PHYS-C0252 - Quantum Mechanics Part 2
Sections 2-3

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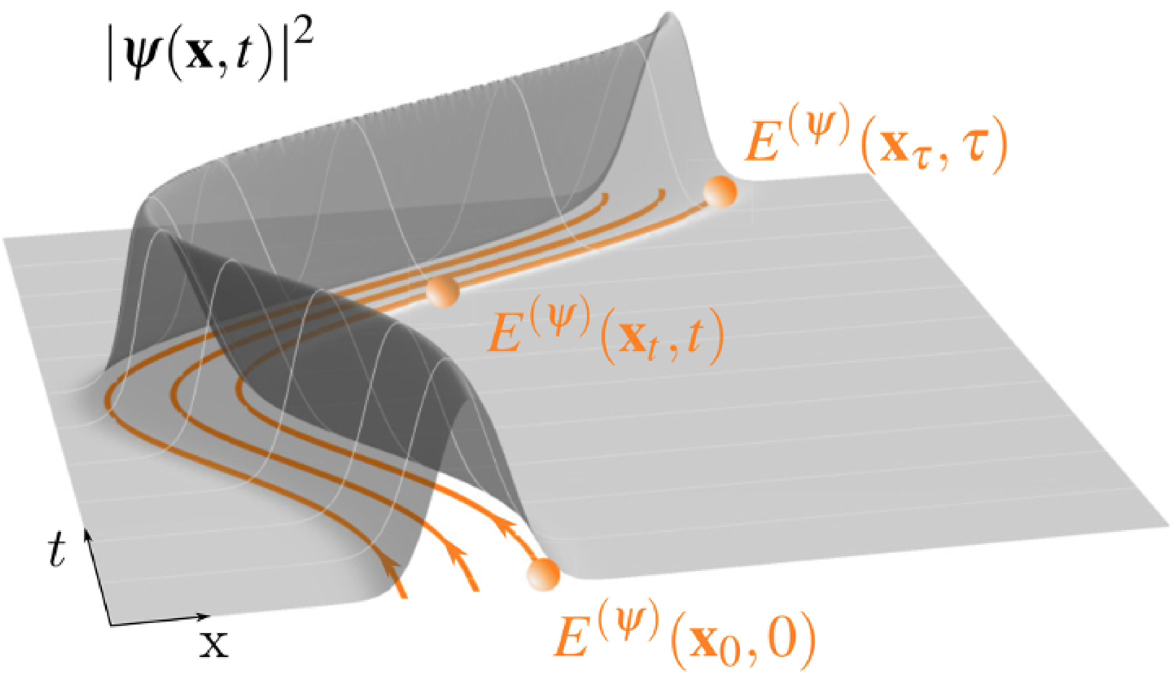

2. Free Particles and Plane Waves

- Consider a free particle that does not experience any external potential in space. Going back to the symbolic notation, let us define the position and momentum (right) eigenvectors as

$$\hat{q}|x\rangle = x|x\rangle; \quad \hat{p}|p\rangle = p|p\rangle$$

and correspondingly the left eigenvectors

$$\langle x|\hat{q} = \langle x|x; \quad \langle p|\hat{p} = \langle p|p$$

The left and right eigenvectors are orthonormal:

$$\langle x'|x\rangle = \delta(x' - x); \quad \langle p'|p\rangle = \delta(p' - p)$$

and they form complete orthonormal sets that can be inserted between any states when necessary:

$$\int dx|x\rangle\langle x| = 1; \quad \int dp|p\rangle\langle p| = 1$$

Now we can define the *momentum operator in the position basis* as (can be derived by inserting *p* basis and partial integration)

$$\langle x'|\hat{p}|x\rangle = \delta(x'-x)\frac{\hbar}{\imath}\frac{d}{dx}$$

and the *position operator in the momentum basis* as

$$\langle p'|\hat{q}|p\rangle = -\delta(p'-p)\frac{\hbar}{\imath}\frac{d}{dp}$$

Now we can solve for the *momentum eigenstate in the position basis* $\psi_p(x) = \langle x|p\rangle$ from the DE

$$\langle x|\hat{p}|p\rangle = \frac{\hbar}{\imath}\frac{d}{dx}\langle x|p\rangle = p\langle x|p\rangle$$

This equation comes about from the fact that

$$\langle x|\hat{p}|p\rangle = \int dx'\langle x|p|x'\rangle\langle x'|p\rangle = \int dx'\delta(x-x')\frac{\hbar}{\imath}\frac{d}{dx'}\langle x'|p\rangle$$

$$= \frac{\hbar}{\imath}\frac{d}{dx}\langle x|p\rangle$$

Thus we have the differential equation

$$\frac{\hbar}{\imath}\frac{d}{dx}\langle x|p\rangle = p\langle x|p\rangle$$

The solution to this DE is a simple plane wave

$$\langle x|p\rangle = \sqrt{\frac{1}{2\pi\hbar}}\; e^{\imath px/\hbar}$$

where $p = \hbar k$. The corresponding *momentum eigenstate in the momentum basis* is

$$\psi_p(p') = \langle p'|p\rangle = \delta(p - p')$$

The normalized *position eigenstate in the momentum basis* can be obtained from

$$\langle p|x\rangle = \langle x|p\rangle^\dagger = \sqrt{\frac{1}{2\pi\hbar}}\, e^{-\imath px/\hbar}$$

and the normalized *position eigenstate in the position basis* is (of course)

$$\psi_x(x) = \langle x'|x\rangle = \delta(x - x')$$

Note that these results can be immediately obtained from the Schrödinger equation (in the position basis)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = 0$$

- Let us next look at the time dependence of the plane waves in the position basis. The time-dependent Schrödinger equation is

$$\imath\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{d^2\Psi(x,t)}{dx^2}$$

We look for separable solutions of the form

$$\Psi(x,t) = f(t)\psi(x)$$

This gives

$$\frac{\imath\hbar}{f(t)}\frac{\partial f(t)}{\partial t} = -\frac{1}{\psi(x)}\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}$$

Both sides must be constant (why?) and thus

$$f(t) = e^{-\imath Et/\hbar}$$

The solutions

$$\Psi(x,t) = e^{-\imath Et/\hbar}\psi_E(x)$$

are called *stationary states* because

$$\partial_t|\Psi(x,t)|^2 = 0$$

The general solution of the time-dependent wave equation can be obtained from the superposition principle

$$\Psi(x,t) = \sum_E c_E e^{-\imath Et/\hbar} e^{\imath px/\hbar}$$

for the case of free particles (plane waves)

- An important generalization of this result is that if we use the completeness of the energy eigenfunctions to expand in terms of them $\psi_E(x)$, then in general

$$\Psi(x,t) = \sum_E c_E e^{-\imath Et/\hbar} \psi_E(x)$$

3. Particle in a Periodic Box

- Consider a free particle that traverses a box of finite (linear) size *L*, but no external *V*(*x*) (periodic boundary conditions) s.t. $\psi_k(x) = \psi_k(x+L)$

  The solution is that of the free particle, but now

$$k = \frac{2\pi n}{L}, \ n \in \mathbb{Z}$$

  The completeness relations now read as

$$\int_0^L dx|x\rangle\langle x| = 1; \ \sum_{n=-\infty}^{\infty} |k\rangle\langle k| = 1$$

The wavevector eigenstate in the position basis is then simply

$$\psi_k(x) = \langle x|k\rangle = \langle k|x\rangle^\dagger = \frac{1}{L^{1/2}}\ e^{\imath kx}$$