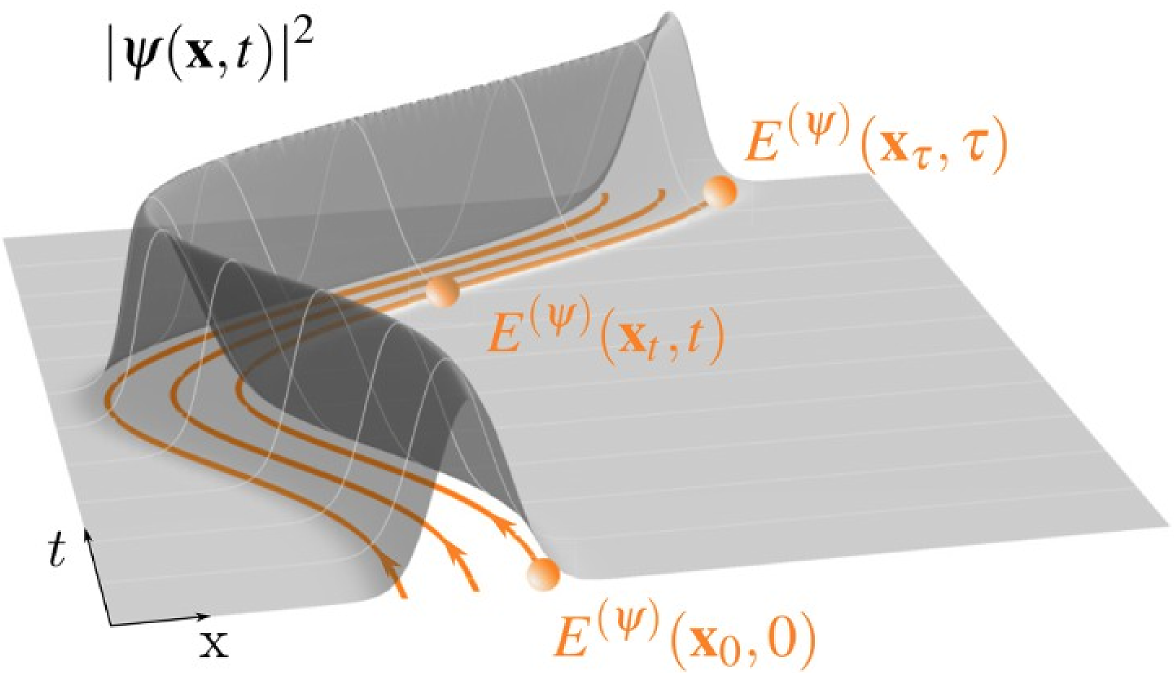
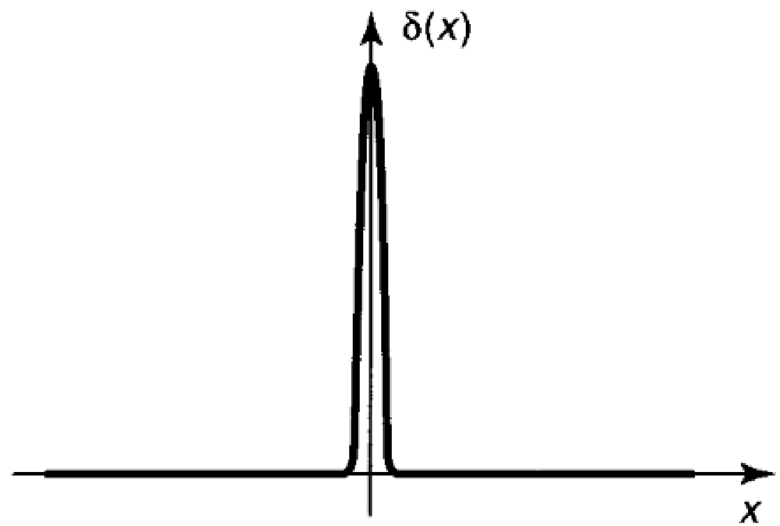
PHYS-C0252 - Quantum Mechanics Part 2
Section 4.4

Tapio.Ala-Nissila@aalto.fi



4.4 Delta-Function Potential

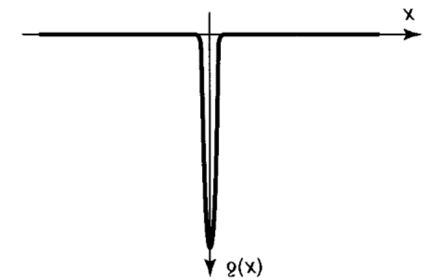
- The last case to consider here is that of a (non-analytic) delta-function potential at $x = 0$:



$$\delta(x) = \left\{ \begin{array}{ll} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{array} \right\}, \text{ with } \int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$

