# Problem Set 5

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# Exercise 1 - PS5

Let $f(x, y, z) = x + 2z$ be a function defined over $\mathbb{R}^3$. In addition, let $g_1(x, y, z) = x + y + z$ and $g_2(x, y, z) = x^2 + y^2 + z$ be two additional functions defined over $\mathbb{R}^3$. Consider the constrained maximization problem:

$$\begin{aligned} \max_{x,y,z} \quad & f(x, y, z) \\ \text{s.t.} \quad & g_1(x, y, z) = 1 \\ & g_2(x, y, z) = \frac{7}{4}. \end{aligned}$$

(a) Check if the NDCQ is satisfied.

(b) Solve the maximization problem.

# Exercise 1 - Solution

$$L = x + 2z - \lambda_1 (x + y + z - 1) - \lambda_2 \left( x^2 + y^2 + z - \frac{7}{4} \right).$$

The rank of the Jacobian of the two constraint functions is equal to 2 unless $x = y = \frac{1}{2}$, in which case the rank is equal to 1. However, there is no point in the constraint set in which $x = y = \frac{1}{2}$. Therefore, the NDCQ is satisfied.

The first order conditions are

$$1 - \lambda_1 - 2\lambda_2 x = 0 \tag{1}$$

$$-\lambda_1 - 2\lambda_2 y = 0 \tag{2}$$

$$2 - \lambda_1 - \lambda_2 = 0 \tag{3}$$

$$x + y + z = 1 \tag{4}$$

$$x^2 + y^2 + z = \frac{7}{4}. \tag{5}$$

# Exercise 1 - Solution

From (3) we obtain $\lambda_2 = 2 - \lambda_1$, which inserted into (2) gives $\lambda_1(2y-1) = 4y$. This equation implies that $y \neq \frac{1}{2}$, so $\lambda_1 = \frac{4y}{2y-1}$. Inserting this into (1) with $\lambda_2 = 2 - \lambda_1$ eventually yields $y = 2x - \frac{1}{2}$. Inserting the last expression into the two constraints yields $3x + z = \frac{3}{2}$ and $5x^2 - 2x + z = \frac{3}{2}$. These two equations combined give $z = \frac{3}{2} - 3x$ and $5x(x-1) = 0$. Thus, $x = 0$ or $x = 1$. If $x = 0$, we obtain $y = -\frac{1}{2}$ , $z = \frac{3}{2}$, and $\lambda_1 = \lambda_2 = 1$. If $x = 1$, we get $y = \frac{3}{2}$, $z = -\frac{3}{2}$, $\lambda_1 = 3$, and $\lambda_2 = -1$. Evaluating $f$ at these two critical points we get $f(0, -1/2, 3/2) = 3$ and $f(1, 3/2, -3/2) = -2$. Thus the only candidate for a solution is $(0, -1/2, 3/2)$. Given $\lambda_1 = \lambda_2 = 1$, the Lagrangian is a concave function in $(x, y, z)$. So we can conclude that $(0, -1/2, 3/2)$ is the solution to this maximization problem.

# Exercise 2

Let $u(x, y) = x^a y^{1-a}$ be a Cobb-Douglas utility function, where $0 < a < 1$. Use the Kuhn-Tucker formulation to solve the following utility maximization problem:

$$\begin{aligned} \max_{x,y} \quad & u(x, y) \\ \text{s.t.} \quad & p_x x + p_y y \leq w \\ & x \geq 0 \\ & y \geq 0, \end{aligned}$$

where $p_x > 0$ and $p_y > 0$ are commodity prices, and $w > 0$ is income.

# Exercise 2 - Solution

First of all, a solution exists by Weierstrass's Theorem. The Kuhn-Tucker Lagrangian is

$$\tilde{L} = x^a y^{1-a} - \mu \left( p_x x + p_y y - w \right).$$

The first order conditions are

$$a x^{a-1} y^{1-a} - \mu p_x \leq 0 \tag{6}$$

$$(1-a) x^a y^{-a} - \mu p_y \leq 0 \tag{7}$$

$$x \left( a x^{a-1} y^{1-a} - \mu p_x \right) = 0 \tag{8}$$

$$y \left( (1-a) x^a y^{-a} - \mu p_y \right) = 0 \tag{9}$$

$$p_x x + p_y y \leq w \tag{10}$$

$$\mu \left( p_x x + p_y y - w \right) = 0 \tag{11}$$

$$x \geq 0, y \geq 0, \mu \geq 0. \tag{12}$$

Notice that a solution must be such that $x > 0$ and $y > 0$. If not, total utility is zero. But then it would be feasible to attain strictly positive utility by choosing positive quantities of both commodities while satisfying the budget constraint.

