ELEC-E4130

Lecture 20: Rectangular Waveguides + circular Waveguides - Ch. 10

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ELEC-E4130 / Taylor

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Conductor and dielectric loss

Waveguide Loss Setup: Taylor Expansion

In a lossy waveguide, as power decays according to the factor $\boxed{e^{-2\alpha z}}$

$$\frac{P_0 - P_\ell}{P_0} = e^{-2\alpha\ell}$$

$$1 - \frac{P_\ell}{P_0} = e^{-2\alpha\ell}$$

$$\frac{P_\ell}{P_0} = 1 - e^{-2\alpha\ell} \approx 2\alpha\ell$$

when loss is small

Define the power dissipated after traveling length in waveguide as P_ℓ and the incident power as P_0, the ratio between the power observed at the unit length away and the original incident power is

Let ℓ = 1

The attenuation constant per unit length can thus be determined by:

$$\boxed{\alpha = \frac{P_1}{2P_0} \quad (Np/m)}$$

In general, power dissipation in a non-ideal waveguide may be attributed to both conductor loss and dielectric loss

conductor loss

dielectric loss

$$\alpha = \frac{P_{\ell c} + P_{\ell d}}{2P_0} = \alpha_c + \alpha_d$$

attenuation constant due to conductor loss

attenuation constant due to dielectric loss

Waveguide Loss Setup: Transmission lines

Forward traveling waves

$$V(z) = V_0^+ e^{-\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z}$$

$$\gamma = \alpha + j\beta = \sqrt{(\underbrace{R}_{\text{Conductor loss}} + j\omega L)(\underbrace{G}_{\text{Dielectric loss}} + j\omega C)}$$

$$V(z) = V_0^+ e^{-(\alpha + j\beta) z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-(\alpha + j\beta) z}$$

Time–average power propagated long the line at any z

$$P(z) = \frac{1}{2} \mathbb{R}\mathrm{e}\{V(z) I^*(z)\} = \frac{(V_0^+)^2}{2|Z_0|^2} R_0 e^{-2\alpha z}$$

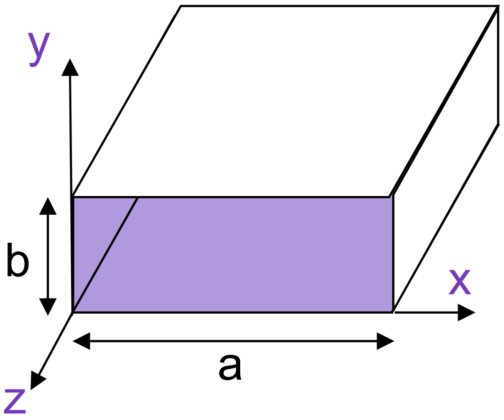
Rate of decrease in P(z) along z equals the time-average power loss along z

$$\underbrace{-\frac{\partial P(z)}{\partial z}}_{\text{Drop from initial power per unit length}} = \underbrace{P_L(z)}_{\text{Loss per unit length}} = -\big(-2\alpha P(z)\big)$$

The attenuation constant per unit length can thus be determined by:

$$\boxed{\alpha = \frac{P_L(z)}{2P(z)} \quad (\mathrm{Np/m})}$$

TE_{10} mode Equations



Surface Current Density

Wall (x=0): $\mathbf{J_s} = -\mathbf{a_y} A_{10} e^{-j\beta z}$ Wall (x=a): $\mathbf{J_s} = -\mathbf{a_y} A_{10} e^{-j\beta z}$

Floor (y=0): $\mathbf{J_s} = \left(+\mathbf{a_x} A_{10} \cos\left(\frac{\pi x}{a}\right) - \mathbf{a_z} A_{10} \frac{j\beta_{10} a}{\pi} \sin\left(\frac{\pi x}{a}\right) \right) e^{-j\beta z}$

Ceiling (y=b): $\mathbf{J_s} = \left(-\mathbf{a_x} A_{10} \cos\left(\frac{\pi x}{a}\right) + \mathbf{a_z} A_{10} \frac{j\beta_{10} a}{\pi} \sin\left(\frac{\pi x}{a}\right) \right) e^{-j\beta z}$

Fields

$$E_y = \frac{-j\omega\mu a}{\pi} B_{10} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z}$$

$$H_x = \frac{j\beta a}{\pi} B_{10} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z}$$

$$H_z = B_{10} \cos\left(\frac{\pi x}{a}\right) e^{-j\beta z}$$

$$E_z = E_x = H_y = 0$$

Phase constant

$$\beta_{10} = \sqrt{k^2 - k_{c,10}^2} = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2} = k\sqrt{1 - \left(\frac{\lambda}{2a}\right)}$$

Cutoff wavelength

$$\lambda_{c,10} = \frac{2\pi}{k_c} = \frac{2\pi}{\pi/a} = 2a$$

Cutoff frequency

$$f_{c,10} = \frac{c/\sqrt{\varepsilon_r}}{\lambda_{c,10}} = \frac{c/\sqrt{\varepsilon_r}}{2a} = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

Average power flow in the TE_{10} mode

Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}^* \quad \rightarrow \quad \mathbf{S}_{\mathrm{AVE}} = \frac{1}{2}\mathbb{R}\mathrm{e}\{\mathbf{E} \times \mathbf{H}^*\}$$

Power subtended by an area

$$\mathrm{P} = \int_{\mathrm{A}} \mathbf{S} \cdot \mathbf{dA} \quad \rightarrow \quad \mathrm{P}_{\mathrm{AVE}} = \frac{1}{2}\mathbb{R}\mathrm{e}\left\{\int_{\mathrm{A}} \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{dA}\right\}$$

$$\boxed{\begin{aligned} \mathrm{E_y} &= \frac{-\mathrm{j}\omega\mu\mathrm{a}}{\pi}\mathrm{B}_{10}\sin\left(\frac{\pi\mathrm{x}}{\mathrm{a}}\right)\mathrm{e}^{-\mathrm{j}\beta\mathrm{z}} \\ \mathrm{H_x} &= \frac{\mathrm{j}\beta\mathrm{a}}{\pi}\mathrm{B}_{10}\sin\left(\frac{\pi\mathrm{x}}{\mathrm{a}}\right)\mathrm{e}^{-\mathrm{j}\beta\mathrm{z}} \end{aligned}}$$

Average power flowing in the TE_{10} mode

$$\mathrm{P}_{10} = \frac{1}{2}\mathbb{R}\mathrm{e}\left\{\int_{\mathrm{x}=0}^{\mathrm{a}}\int_{\mathrm{y}=0}^{\mathrm{b}} \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{a_z}\mathrm{dydx}\right\}$$

$$\mathrm{P}_{10} = \frac{1}{2}\mathbb{R}\mathrm{e}\left\{\int_{\mathrm{x}=0}^{\mathrm{a}}\int_{\mathrm{y}=0}^{\mathrm{b}} \mathrm{E_y}\mathrm{H_x^*}\mathrm{dydx}\right\}$$

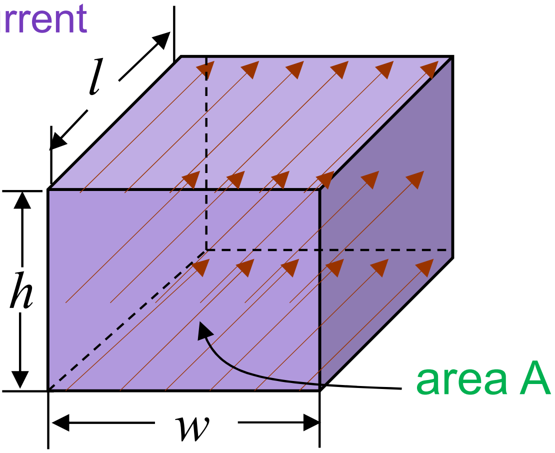
$$\mathrm{P}_{10} = \frac{\omega\mu\mathrm{a}^2}{2\pi^2}\mathbb{R}\mathrm{e}(\beta)|\mathrm{A}_{10}|^2 \underbrace{\int_{\mathrm{x}=0}^{\mathrm{a}}\int_{\mathrm{y}=0}^{\mathrm{b}} \sin^2\left(\frac{\pi\mathrm{x}}{\mathrm{a}}\right)\mathrm{dydx}}_{\frac{\mathrm{ab}}{2}}$$

Average power flowing in the TE_{10} mode

$$\boxed{P_{10} = \frac{\omega\mu a^3 b}{4\pi^2}\mathbb{R}\mathrm{e}(\beta)|A_{10}|^2 = \frac{\omega\mu a^3 b\beta}{4\pi^2}|A_{10}|^2}$$

Current Flow in Good Conductor

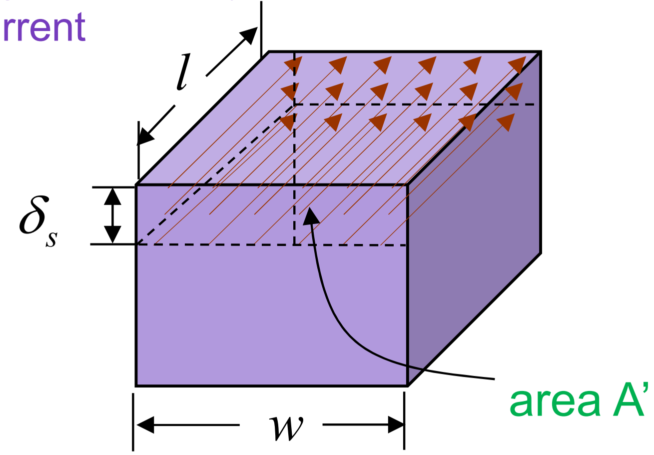
DC current



Current flows inside the conductor uniformly. The resistance of the conductor is given by Ohm's law,

$$R = \frac{1}{\sigma}\frac{l}{A} = \frac{1}{\sigma}\frac{l}{w \cdot h}$$

High Frequency current



Due to the skin effect, current flows within a very thin layer of conductor close to the surface. The resistance of the conductor is thus given by,

$$R = \frac{1}{\sigma}\frac{l}{A'} = \frac{1}{\sigma}\frac{l}{w \cdot \delta_s} = \overbrace{\frac{1}{\sigma\delta_s}}^{R_s}\frac{l}{w}$$

$$\delta_s = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

Surface impedance is thus defined as,

$$\boxed{R_s = \frac{1}{\sigma\delta_s} = \sqrt{\frac{\pi f \mu}{\sigma}}}$$

Conductor Loss

Consider the power dissipated over length ℓ

$$P_\ell = \ell \frac{R_s}{2}\int_A \mathbf{J_s}\cdot\mathbf{J_s^*}dS$$

Area reduces to contour integral due to sheet current

$$P_1 = \frac{P_\ell}{\ell} = \frac{R_s}{2}\int_A |\mathbf{J_s}|^2 ds = \frac{R_s}{2}\oint_C |\mathbf{J_s}|^2 dl$$

