ELEC-E4130

Lecture 19: Rectangular Waveguides

Ch. 10

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School of Electrical
Engineering

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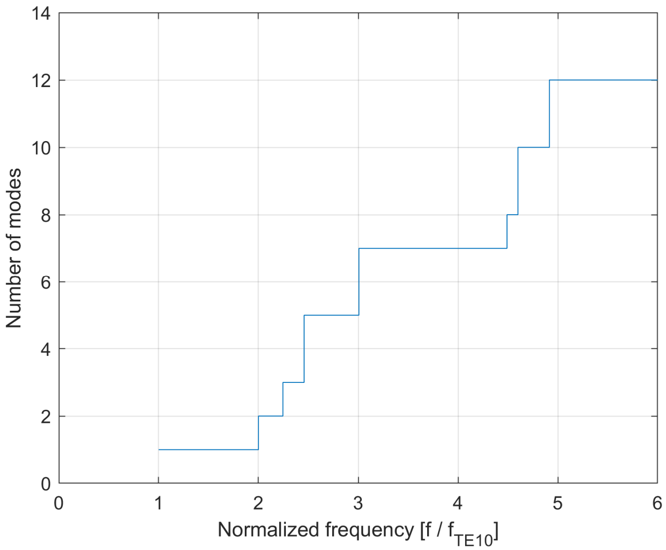
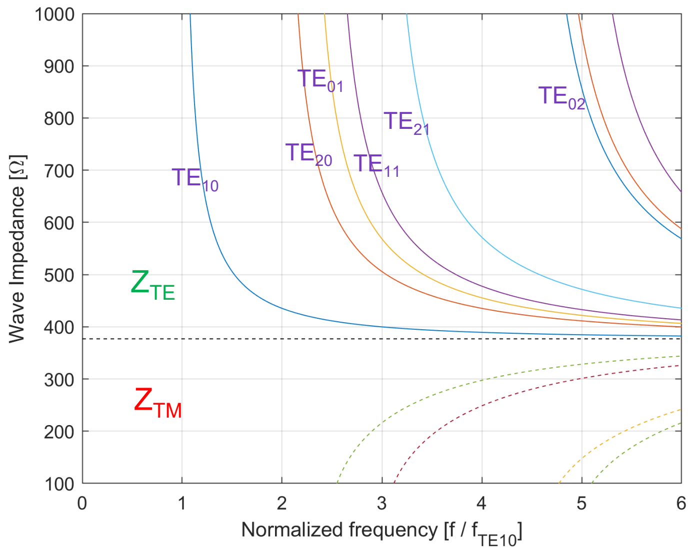
Recap of modes