# Exercise 2 - Solution

Since we must have $x > 0$ and $y > 0$, (8) and (9) imply that both (6) and (7) hold with equality and, consequently, $\mu > 0$, which in turn implies that the budget constraint is binding via (11). Combining (6) and (7), we get $y = \frac{p_1}{p_2}\frac{(1-a)}{a}x$. Combining the latter expression with the budget constraint we obtain $x = \frac{aw}{p_x}$ and $y = \frac{(1-a)w}{p_y}$, which is the unique solution.

# Exercise 3

Consider the following maximization problem:

$$\begin{aligned} \max_{x,y} \quad & x^2 - y^2 \\ \text{s.t.} \quad & x^2 + y^2 \leq 9 \\ & x \geq -2 \\ & y \geq -1. \end{aligned}$$

(a) Check if the NDCQ is satisfied.

(b) Form the Lagrangian and solve the problem.

# Exercise 3 - Solution

(a) we check 8 cases of different values of $\lambda_1$, $\lambda_2$, and $\lambda_3$ (whether the 3 constraints are active or not)

Case 1: $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = 0$

The rank of Dg(x,y) is 0. NDCQ holds.

Case 2: $\lambda_1 > 0$, $\lambda_2 = 0$, and $\lambda_3 = 0$

Dg(x,y) = (2x 2y). NDCQ holds because $x^2 + y^2 = 9$.

Case 3: $\lambda_1 = 0$, $\lambda_2 > 0$, and $\lambda_3 = 0$

Dg(x,y) = (-1 0). NDCQ holds.

Case 4: $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 = 0$

Dg(x,y) = $\begin{pmatrix} 2x & 2y \\ -1 & 0 \end{pmatrix}$. NDCQ holds because $x^2 + y^2 = 9$ and $x = -2$.

# Exercise 3 - Solution

(a) we check 8 cases of different values of $\lambda_1$, $\lambda_2$, and $\lambda_3$ (whether the 3 constraints are active or not)
Case 5: $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 > 0$
Dg(x,y) = (0 -1). NDCQ holds.
Case 6: $\lambda_1 > 0$, $\lambda_2 = 0$, and $\lambda_3 > 0$
Dg(x,y) = $\begin{pmatrix} 2x & 2y \\ 0 & -1 \end{pmatrix}$. NDCQ holds because $x^2 + y^2 = 9$ and $y = -2$.
Case 7: $\lambda_1 = 0$, $\lambda_2 > 0$, and $\lambda_3 > 0$
Dg(x,y) = $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. NDCQ holds.
Case 8: $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 > 0$
$x^2 + y^2 = 5 \neq 9$, point (-2,-1) violates the constraint set in this case. NDCQ holds.

# Exercise 3 - Solution

(b) The Lagrangian is

$$L = x^2 - y^2 - \lambda_1(x^2 + y^2 - 9) - \lambda_2(-2 - x) - \lambda_3(-1 - y)$$

The first order conditions are

$$2x - 2x * \lambda_1 + \lambda_2 = 0 \tag{13}$$

$$-2y - 2y * \lambda_1 + \lambda_3 = 0 \tag{14}$$

$$\lambda_1(x^2 + y^2 - 9) = 0 \tag{15}$$

$$\lambda_2(-2 - x) = 0 \tag{16}$$

$$\lambda_3(-1 - y) = 0 \tag{17}$$

$$\lambda_1 \geq 0, \lambda_1 \geq 0, \lambda_3 \geq 0 \tag{18}$$

$$x^2 + y^2 - 9 \leq 0, -2 - x \leq 0, -1 - y \leq 0. \tag{19}$$

# Exercise 3 - Solution

(b) We can check 8 cases of different values of $\lambda_1$, $\lambda_2$, and $\lambda_3$ (whether the 3 constraints are active or not). Alternatively, it might be faster to go through 4 separate cases: 1) $x = -2$ and $y = -1$; 2) $x = -2$ and $y > -1$; 3) $x > -2$ and $y = -1$ and 4) $x > -2$ and $y > -1$.

Case 1: $x = -2$ and $y = -1$

We have $\lambda_1 = 0$, $\lambda_2 = 4$, $\lambda_3 = -2$.

Eliminate this case since $\lambda_3 < 0$

Case 2: $x = -2$ and $y > -1$

We have $\lambda_2 > 0$, and $\lambda_3 = 0$.

If $\lambda_1 = 0$, then y=0, $\lambda_2 = 4$, (-2,0) is a candidate.

If $\lambda_1 > 0$, then $y^2 = 5, (-2, \sqrt{5})$ and $(-2, -\sqrt{5})$ are candidates.