Sheet current, 2D integral reduces to 1D contour integral

Consider the power dissipated over length ℓ

Wall (x=0): Wall (x=a): Floor (y=0): Ceiling (y=b):

$$P_1 = \frac{R_s}{2}\int_{y=0}^{b}\left|\mathbf{J_{s,y}}\right|^2 dy + \frac{R_s}{2}\int_{y=0}^{b}\left|\mathbf{J_{s,y}}\right|^2 dy + \frac{R_s}{2}\int_{x=0}^{a}\left[\left|\mathbf{J_{s,x}}\right|^2 + \left|\mathbf{J_{s,z}}\right|^2\right] dx + \frac{R_s}{2}\int_{x=0}^{a}\left[\left|\mathbf{J_{s,x}}\right|^2 + \left|\mathbf{J_{s,z}}\right|^2\right] dx$$

$$P_1 = R_s|A_{10}|^2\left(b + \frac{a}{2} + \frac{\beta^2 a^3}{2\pi^2}\right)$$

Significant algebra

$$\boxed{\alpha_c = \frac{P_1}{2P_0} = \frac{R_s\left(1 + \left(\frac{2b}{a}\right)\left(\frac{f_c}{f}\right)^2\right)}{\eta b\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}}$$

Dielectric Loss

If the waveguide is complete filled in with a homogenous lossy medium, the complex propagation constant is,

$$\gamma = \alpha_d + j\beta = \sqrt{k_c^2 - k_{complex}{}^2}$$

$$= \sqrt{k_c^2 - \omega^2\mu(\varepsilon' - j\varepsilon")}$$

$$= \sqrt{k_c^2 - \omega^2\mu\varepsilon(1 - j\tan\delta)}$$

Define loss tangent as the ratio between the imaginary part and real part of the complex permittivity

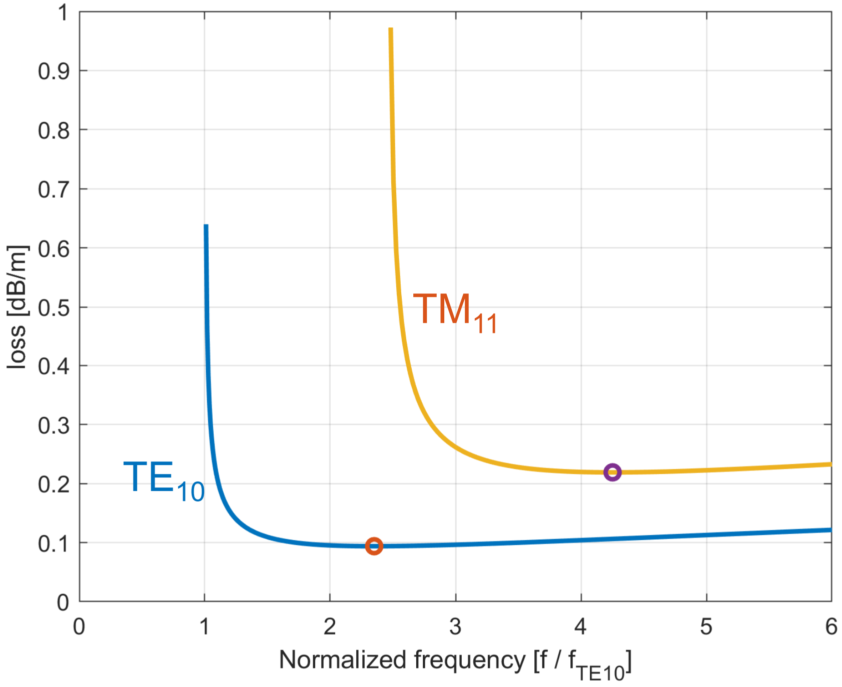
$$\tan\delta = \frac{\varepsilon"}{\varepsilon'}$$

Binomial expansion, **significant algebra** (see your book, page 544)

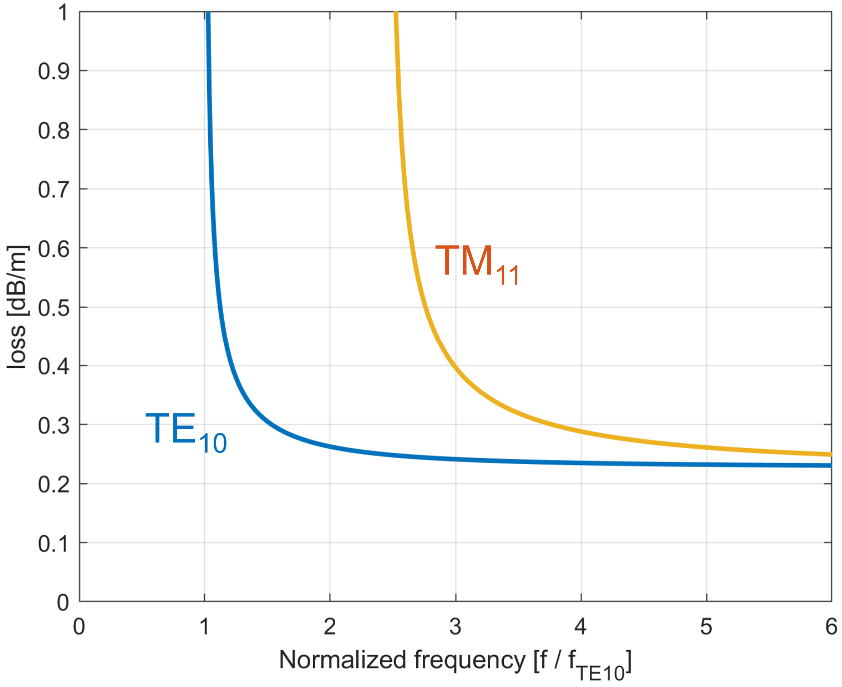
$$\boxed{\alpha_d = \frac{\eta\sigma}{2\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \left(\frac{\varepsilon"}{\varepsilon'}\right)\left(\frac{\pi}{\lambda}\right)\left(\frac{\lambda_g}{\lambda}\right)} \longleftarrow \varepsilon" = \frac{\sigma}{\omega}$$

Waveguide loss

Conductor loss



Dielectric loss



$$\alpha_{c,d}(dB/m) = 8.686\alpha_{c,d}(Np/m)$$

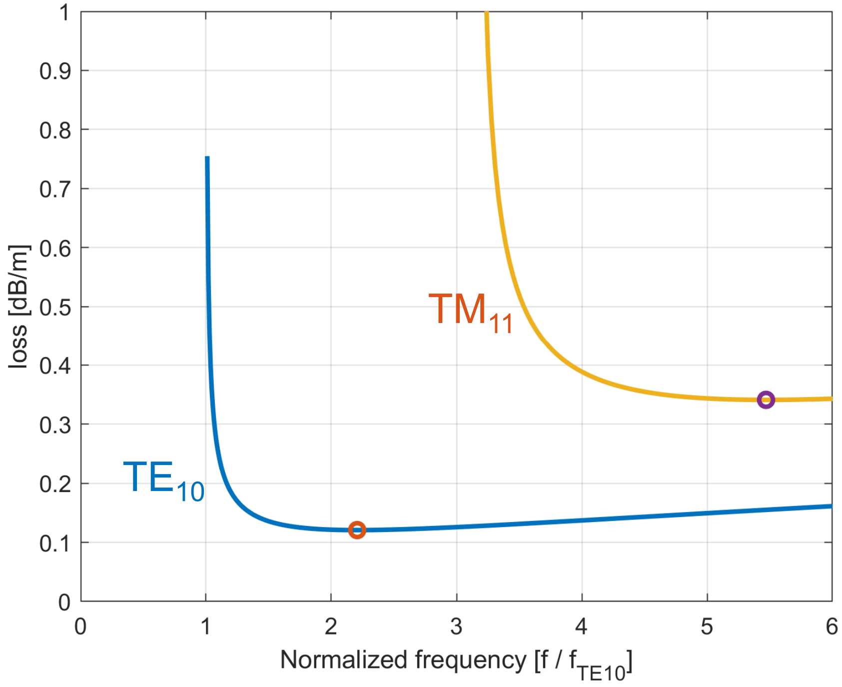
a = 2.29 cm
b = 1.02 cm
$f_c(TE_{10})$ = 6.55 GHz

a = 2.29 cm
b = 0.76 cm
$f_c(TE_{10})$ = 6.55 GHz

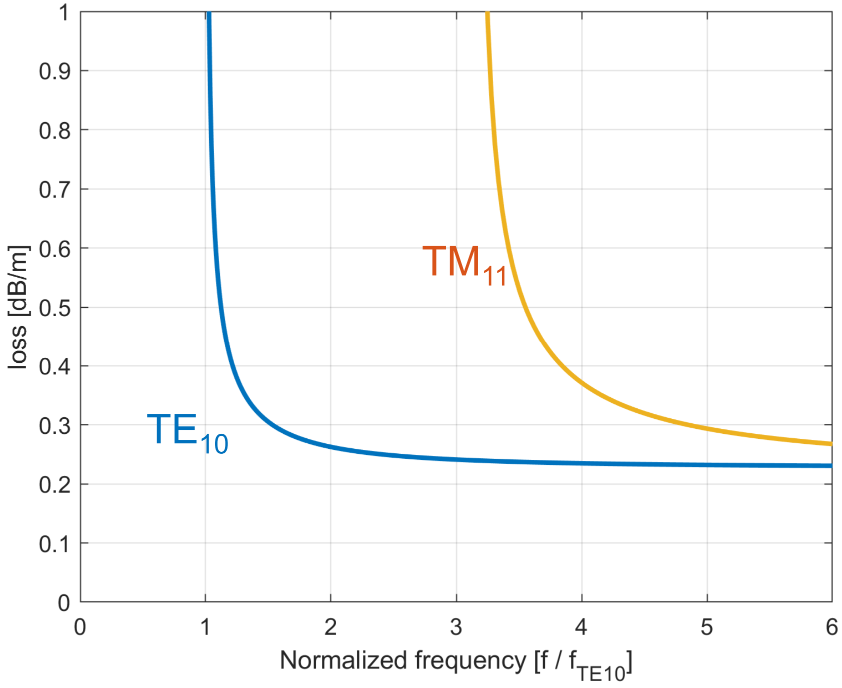
a = 2.29 cm
b = 1.53 cm
$f_c(TE_{10})$ = 6.55 GHz