If the actual *potential well* has strength -α, the SE reads

$$-\frac{\hbar^2}{2m}\psi''(x) - \alpha\delta(x)\psi(x) = E\psi(x)$$



- The delta-function potential well supports both *bound* ($E < 0$) and *scattering* ($E > 0$) states

For *bound* states when $x < 0$:

$$\psi''(x) = \kappa^2\psi(x),\ \ \kappa = \frac{\sqrt{-2mE}}{\hbar}$$

where the solution is

$$\psi(x) = Ae^{-\kappa x} + Be^{\kappa x} = Be^{\kappa x}$$

Correspondingly, in the other half of the plane

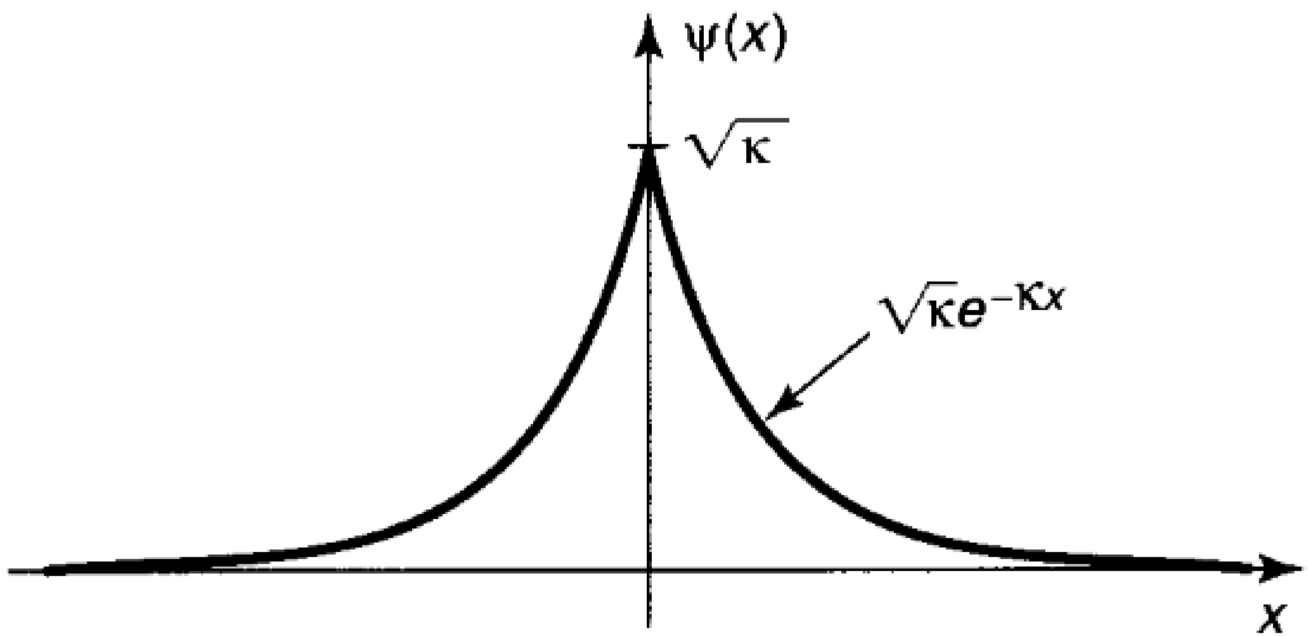
$$\psi(x) = Fe^{-\kappa x}$$

From the previous examples we have learned that

$$\begin{cases} 1. \ \psi \text{ is always continuous, and} \\ 2. \ d\psi/dx \text{ is continuous except at points} \\ \quad \text{where the potential is infinite.} \end{cases}$$

The first BC is easily satisfied with *F* = *B*

$$\psi(x) = \begin{cases} Be^{\kappa x}, & (x \leq 0), \\ Be^{-\kappa x}, & (x \geq 0). \end{cases}$$



Bound state wave function for $E < 0$

The contradiction here is that the delta-function potential does not enter the result. To examine this we must look at the derivative at $x = 0$:

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{+\epsilon}\frac{d^2\psi}{dx^2}\,dx+\int_{-\epsilon}^{+\epsilon}V(x)\psi(x)\,dx=E\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx$$