Waveguide modes

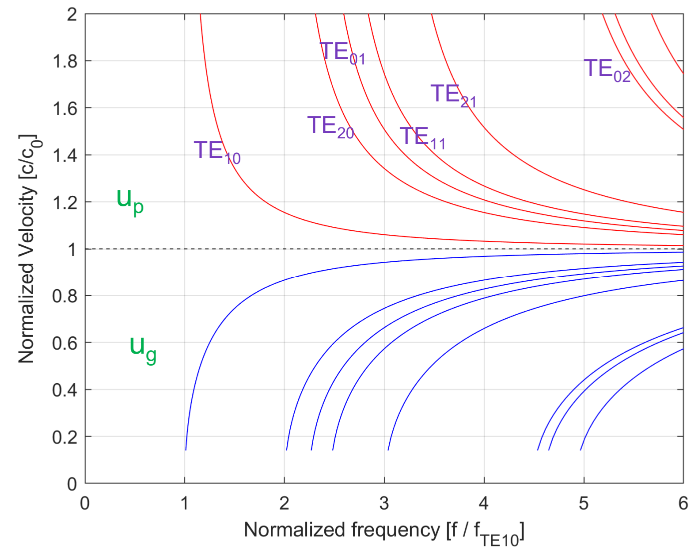
Number of propagating modes



Wave impedance



Phase/Group Velocity



a = 2.29 cm
b = 1.02 cm
$f_c(TE_{10})$ = 6.55 GHz

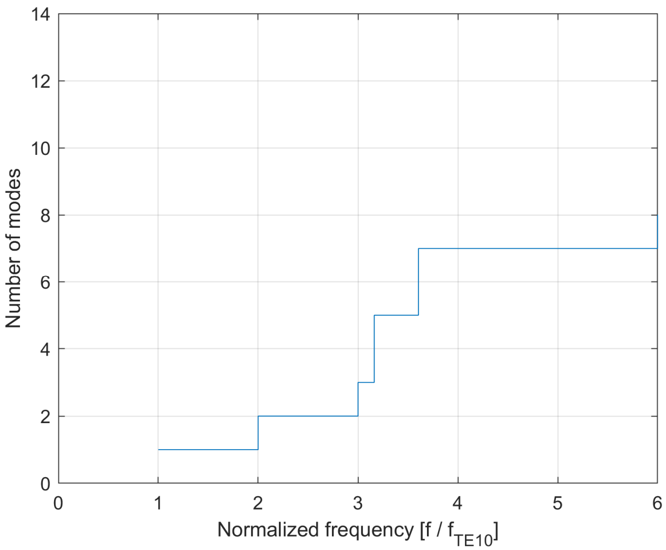
a = 2.29 cm
b = 0.76 cm
$f_c(TE_{10})$ = 6.55 GHz

a = 2.29 cm
b = 1.53 cm
$f_c(TE_{10})$ = 6.55 GHz

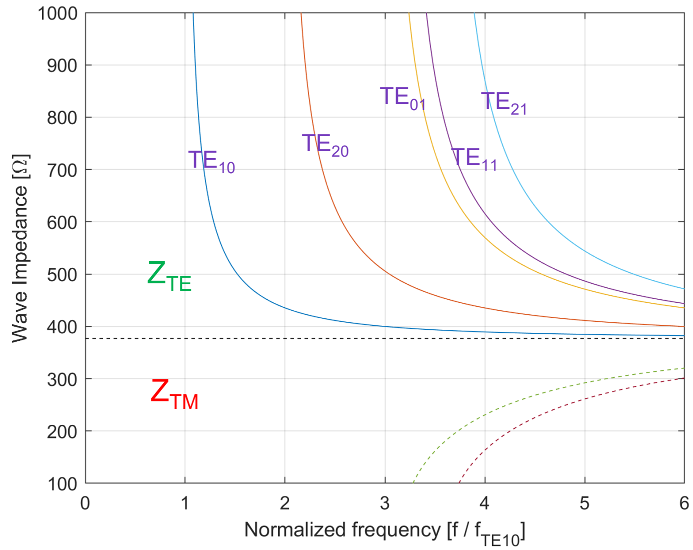
- Standard waveguide size (**WR-90**)