Waveguide loss

Conductor loss



Dielectric loss



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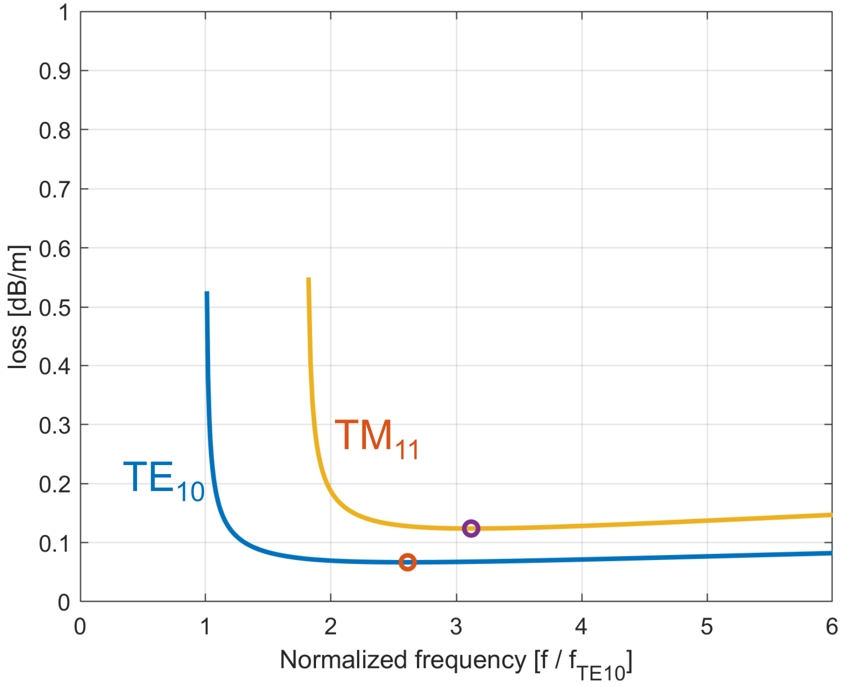
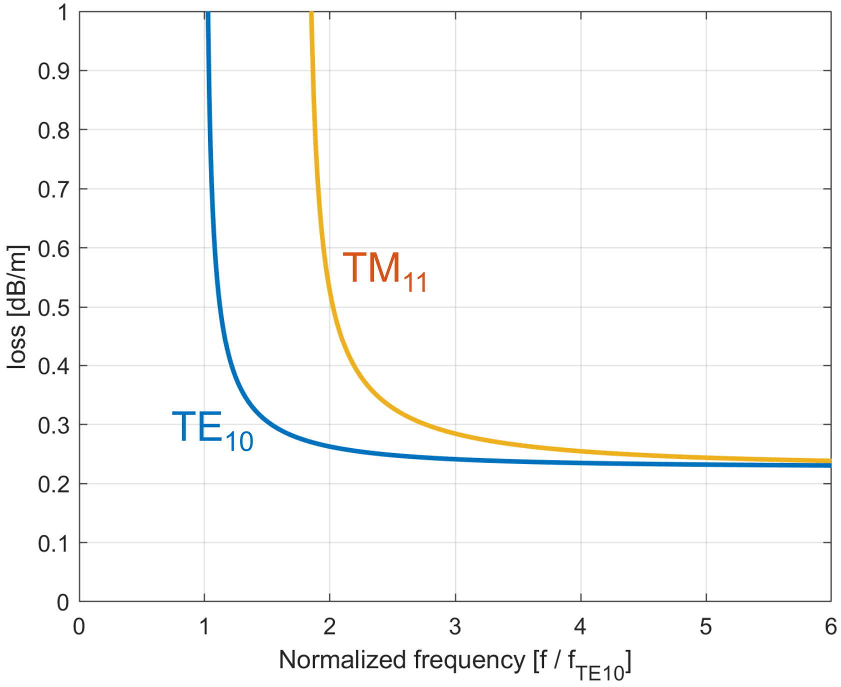
Waveguide loss

Conductor loss



Dielectric loss



$$\alpha_{c,d}(dB/m) = 8.686\alpha_{c,d}(Np/m)$$

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a = 2.29 cm
b = 1.53 cm
$f_c(TE_{10})$ = 6.55 GHz

Waveguide Dimensions

Propagation loss

$$\alpha_c \rightarrow f^{3/2} \quad \text{for} \quad f \gg f_c$$

$$\alpha_d \rightarrow f^{1} \quad \text{for} \quad f \gg f_c$$

$$\boxed{\alpha_c \downarrow \quad \text{for} \quad \frac{a}{b} \downarrow}$$

Bandwidth

next mode occurs @ $f_{c2,0} = 2\frac{f}{f_c}$ $a \geq 2b$

next mode occurs @ $f_{c0,1} < 2\frac{f}{f_c}$ $a < 2b$

- Conductor loss decreases as the waveguide aspect ratio becomes taller
- Normalized waveguide bandwidth = $1f_{c1,0}$ until a<2b
- **a = 2b → loss minimized subject to $1f_{c1,0}$ bandwidth constraint**

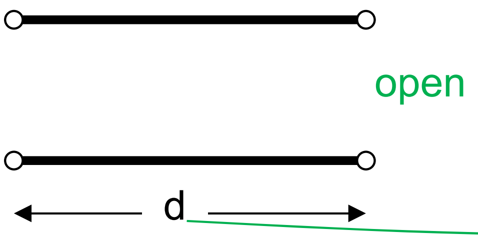
Rectangular Waveguide Cavities

Transmission Line Resonator

Resonance: Self sustaining of electromagnetic energy in waveguide structures at certain discrete frequencies

maximum energy
accumulation & absorption

Open ended transmission line

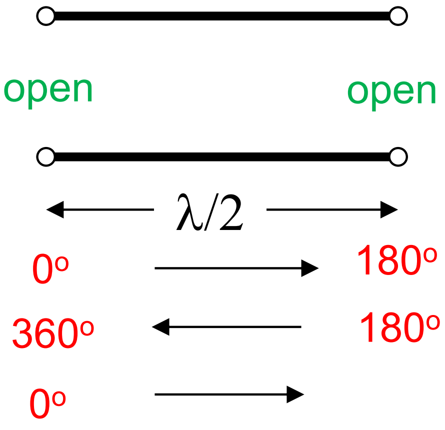

open

open

d

resonance when: $d = \frac{\lambda}{2} \cdot l \quad for \quad l = 1, 2, 3, \ldots$

Equal Phase Analysis


open

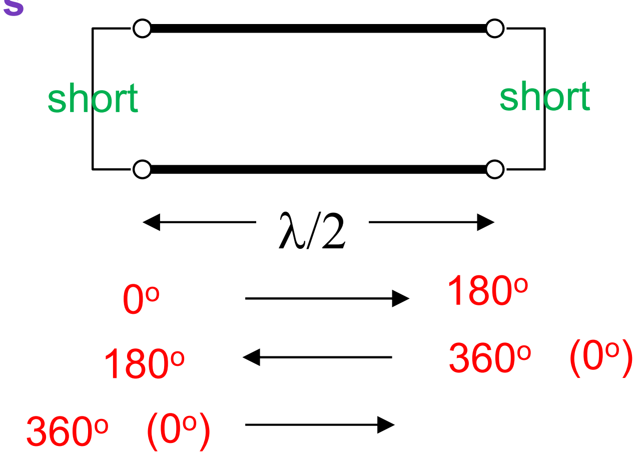
open

$\lambda/2$

0° → 180°

360° ← 180°

0° →


short

short

$\lambda/2$

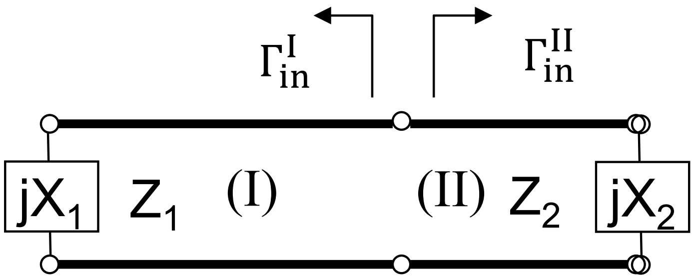
0° → 180°

180° ← 360° (0°)

360° (0°) →

Transmission Line Resonator

General Case



Resonance Condition:

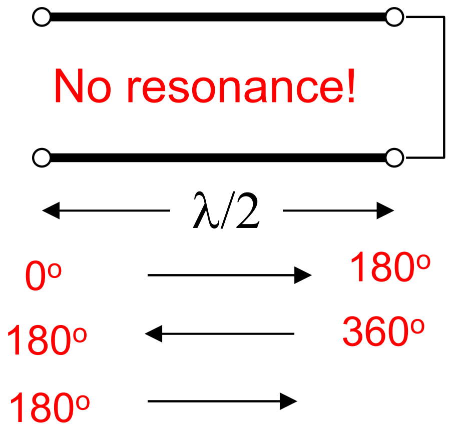
$$\boxed{\Gamma_{in}^{I} = \Gamma_{in}^{II\,*}}$$

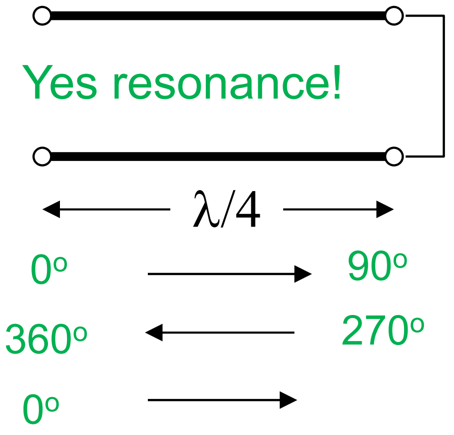
(first order approximation when Z$_1$ and Z$_2$ are not very different)

So the total transfer phase is

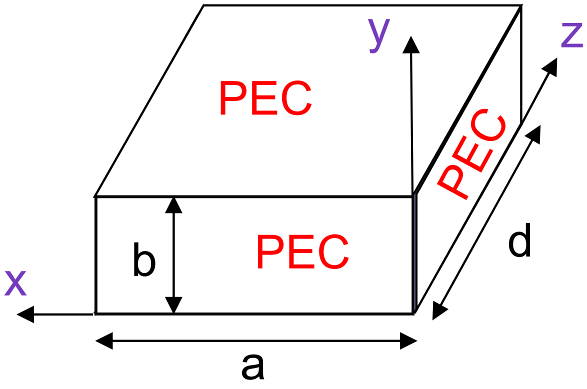
$$\varphi = \angle(\Gamma_{\text{in}}^{\text{I}} \cdot \Gamma_{\text{in}}^{\text{II}}) = \angle\left|\Gamma_{\text{in}}^{\text{I}}\right|^2 = 0^{\text{o}}$$

Examples:





Rectangular Waveguide Cavity



Basically, the rectangular cavity can be considered as TE and TM waves of the waveguide bounce back and forth between the two conductor plates at z=0 and z=d

$$\left.\begin{array}{l}\text{From previous slides, if two ends are}\\ \text{shorted, the resonance should occur at}\end{array}\right\} \quad d = \frac{\lambda_g}{2} \cdot p$$

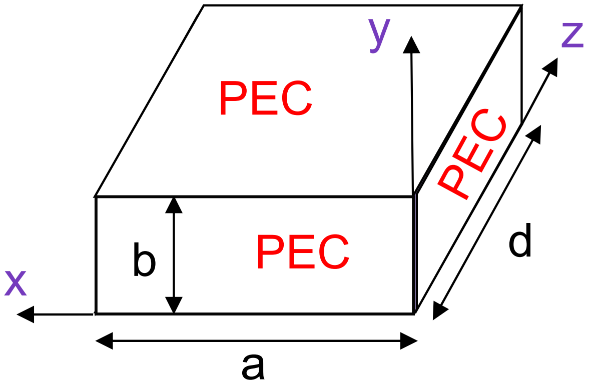
For TE$_{mn}$ mode,

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$H_z(x,y,z) = \Big(\underbrace{A_{mn} e^{-j\beta_{mn} z}}_{\text{Original +z prop. mode}} + \underbrace{B_{mn} e^{j\beta_{mn} z}}_{\text{New -z prop. mode}}\Big) \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

Add a reverse propagating mode

Rectangular Waveguide Cavity



For TE_{mn} mode, $\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$

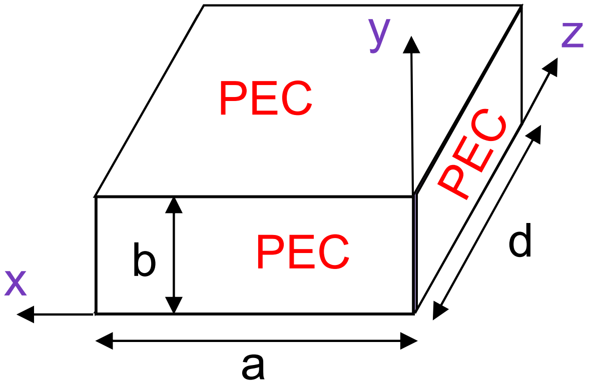
$$H_z(x,y,z) = \Big(\underbrace{A_{mn}e^{-j\beta_{mn}z}}_{\text{Original +z prop. mode}} + \underbrace{B_{mn}e^{j\beta_{mn}z}}_{\text{New -z prop. mode}}\Big)\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

Boundary conditions (in additional to the original waveguide B.C):

$$H_n = H_z = 0 \quad \text{at} \quad \begin{cases} z = 0 \\ z = d \end{cases} \Rightarrow \begin{cases} A_{mn} + B_{mn} = 0 \\ A_{mn}e^{-j\beta_{mn}d} + B_{mn}e^{j\beta_{mn}d} = 0 \end{cases} \Rightarrow \begin{cases} A_{mn} = -B_{mn} \\ \sin(\beta_{mn}d) = 0 \end{cases}$$

Normal component of H is 0 at PEC

Rectangular Waveguide Cavity



$$\sin(\beta_{mn} d) = 0 \Rightarrow \beta_{mn} d = p\pi, \quad p = 1,2,3\ldots$$

Therefore, $$\left\{ \begin{array}{l} \beta_{mn} = \frac{p\pi}{d} \\ \lambda_g = \frac{2\pi}{\beta_{mn}} \end{array} \right\} \Rightarrow d = \frac{\lambda_g}{2} \cdot l$$

agree with TRL model

Longitudinal H: $$H_z(x,y,z) = 2A_{mnp} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{l\pi z}{d}\right)$$

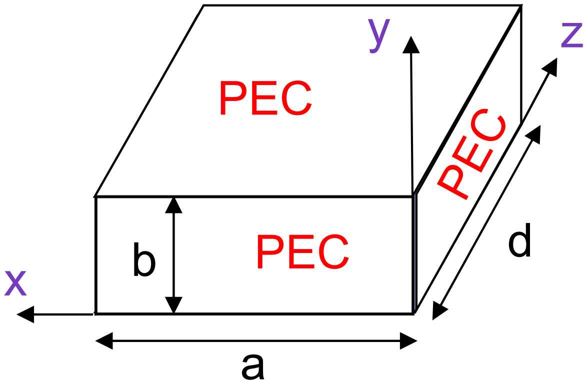
H_z satisfies: $$\beta_{mn} = \frac{p\pi}{d} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

k can only be certain discrete values !

Resonant wave number for the *mnp*th mode is thus given by, ⇒

$$\boxed{k_{r,mnp} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}}$$

Rectangular Waveguide Cavity



Resonant wave number for the *mnp*th mode is thus given by, ⇨

$$k_{r,mnp} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

Resonant Wavelength

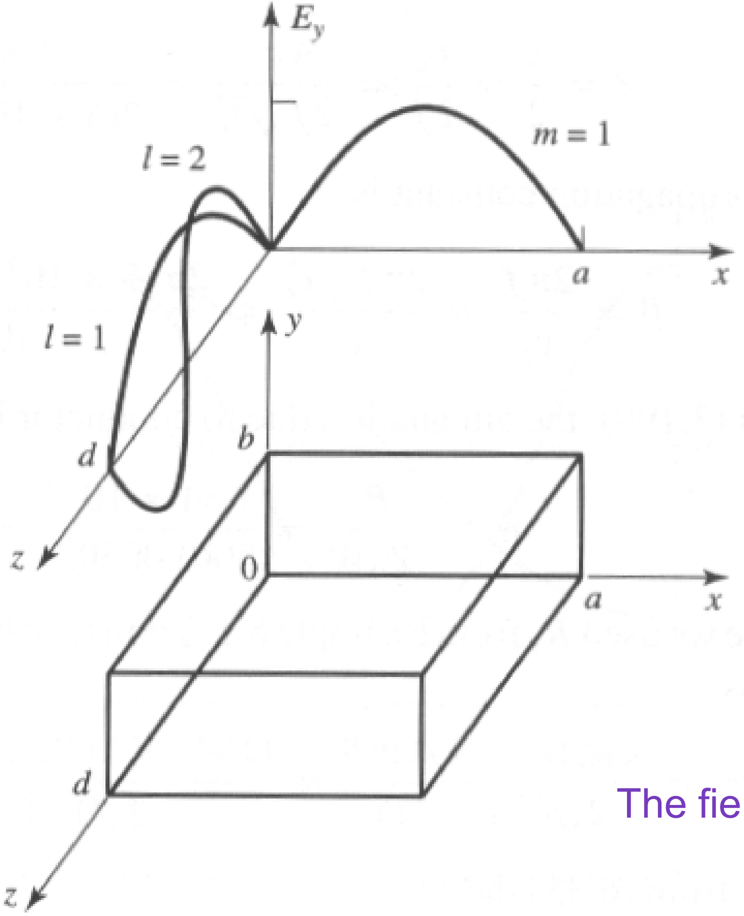
$$\lambda_{mnp} = \frac{2\pi}{k_{r,mnp}} = \frac{2\pi}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}}$$

Resonant Frequency

$$\boxed{f_{mnp} = \frac{c_0/\sqrt{\varepsilon_r}}{\lambda_{mnp}} = \frac{k_{r,mnp}}{2\pi} \cdot \frac{c_0}{\sqrt{\varepsilon_r}}}$$

- Rectangular waveguide with ends capped with PEC will only resonate at discrete frequencies
- Only energy at discrete frequencies can be stored in the cavity

Rectangular Waveguide Cavity



Similarly, for TM_{mn} mode, field is derived the same way and resonant frequencies are the same

$$E_z(x,y,z) = 2A_{mnp}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{l\pi z}{d}$$

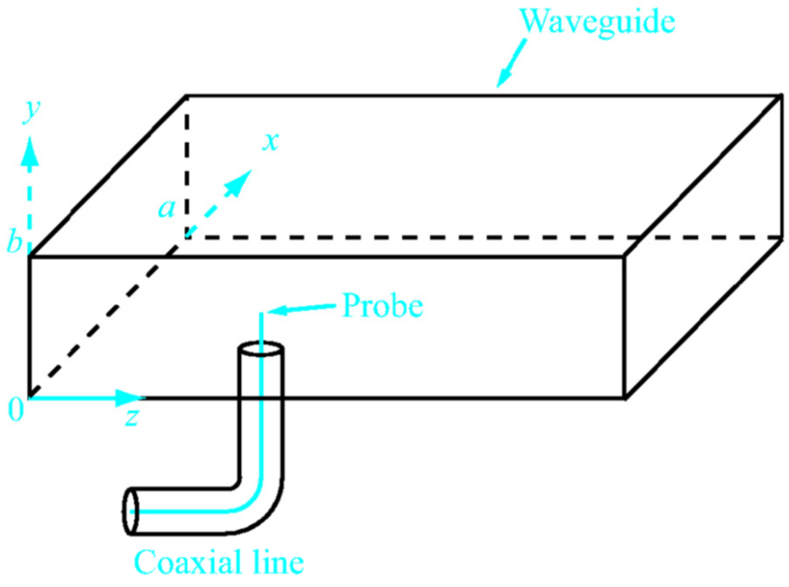
For TE_{101} mode (lowest resonant frequency when a&d >b), the resonant wavelength is,

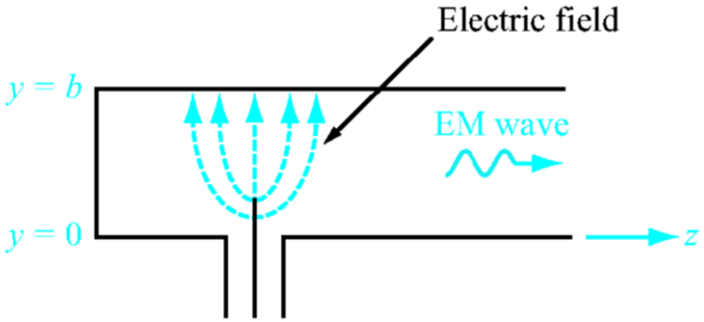
$$\lambda_{101} = \frac{2}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{d}\right)^2}}$$

The field are

$$\begin{cases} H_z = A_{101}\cos\frac{\pi x}{a}\sin\frac{\pi z}{d} \\ H_x = -\frac{a}{d}A_{101}\sin\frac{\pi x}{a}\cos\frac{\pi z}{d} \\ E_y = \frac{-j\omega\mu a}{\pi}A_{101}\sin\frac{\pi x}{a}\sin\frac{\pi z}{d} \end{cases}$$