L.h.s. term gives the jump in the derivative as

$$\Delta\left(\frac{d\psi}{dx}\right)=\frac{2m}{\hbar^2}\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon}V(x)\psi(x)\,dx$$

and due to the delta function

$$\Delta\left(\frac{d\psi}{dx}\right)=-\frac{2m\alpha}{\hbar^2}\psi(0)$$

Here

$$\begin{cases} d\psi/dx = -B\kappa e^{-\kappa x}, \text{ for } (x > 0), & \text{so } d\psi/dx\big|_{+} = -B\kappa, \\ d\psi/dx = +B\kappa e^{+\kappa x}, \text{ for } (x < 0), & \text{so } d\psi/dx\big|_{-} = +B\kappa, \end{cases}$$

and thus

$$E = -\frac{\hbar^2\kappa^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}$$

Normalization gives

$$\int_{-\infty}^{+\infty} |\psi(x)|^2\, dx = 2|B|^2 \int_0^{\infty} e^{-2\kappa x}\, dx = \frac{|B|^2}{\kappa} = 1$$

Thus the main result is that the delta-function potential can support *one and only one bound state*

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}; \qquad E = -\frac{m\alpha^2}{2\hbar^2}.$$

For the scattering states $E > 0$

$$\psi''(x) = -k^2\psi(x), \;\; k = \frac{\sqrt{2mE}}{\hbar}$$

and the general solution for $x < 0$ is

$$\psi(x) = Ae^{\imath kx} + Be^{-\imath kx}$$

and for $x > 0$

$$\psi(x) = Fe^{\imath kx} + Ge^{-\imath kx}$$

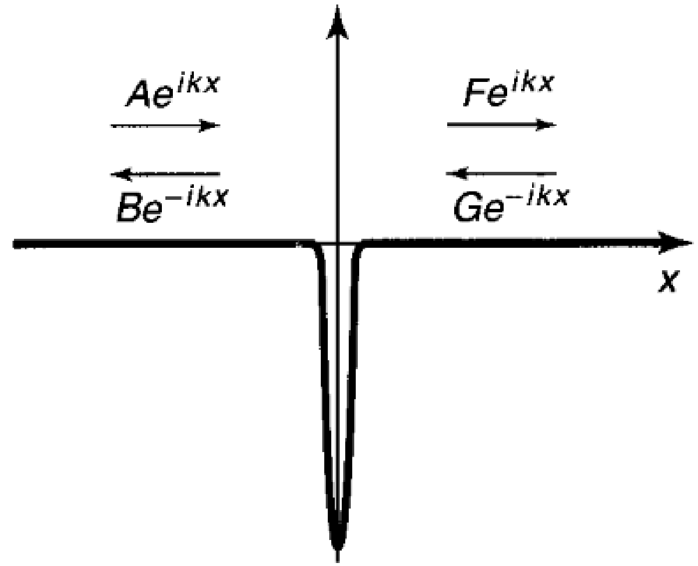
- Continuity requires that $F + G = A + B$ and

$$\begin{cases} d\psi/dx = ik\left(Fe^{ikx} - Ge^{-ikx}\right), \text{ for } (x > 0), & \text{so } d\psi/dx\big|_+ = ik(F - G) \\ d\psi/dx = ik\left(Ae^{ikx} - Be^{-ikx}\right), \text{ for } (x < 0), & \text{so } d\psi/dx\big|_- = ik(A - B). \end{cases}$$

which gives the jump

$$\Delta\psi'|_{x=0} = \imath k(F - G - A + B) = -\frac{2m\alpha}{\hbar^2}\psi(0)$$

- Because the plane waves are *not normalizable in free space*, these equations don't have unique solutions
- We have to assume a wave coming from a given direction, e.g. from left to right

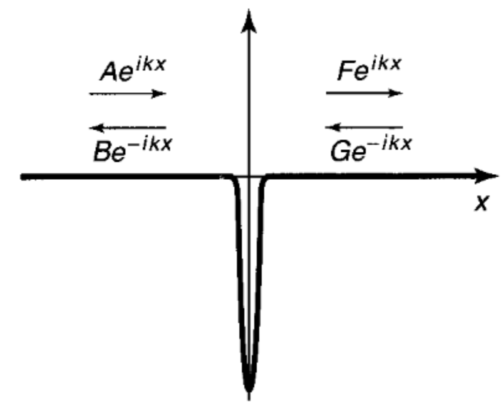


Assuming G = 0 gives B and F as a function of A

$$B = \frac{i\beta}{1 - i\beta} A, \quad F = \frac{1}{1 - i\beta} A.$$

$$\beta \equiv \frac{m\alpha}{\hbar^2 k}$$

These can now be used to refine the corresponding reflection and transmission coefficients $R + T = 1$:



$$R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}; \; T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2}$$