Waveguide modes

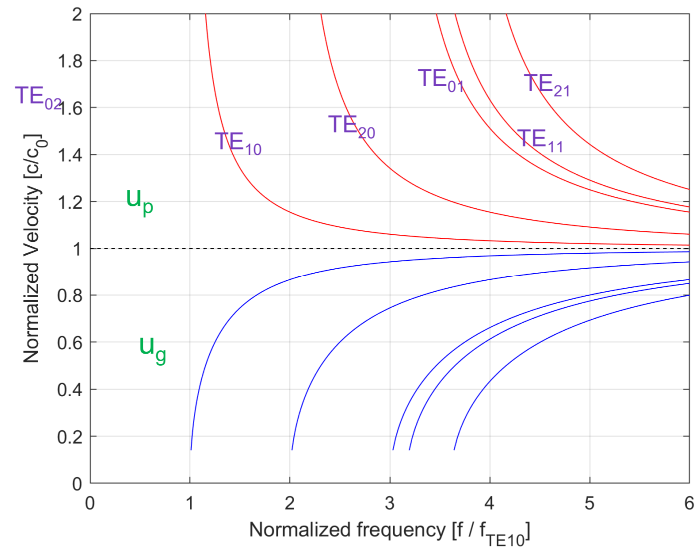
Number of propagating modes



Wave impedance



Phase/Group Velocity



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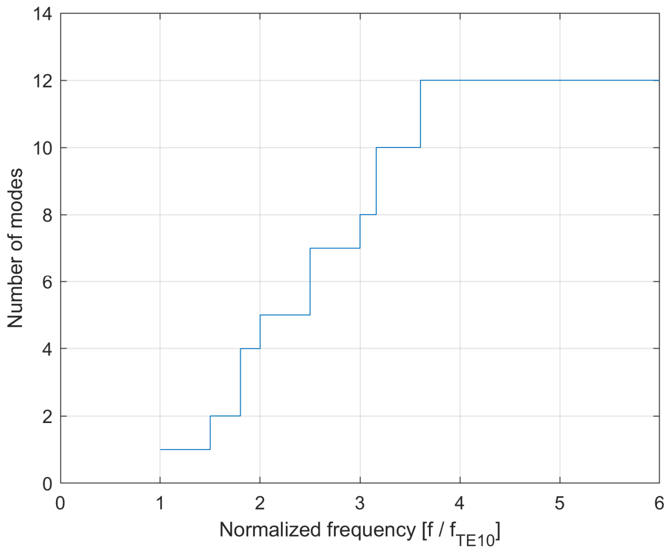
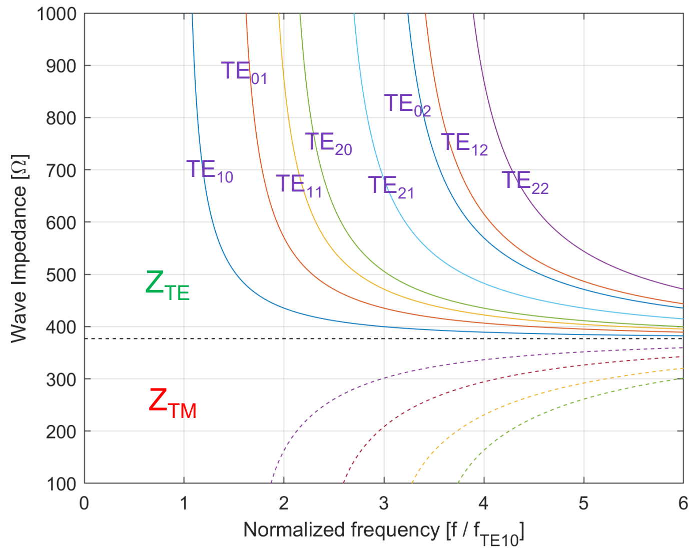
- Making the waveguide shorter spaces the modes out