Excitation of waveguide, cavity





- A current probe can be used to excite electromagnetic field into the waveguide
- The electric field excited by the current probe will resemble the direction of current flow in the probe
- For a fixed amount of current, the maximum power of electromagnetic wave is excited for that mode if the probe is probing at the maximum electric field position of that mode

Resonator Quality factor, TE_{101}

$$W_e = \frac{\varepsilon_0}{4}\int_0^d \int_0^b \int_0^a \left|E_y\right|^2 dv$$

- Electric energy: sum of square electric field subtended by the volume

$$W_m = \frac{\mu_0}{4}\int_0^d \int_0^b \int_0^a \{|H_x|^2 + |H_z|^2\} dv$$

- Magnetic energy: sum of square Magnetic field subtended by the volume

$$P_{loss,ave} = \frac{R_s}{2}\oint_C |J_s|^2 ds$$

- Power loss: total power lost meaning sum (integral) of surface current density in a lossy conductor over walls for length d + sum current density in a lossy conductor at the end caps

Resonator Quality factor, TE_{101}

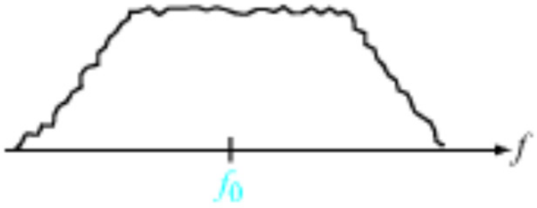
Quality factor Q for a resonator is defined as

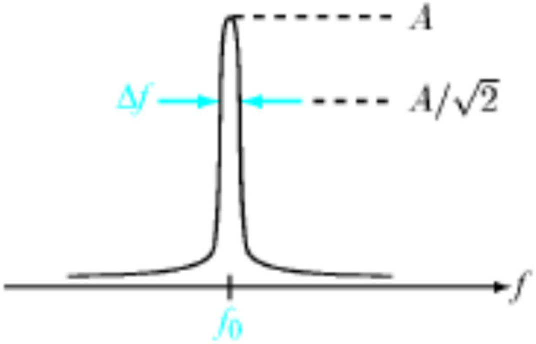
stored magnetic energy

stored electric energy

$$Q = \omega \frac{(\text{average energy stored})}{(\text{energy loss/second})} = \omega \frac{W_m + W_e}{P_{loss,ave}}$$

power dissipation





For most resonators, the Q is inversely proportional to the fractional bandwidth of the resonance

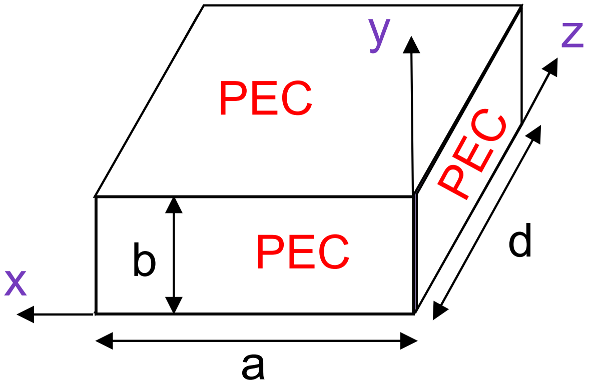
$$Q \approx \frac{f_{mnp}}{\Delta f}$$

For rectangular waveguides, the quality factor for the dominant TE_{101} mode is given by

$$\boxed{Q = \frac{\pi f_{101} \mu_0 abd(a^2 + d^2)}{R_s[2b(a^3 + d^3) + ad(a^2 + d^2)]}} \longrightarrow R_s = \frac{1}{\sigma \delta_s} = \sqrt{\frac{\pi f \mu}{\sigma}}$$

Conductor loss but no dielectric loss

Example



- A square based (a = c) cavity of rectangular corss sectionis constructed of an X-band (8.2 GHz – 12.4 GHz) copper (σ = 5.7 x 10^7 S/m) waveguide that has inner dimensions a = 2.29 cm, b = 1.02 cm. For the dominant TE_{101} mode, determine the Q of the cavity. Assume free space medium inside the cavity.

$$k_{r,101} = \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2}$$

$$\boxed{Q = \frac{\pi f_{101} \mu_0 abd(a^2 + d^2)}{R_s[2b(a^3 + d^3) + ad(a^2 + d^2)]} = 7757.9}$$

$$f_{mnp} = \frac{k_{r,101}}{2\pi} \cdot \frac{c_0}{\sqrt{\varepsilon_r}} = \frac{c_0 k_{r,101}}{2\pi} = 9.28\ GHz$$

Much higher than can be reasonably achieved in practice with lumped element circuits

$$R_s = \frac{1}{\sigma \delta_s} = \sqrt{\frac{\pi f \mu}{\sigma}} = 0.0254\ \Omega$$

Circular, conductor walled waveguide

Recall: Waveguide solutions

transverse variation

$$\mathbf{E}(x, y, z) = [\mathbf{e_t}(x, y) + \mathbf{a_z} e_z(x, y)]e^{-j\beta z}$$

$$\mathbf{H}(x, y, z) = [\mathbf{h_t}(x, y) + \mathbf{a_z} h_z(x, y)]e^{-j\beta z}$$

longitudinal variation

transverse component

longitudinal component

↓

transverse variation

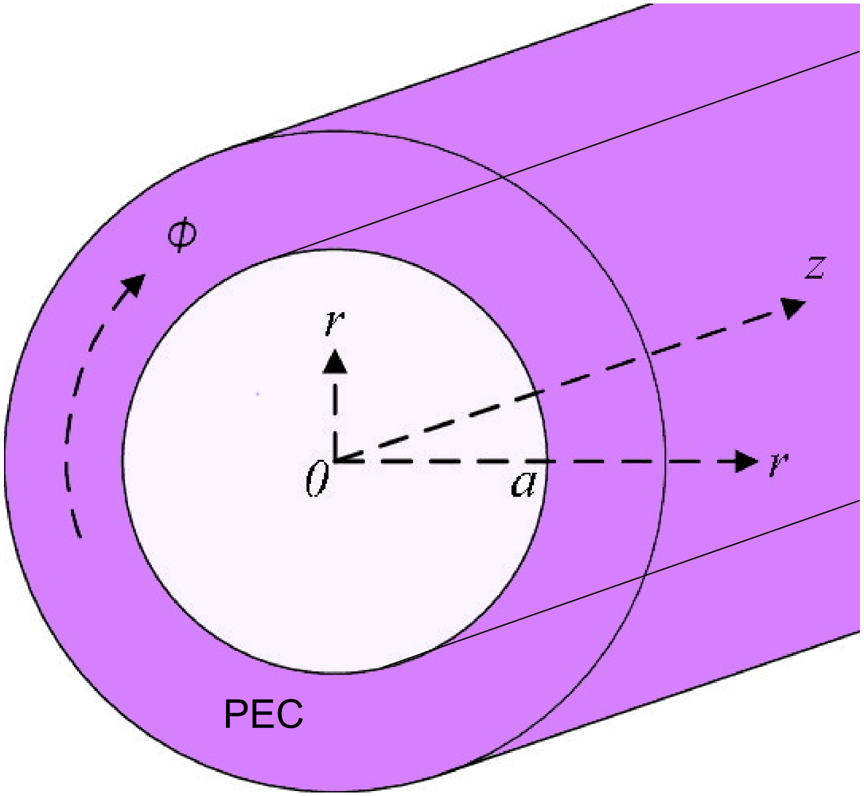
$$\mathbf{E}(r, \phi, z) = [\mathbf{e_t}(r, \phi) + \mathbf{a_z} e_z(r, \phi)]e^{-j\beta z}$$

$$\mathbf{H}(r, \phi, z) = [\mathbf{h_t}(r, \phi) + \mathbf{a_z} h_z(r, \phi)]e^{-j\beta z}$$

longitudinal variation

transverse component

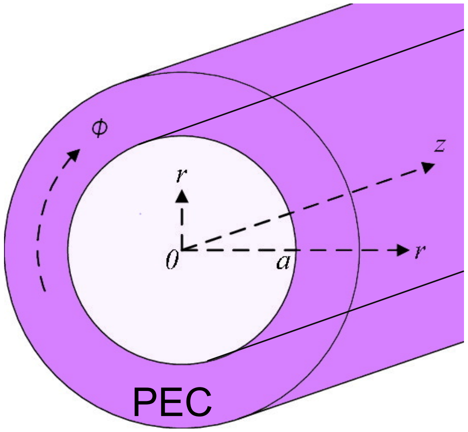
longitudinal component



$$E_z(r, \phi, z) = e_z(r, \phi)e^{-j\beta z}$$

$$H_z(r, \phi, z) = h_z(r, \phi)e^{-j\beta z}$$

Focus on E_z



Rectangular Waveguide Notation

$$E_z(r,\phi,z) = e_z(r,\phi)e^{-j\beta z}$$

$$\longrightarrow$$

Book Notation

$$E_z(r,\phi,z) = E_z^0(r,\phi)e^{-\gamma z}$$

Vector wave equation

$$\nabla_T^2 E_z^0 + (\gamma^2 + k^2)E_z^0 = 0$$

$$\xrightarrow[h^2 = (\gamma^2+k^2)]{\mathbf{T} \to \mathbf{r\phi}}$$

Explicit Trans. Variables

$$\nabla_{\mathbf{r\phi}}^2 E_z^0 + h^2 E_z^0 = 0$$

Cylindrical coordinates

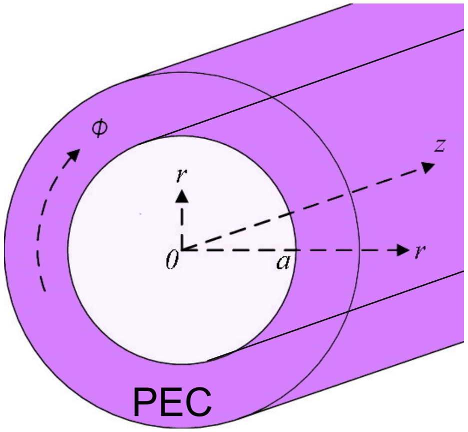
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z^0}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z^0}{\partial \phi^2} + h^2 E_z^0 = 0$$

$$\xrightarrow[\text{Assume separable}]{}$$

Cylindrical coordinates

$$E_z^0(r,\phi) = R(r)\Phi(\phi)$$

Focus on E_z



Cylindrical coordinates

$$\underbrace{\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + h^2r^2}_{n^2} + \underbrace{\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}}_{-n^2} = 0$$