# Exercise 3 - Solution

Case 3: $x > -2$ and $y = -1$
We have $\lambda_2 = 0$, and $\lambda_3 > 0$.
If $\lambda_1 = 0$, then x=0, $\lambda_3 = -2$. Eliminate this case since $\lambda_3 < 0$.
If $\lambda_1 > 0$, then $x^2 = 8, (\sqrt{8}, -1)$ and $(-\sqrt{8}, -1)$ are candidates.
Case 4: $x > -2$ and $y > -1$
We have $\lambda_2 = 0$, and $\lambda_3 = 0$.
If $\lambda_1 = 0$, then $x = y = 0, (0, 0)$ is a candidate.
If $\lambda_1 > 0$, then $2x(1 - \lambda_1) = 0$ and $-2y(1 + \lambda_1) = 0$. Therefore, $\lambda_1 = 1$ and $y = 0$. Point $(3, 0)$ is a candidate.
Checking all candidates, the unique solution is $(3, 0)$.

# Exercise 4

Let $f(x,y) = x$ and $g(x,y) = y^5 - x^4$ be two functions defined over $\mathbb{R}^2$. Consider the following constrained problem:

$$\begin{aligned} \min_{x,y} \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) = 0. \end{aligned}$$

(a) Form the Lagrangian and show that it does not have any critical points.

(b) Find all the points where the NDCQ fails.

(c) Solve the constrained minimization problem.

# Exercise 4 - Solution

(a) The Lagrangian is

$$L = x - \mu(y^5 - x^4).$$

The first order conditions are

$$1 + 4\mu x^3 = 0 \tag{20}$$

$$5\mu y^4 = 0 \tag{21}$$

$$y^5 - x^4 = 0. \tag{22}$$

From (21) we have either $\mu = 0$ or $y = 0$. If $\mu = 0$, (20) cannot hold. If $y = 0$, then $x = 0$ by (22) and, consequently, (20) cannot hold. Thus the system (20)-(22) does not admit any solution.

(b) The NDCQ fails when $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$. That is, $5y^4 = 4x^3 = 0$, which holds only at the point $(x, y) = (0, 0)$. Notice that $(0, 0)$ belongs to the constraint set.

# Exercise 4 - Solution

(c) The point where the NDCQ fails is just a solution candidate but not the true solution. We do not have a solution for the constrained minimization problem because x can go to negative infinity.

# Exercise 5

Consider the following constrained minimization problem:

$$\begin{aligned} \min_{x,y,z} \quad & f(x,y,z) = x^2 + y^2 + z^2 \\ \text{s.t.} \quad & g_1(x,y,z) = x + 2y + z = 30 \\ & g_2(x,y,z) = 2x - y - 3z = 10. \end{aligned}$$

(a) Find the unique point that satisfies the first order conditions for optimality.

(b) Use second order conditions to show that the point you found in (a) is a local minimizer of $f$ subject to $g_1(x,y,z) = 30$ and $g_2(x,y,z) = 10$.

# Exercise 5 - Solution

The Lagrangian is

$$L = x^2 + y^2 + z^2 - \mu_1(x + 2y + z - 30) - \mu_2(2x - y - 3z - 10).$$

The first order conditions are

$$\begin{aligned} 2x - \mu_1 - 2\mu_2 &= 0 \\ 2y - 2\mu_1 + \mu_2 &= 0 \\ 2z - \mu_1 + 3\mu_2 &= 0 \\ x + 2y + z - 30 &= 0 \\ 2x - y - 3z - 10 &= 0. \end{aligned}$$

The above is a system of 5 linear equations in 5 unknowns. The unique solution is $(x, y, z, \mu_1, \mu_2) = (10, 10, 0, 12, 4)$.

# Exercise 5 - Solution

The bordered Hessian is:

$$H = \begin{pmatrix} 0 & 0 & \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ 0 & 0 & \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \\ \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} & L''_{xx} & L''_{xy} & L''_{xz} \\ \frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y} & L''_{yx} & L''_{yy} & L''_{yz} \\ \frac{\partial g_1}{\partial z} & \frac{\partial g_2}{\partial z} & L''_{zx} & L''_{zy} & L''_{zz} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 & -3 \\ 1 & 2 & 2 & 0 & 0 \\ 2 & -1 & 0 & 2 & 0 \\ 1 & -3 & 0 & 0 & 2 \end{pmatrix}.$$

In this problem, we have $n = 3$ variables and $m = 2$ constraints. We have to check the sign of the last $n - m$ leading principal minors. That is, we only need to check the sign of the determinant of the whole matrix $H$. This determinant is equal to 150. Since $(-1)^m = 1$ and $(-1)^n = -1$, and since $\det(H) > 0$, we conclude that $H$ is positive definite on the constraint set. Therefore, $(10, 10, 0)$ is a strict local minimizer of $f$ over the given constraint set.