Waveguide modes

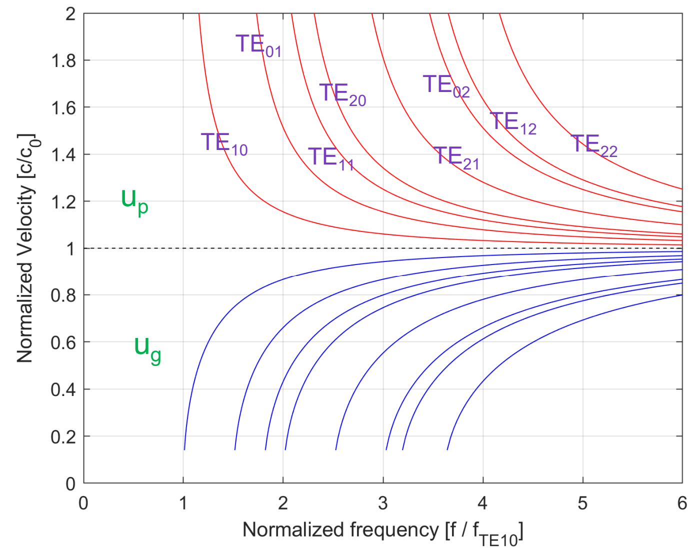
Number of propagating modes



Wave impedance



Phase/Group Velocity



a = 2.29 cm
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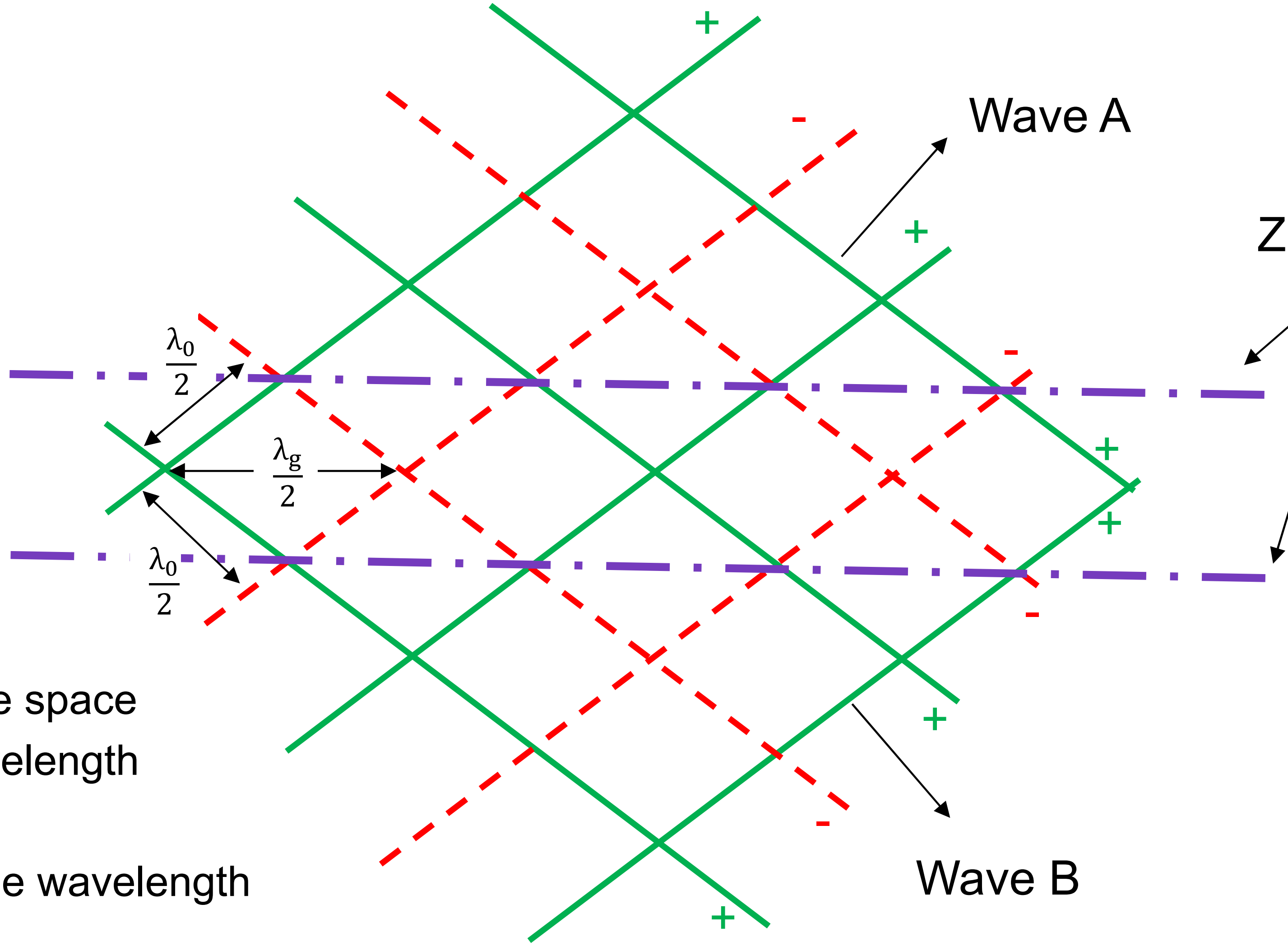
a = 2.29 cm
b = 1.53 cm
$f_c(TE_{10})$ = 6.55 GHz

- Making the waveguide taller makes the modes more dense and also reorders where the fall in frequency space

Phase velocity, group velocity, and dispersion

Plane Wave View of Non-TEM Waves



λ_0: Free space wavelength

λ_g: Guide wavelength

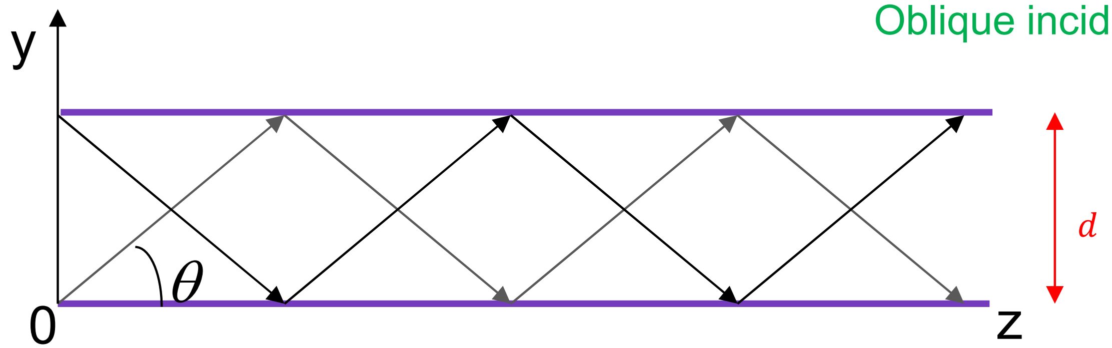
Plane Wave View of Non-TEM Waves

Consider TE_{10} mode, $\beta_1 = \sqrt{k^2 - (\pi/d)^2}$

$$E_x(x,y,z) = A_1 \sin\frac{\pi y}{d} e^{-j\beta_1 z} = \frac{A_1}{2j}\{e^{j\pi y/d - j\beta_1 z} - e^{-j\overbrace{\pi y/d}^{k_y y} - j\overbrace{\beta_1 z}^{k_z z}}\}$$

Oblique incidence

Combination of two waves!