These terms vary independently and therefore must equal the same constant

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + h^2r^2 = n^2$$

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = -n^2$$

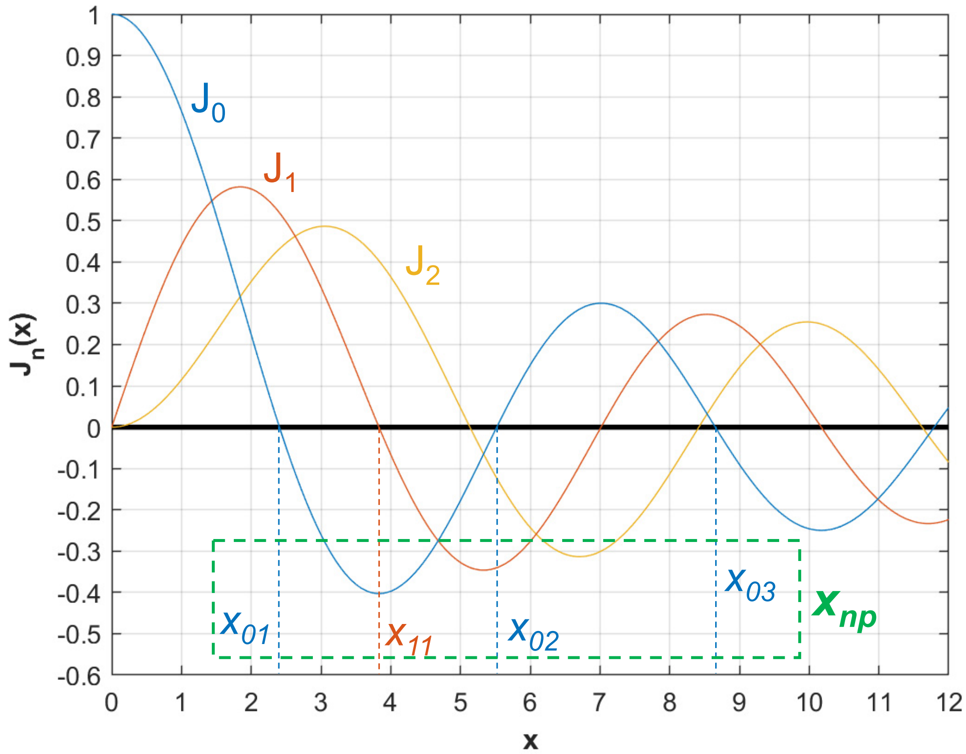
Cylindrical coordinates

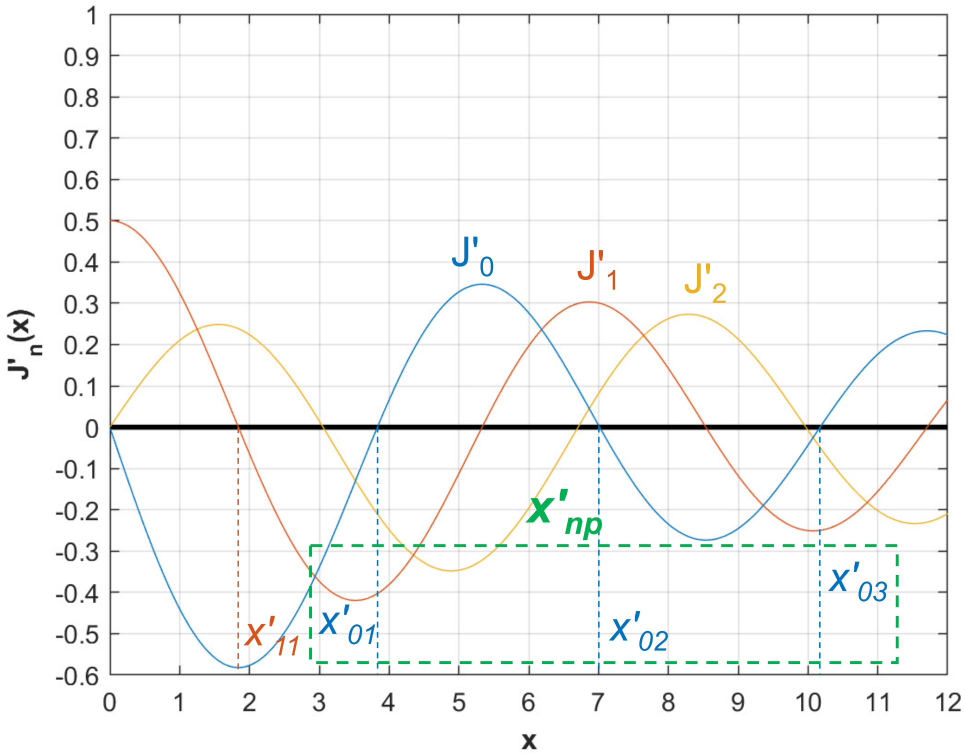
$$E_z^0(r,\phi) = R(r)\Phi(\phi)$$

$$\boxed{\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left(h^2 - \frac{n^2}{r^2}\right)R(r) = 0}$$

$$\boxed{\frac{d^2\Phi(\phi)}{d\phi^2} + n^2\Phi(\phi) = 0}$$

Bessel functions





Bessel's Differential equation

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left(h^2 - \frac{n^2}{r^2}\right)R(r) = 0$$

Bessel function of the first kind

$$R(r) = C_n J_n(hr) \quad \rightarrow \quad J_n(hr) = \sum_{m=0}^{\infty} \frac{(-1)^m (hr)^{n+2m}}{m!\,(n+m)!\,(2)^{n+2m}}$$

Arbitrary Constant

TE vs TM modes

TM modes $H_z^0(r,\phi) = 0$

$$E_z^0(r,\phi) = C_n J_n(hr)\cos(n\phi)$$

$$E_r^0(r,\phi) = -\frac{j\beta}{h} C_n J'_n(hr)\cos(n\phi)$$

$$E_\phi^0(r,\phi) = \frac{j\beta n}{h^2 r} C_n J_n(hr)\sin(n\phi)$$

$$H_r^0(r,\phi) = -\frac{j\omega\varepsilon n}{h^2 r} C_n J_n(hr)\sin(n\phi)$$

$$H_\phi^0(r,\phi) = -\frac{j\omega\varepsilon}{h} C_n J'_n(hr)\cos(n\phi)$$

$$E_z^0(r = a,\phi) = 0 \quad \rightarrow \quad J_n(ha) = 0$$

$$x_{01} \quad \rightarrow \quad h_{TM01} = \frac{2.405}{a} \quad \rightarrow \quad f_{c,TM01} = \frac{h_{TM01}}{2\pi\sqrt{\mu\varepsilon}}$$

TE modes $E_z^0(r,\phi) = 0$

$$E_r^0(r,\phi) = \frac{j\omega\mu n}{h^2 r} C'_n J_n(hr)\sin(n\phi)$$

$$E_\phi^0(r,\phi) = \frac{j\omega\mu}{h^2 r} C'_n J'_n(hr)\cos(n\phi)$$

$$H_z^0(r,\phi) = C'_n J_n(hr)\cos(n\phi)$$

$$H_r^0(r,\phi) = -\frac{j\beta}{h} C'_n J'_n(hr)\cos(n\phi)$$

$$H_\phi^0(r,\phi) = \frac{j\beta n}{h^2 r} C'_n J_n(hr)\sin(n\phi)$$

$$H_z^0(r = a,\phi) = 0 \quad \rightarrow \quad J'_n(ha) = 0$$

$$x'_{11} \quad \rightarrow \quad h_{TE11} = \frac{1.841}{a} \quad \rightarrow \quad f_{c,TE11} = \frac{h_{TE11}}{2\pi\sqrt{\mu\varepsilon}}$$

Propagation equations

Guide wavelength

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - k_c^2}} = \frac{2\pi}{k\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

TE wave impedance

$$Z_{TE} = \frac{\eta_0/\sqrt{\varepsilon_r}}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

