling interval is $T = 1$. The system evolves with constant velocity and the state equation is:

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} T^2/2 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{w}(k)$$

and the measurement equation is

$$y(k) = [1 \quad 0]\mathbf{x}(k) + v(k)$$

Do the tracking over 200 time steps using the following assumptions:

- initial "true" state $\mathbf{x}(0) = [0.5 \quad 10.0]^T$
- initial state estimate $\hat{\mathbf{x}}(0|0) = [0.5 \quad 11.0]^T$
- variance of the state noise $\sigma_w^2 = 9.0$
- variance of the measurement noise $\sigma_v^2 = 1.0$
- all noises are zero mean, white, Gaussian

The initial state covariance is

$$\mathbf{P}(0|0) = \begin{bmatrix} \sigma_v^2 & \sigma_v^2/T \\ \sigma_v^2/T & 2\sigma_v^2/T^2 \end{bmatrix}$$

where $T = 1$. In the simulation each new "true state" is generated by

$$\mathbf{x}(k) = \mathbf{F}\mathbf{x}(k-1) + \mathbf{G}\mathbf{w}(k)$$

and the new measurement

$$y(k) = \mathbf{H}\mathbf{x}(k) + v(k).$$

As an example, for the first step they are generated by $\mathbf{x}(1) = \mathbf{F}\mathbf{x}(0) + \mathbf{G}\mathbf{w}(1)$ and $y(1) = \mathbf{H}\mathbf{x}(1) + v(1)$.

Requirements:

Plot the tracking results as follows:

a) the position and the velocity of the target and the corresponding tracking result, in the same plot (position on the horizontal axis, velocity on the vertical axis).

b) the predicted and the estimated velocity error variances as a function of time, on the same plot

c) the predicted and the estimated position error variances as a function of time, on the same plot

d) the Kalman gains as a function of time

e) in your answer, include also the Matlab codes.

3. DoA estimation experiment

Write a Matlab program implementing the following DoA (Direction of Arrival) estimation methods: the classical beamformer, the MVDR (Capon) beamformer and the MUSIC algorithm. Test your function by using a 6-element uniform linear antenna array with $\lambda/2$ inter-element spacing. Consider two mutually uncorrelated QPSK signal sources in additive white Gaussian noise. The signal-to-noise ratio is assumed to be 20 dB. The signals arrive from directions:
(a) 38 and 93 degrees
(b) 88 and 93 degrees.
A total of 512 snapshots are collected. Estimate the DoA's by using the three methods mentioned above. Compare the results from 25 independent experiments, by plotting the spatial spectrum estimates in dB. How do they compare in terms of angular resolution? In your simulation assume the number of signals to be known. Repeat the same simulations at SNR 2 dB. How did SNR impact the results?

In your answer include the Matlab codes as well.

4. BONUS: DOA estimation on a real-world data

Apply the either the MUSIC method to the real-world data in the file *submarine.mat*, which can be found at the course MyCourses page under Assignments. These data are underwater measurements collected by the Swedish Defence Agency in the Baltic Sea. The 6-element array of hydrophones used in the experiment can be assumed to be a ULA with inter-spacing element equal to 0.9m. The wavelength of the signals is approximately 5.32m. Can you find the submarine(s)?