$$\begin{cases} k_y = k\sin\theta = \dfrac{\pi}{d} \\ k_z = k\cos\theta = \beta_1 \end{cases}$$

Cutoff wave number k_c

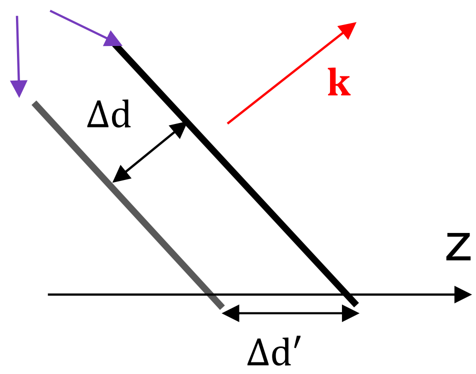
Phase constant

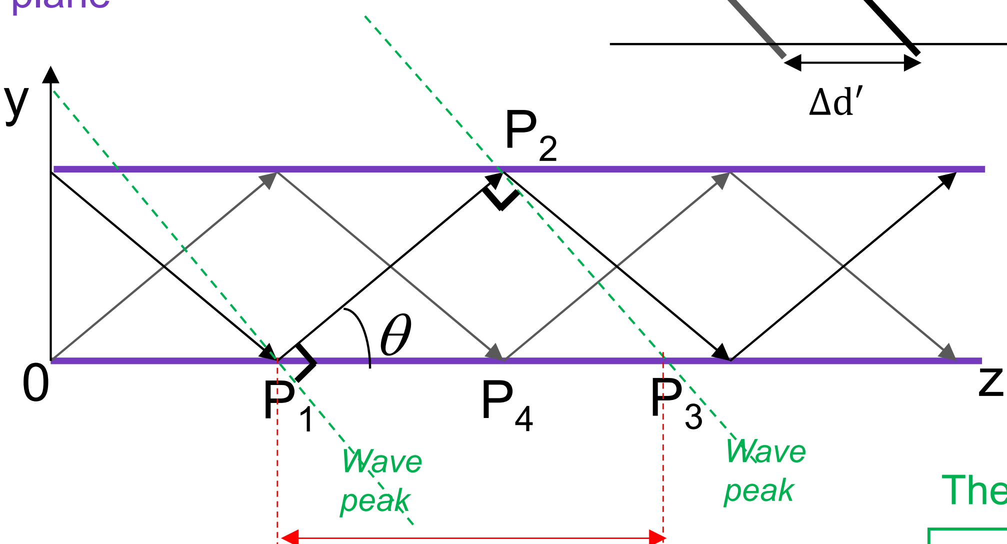
$$\boxed{k_y^2 + k_z^2 = k_c^2 + \beta^2 = k^2}$$

Observation: When the frequency is very close to the cutoff, θ approaches to 90 degree.

Phase Velocity

Phase Velocity: Velocity of the wave front or equal phase plane





How long did it take to travel this distance?

According to the definition, the phase velocity is,

$$v_p = \frac{\Delta d'}{\Delta t} \quad \text{(observe in z direction)}$$

Since $\begin{cases} C = \dfrac{\Delta d}{\Delta t} \\ \Delta d' > \Delta d \end{cases} \Rightarrow$ u_p could be greater > c

$$u_p = f\lambda_g = \frac{\omega}{\beta}$$

Plane wave traveling speed: $\dfrac{c}{\sqrt{\varepsilon_r}} = \dfrac{\overline{P_1P_2}}{\Delta t}$