Propagation constant

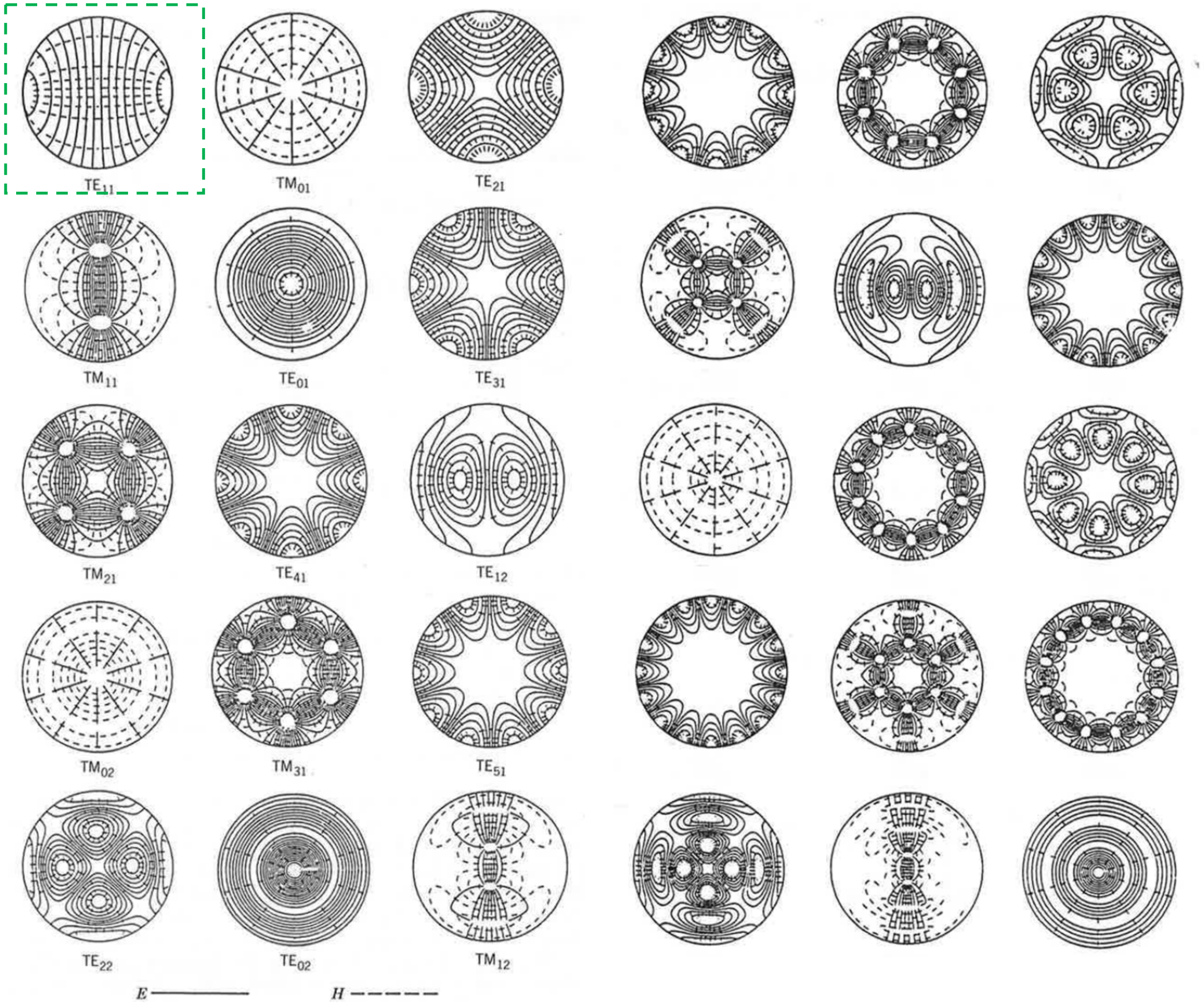
$$\beta = \frac{2\pi}{\lambda_g}$$

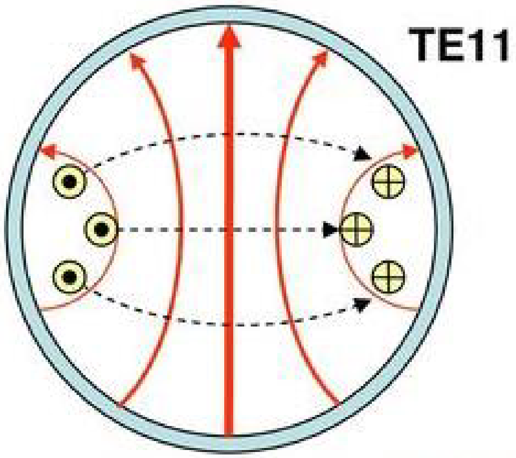
TM wave impedance

$$Z_{TM} = \frac{\eta_0}{\sqrt{\varepsilon_r}}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

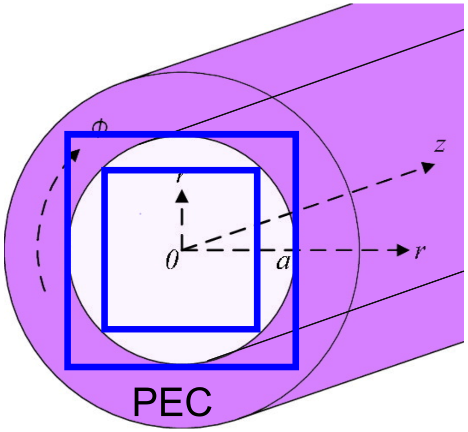
- The same equations apply as to circular waveguides as did for rectangular waveguides
- The cutoff frequencies are defined by the geometry and dimensions.
- Once the cutoff frequency is determined, then everything else is determined

Circular Waveguide modes





In class exercise 1

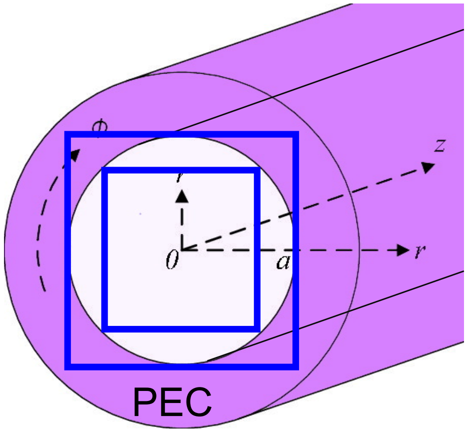


Cutoff frequency of rectangular waveguide mode

$$f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{c}{2\sqrt{\varepsilon_r}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

- A circular waveguide of radius a = 3 cm that is filled with polystyrene (ε_r = 2.56) is used at a frequency of 2 GHz. For the dominant TE_{mn} mode determine the following
  - (a) Cutoff frequency
  - (b) guide wavelength in cm
  - (c) Phase constant beta
  - (d) Wave impedance ZTE
  - (e) Compare the cutoff frequency to the fundamental mode of a square (a = b) waveguide whose **diagonal** is equal to the circular waveguide diameter
  - (f) Compare the cutoff frequency to the fundamental mode of a square (a = b) waveguide whose **side** is equal to the circular waveguide diameter

In class exercise 1



(a) Cutoff frequency

$$\boxed{f_{cTE11} = \frac{1.841}{2\pi a\sqrt{\mu\varepsilon}} = 1.835\text{ GHz}}$$

(b) guide wavelength in cm

$$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = 9.357\text{ cm}$$

(c) Phase constant beta

$$\beta = \frac{2\pi}{\lambda_g} = 0.2692\, rad/s$$

(d) Wave impedance ZTE

$$Z_{TE} = \frac{\eta_0/\sqrt{\varepsilon_r}}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = 586.56\ \Omega$$

Cutoff frequency of rectangular waveguide mode

$$f_{c,rect} = \frac{c}{2\sqrt{\varepsilon_r}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

(e) Cutoff frequency

$$\sqrt{2}a_{rect} = 2a_{circ}$$

$$a_{rect} = \sqrt{2}a_{circ}$$

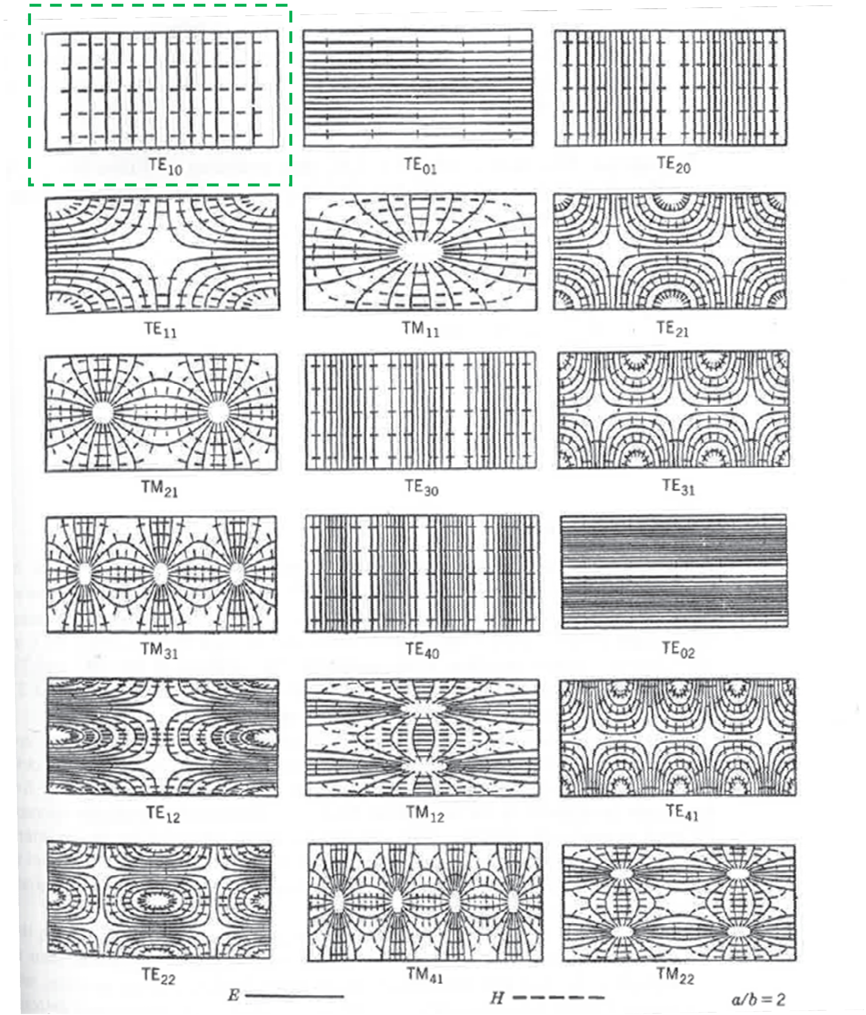
$$\boxed{f_{c,rect} = \frac{c}{2a_{circ}\sqrt{2\varepsilon_r}} = 2.21\text{ GHz}}$$

(f) Cutoff frequency

$$a_{rect} = 2a_{circ}$$

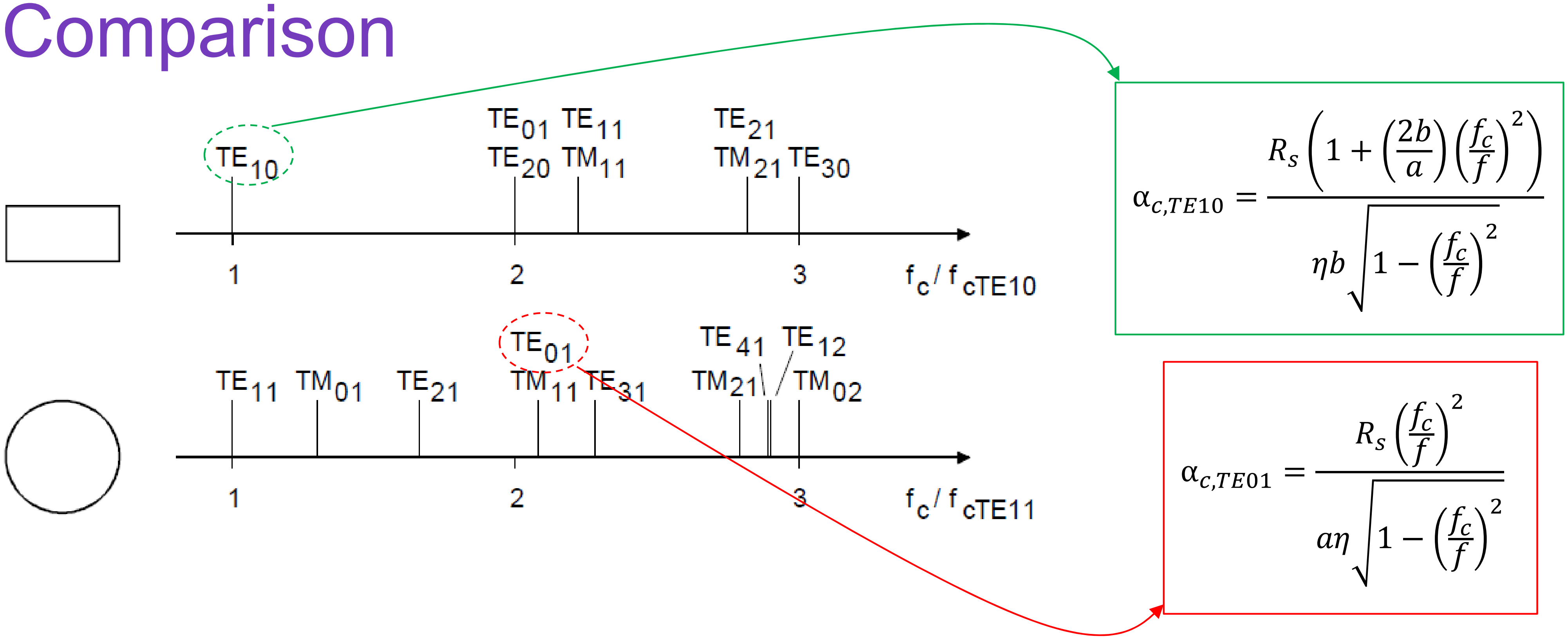
$$\boxed{f_{c,rect} = \frac{c}{4a_{circ}\sqrt{\varepsilon_r}} = 1.56\text{ GHz}}$$