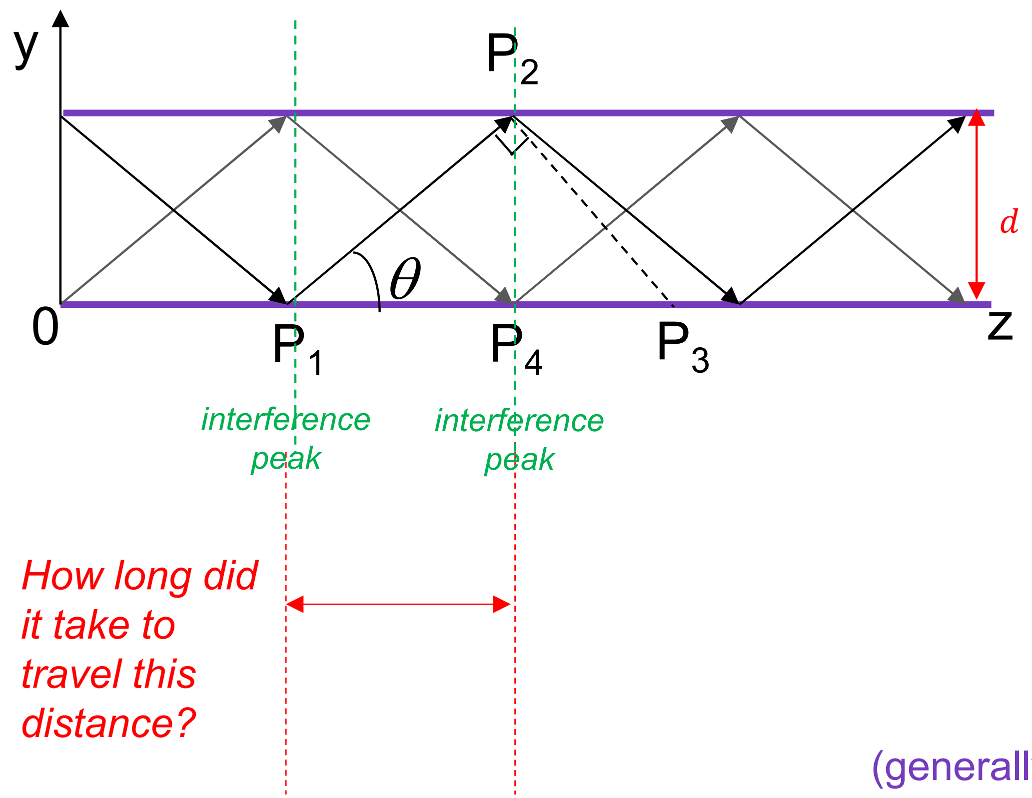
The phase velocity is given by,

$$\frac{1}{\sqrt{\mu_0\varepsilon_0}} \qquad \omega\sqrt{\mu\varepsilon}$$

$$\boxed{u_p = \frac{\overline{P_1P_3}}{\Delta t} = \frac{\overline{P_1P_2}/\cos\theta}{\Delta t} = \frac{c}{\sqrt{\varepsilon_r}}\frac{1}{\cos\theta} = \frac{c}{\sqrt{\varepsilon_r}}\frac{k}{\beta} = \frac{\omega}{\beta}}$$

Group Velocity

Group Velocity: Energy propagation velocity of the wave: (The true velocity!!!)



The group velocity is,

$$u_g = \frac{\overline{P_1P_4}}{\Delta t} = \frac{\overline{P_1P_2}\cos\theta}{\Delta t} = \frac{c}{\sqrt{\varepsilon_r}}\cos(\theta) = \frac{c}{\sqrt{\varepsilon_r}}\frac{\beta}{k}$$

$$\cos(\theta) = \frac{\beta_1}{k}$$

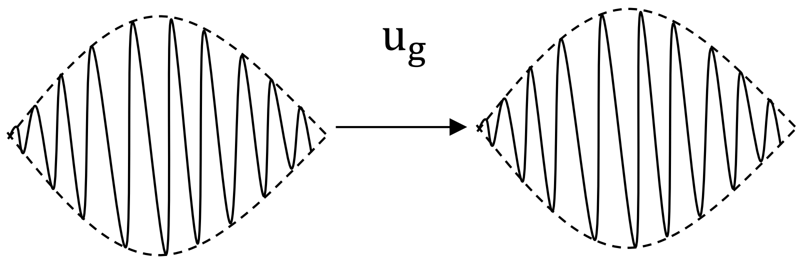
We thus have,

$$u_p \cdot u_g = \frac{c^2}{\varepsilon_r}$$

(generally applicable to any uniformly filled waveguides)

"Information theory" view of u_p vs u_g

Group velocity: Propagation speed of an envelop (signal group) modulated on a carrier



Original modulated signal:

$$e^{j(\omega_0+\Delta\omega)t} = \underbrace{e^{j\omega_0 t}}_{\text{carrier}} \cdot \underbrace{e^{j\Delta\omega t}}_{\text{envelop}}$$

At a distance z, the received signal is:

$$e^{j[(\omega_0+\Delta\omega)t-\beta(\omega_0+\Delta\omega)z]} = e^{j\left\{\omega_0 t+\Delta\omega t-\left[\beta(\omega_0)+\Delta\omega\frac{\Delta\beta}{\Delta\omega}\right]z\right\}}$$

$$= \underbrace{e^{j(\omega_0 t-\beta z)}}_{\text{carrier}} \cdot \underbrace{e^{j\Delta\omega(t-\frac{\Delta\beta}{\Delta\omega}z)}}_{\text{envelop}}$$

time delay

The time delay for the envelop is,

$$\tau = \frac{\Delta\beta}{\Delta\omega}z = \frac{d\beta}{d\omega}z$$ (group delay)

Therefore the group velocity is obtained as,

$$u_g = \frac{z}{\tau} \Rightarrow \frac{1}{u_g} = \frac{d\beta(\omega)}{d\omega} \Rightarrow \boxed{u_g = \frac{d\omega}{d\beta}} \Rightarrow$$ recall $$u_p = \frac{\omega}{\beta}$$

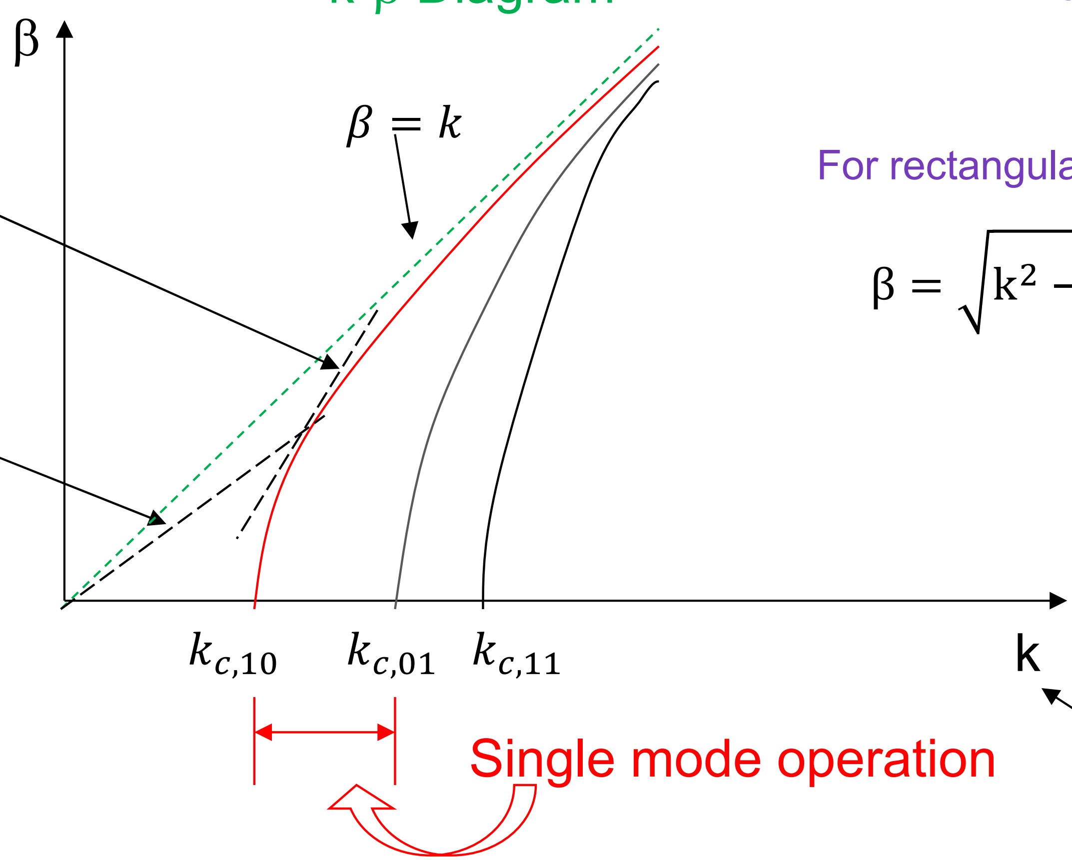
Dispersion relationship in a Waveguide

For each mode, β is a function of the wave number k, this relationship is called **dispersion relationship**

k-β Diagram ← For general metallic waveguides

$$\frac{1}{v_g} = \frac{d\beta}{d\omega} = \frac{1}{c}\frac{d\beta}{dk}$$

$$\frac{1}{v_p} = \frac{\beta}{\omega} = \frac{1}{c}\frac{\beta}{k}$$



$$k = \left(\frac{\omega}{c/\sqrt{\varepsilon_r}}\right)$$

For rectangular waveguides,

$$\beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Wave velocities in presence of dispersion