Compare to Rectangular Waveguide modes



- Rectangular waveguide modes less densely distributed than circular waveguide modes

Comparison

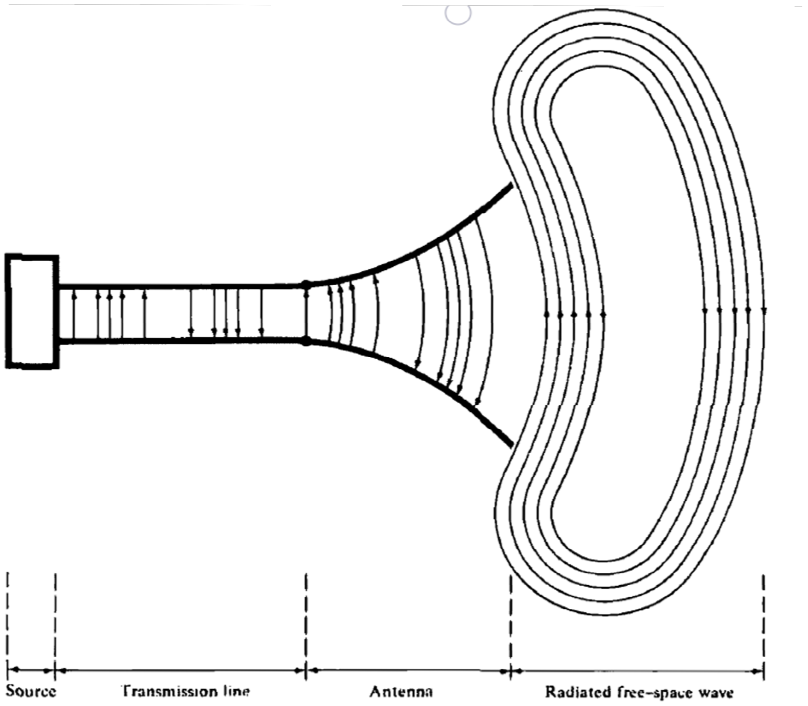


$$\alpha_{c,TE10} = \frac{R_s\left(1+\left(\frac{2b}{a}\right)\left(\frac{f_c}{f}\right)^2\right)}{\eta b\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\alpha_{c,TE01} = \frac{R_s\left(\frac{f_c}{f}\right)^2}{a\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

- Rectangular waveguides have broadest single mode operation
- In an oversized circular metal waveguide, a very low-loss TE01 mode can propagate.
  - The cut-off wavelength of this mode is $\lambda_c = 1.64a$

Why should we care about circular waveguides?



- Waveguides are often used as a transmission line between and device and antenna
- Good beam patterns are obtained with radially symmetric aperture cross sections
- How do you couple a rectangular waveguide to a circular mode?

Why should we care about circular waveguides?

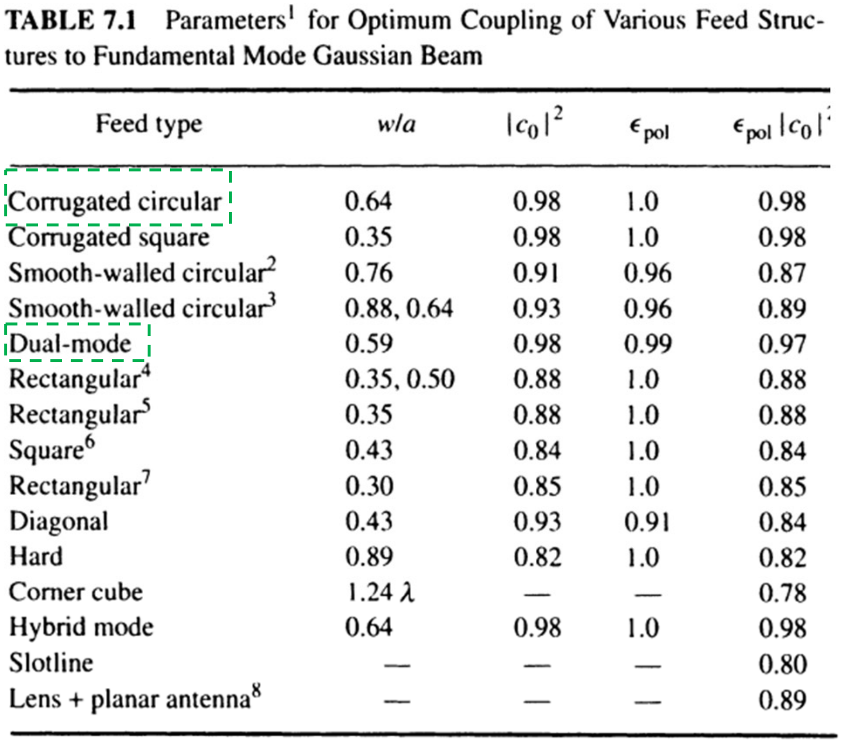
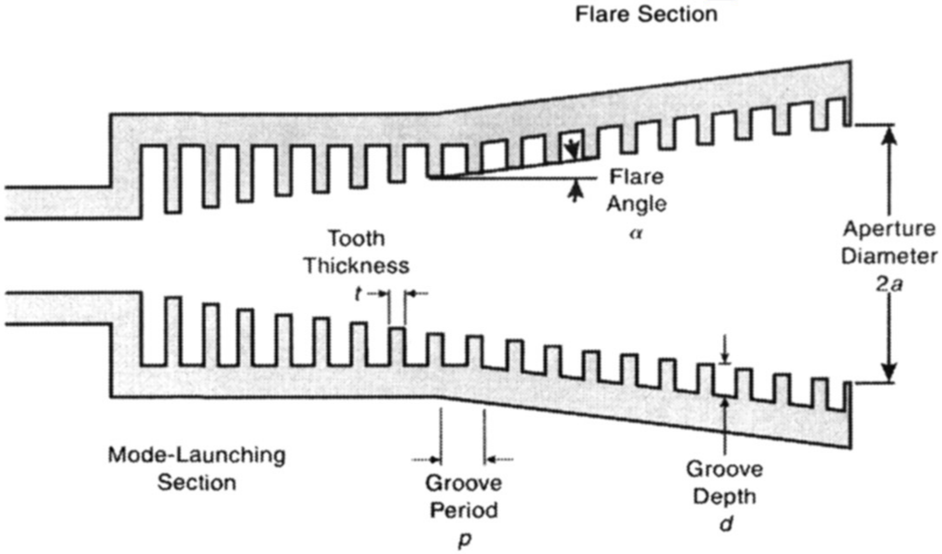
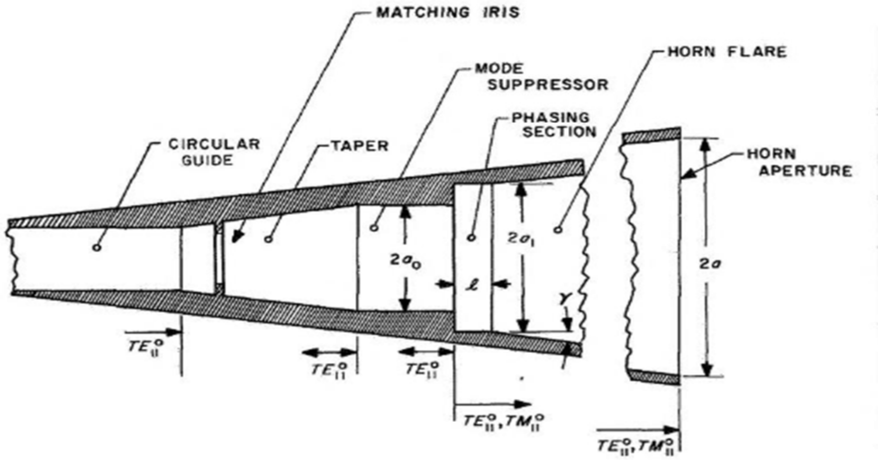


TABLE 7.1 Parameters[1] for Optimum Coupling of Various Feed Structures to Fundamental Mode Gaussian Beam

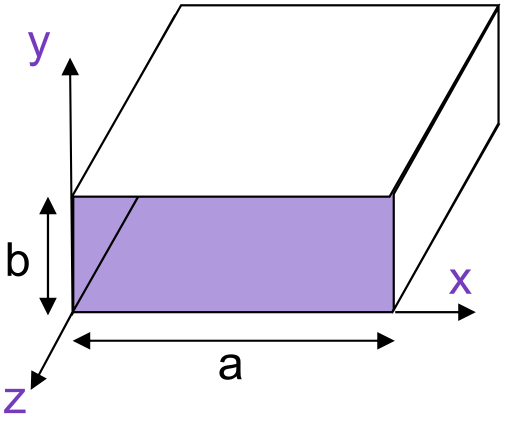
Feed type	w/a	$\lvert c_0\rvert^2$	ϵ_{pol}	$\epsilon_{\text{pol}}\lvert c_0\rvert^2$
Corrugated circular	0.64	0.98	1.0	0.98
Corrugated square	0.35	0.98	1.0	0.98
Smooth-walled circular[2]	0.76	0.91	0.96	0.87
Smooth-walled circular[3]	0.88, 0.64	0.93	0.96	0.89
Dual-mode	0.59	0.98	0.99	0.97
Rectangular[4]	0.35, 0.50	0.88	1.0	0.88
Rectangular[5]	0.35	0.88	1.0	0.88
Square[6]	0.43	0.84	1.0	0.84
Rectangular[7]	0.30	0.85	1.0	0.85
Diagonal	0.43	0.93	0.91	0.84
Hard	0.89	0.82	1.0	0.82
Corner cube	1.24 λ	—	—	0.78
Hybrid mode	0.64	0.98	1.0	0.98
Slotline	—	—	—	0.80
Lens + planar antenna[8]	—	—	—	0.89





Conclusions

Propagation



$$\beta \neq k$$

- The one conductor geometry supports TE/TM operation
- The longitudinal phase variation of a TE/TM is not equal to the free space (plane wave) TEM phase variation
  - β → rectangular waveguide longitudinal phase variation
  - k → free space longitudinal phase variation
- β is a strong function of frequency and geometry