Center frequency

$$f_0 = 9\ \text{GHz}$$

Spectral components

$$f_m = f_0 + m\Delta f \quad m = 0 \ldots 30$$

$$\Delta f = 500\ \text{MHz}$$

Wavenumber k

$$k_m = \frac{2\pi f_m}{c_0}$$

Propagation constant

$$\boxed{\beta_m = (k_m)^{1.25}}$$

Exaggerated

Spectral component phasors

$$E_m(z) = A_m e^{-j\beta_m z}$$

Band limit

$$A_m = e^{-\frac{1}{2}\left(\frac{4.5n}{L-1}\right)^2}$$

Total Field phasor

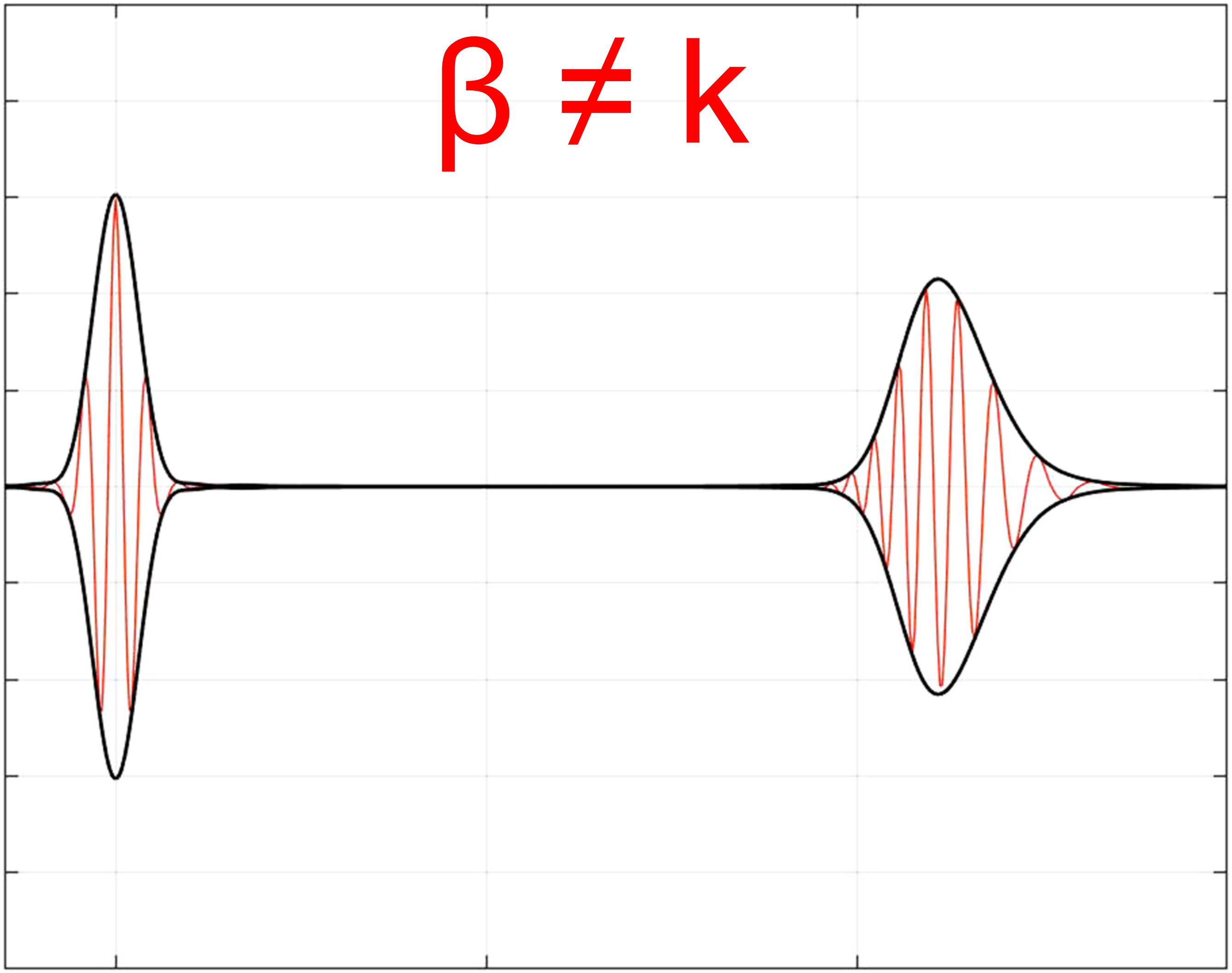
$$E(z) = \sum_{m=0}^{30} E_m(z) = \sum_{m=0}^{30} A_m e^{-j\beta_m z}$$

Total real field

$$E(z,t) = \mathbb{R}\text{e}\left\{\sum_{m=0}^{30} E_m(z) e^{-j\omega t}\right\} = \sum_{m=0}^{30} A_m \cos(\omega t - \beta_m z)$$

- Make a wave packet with 31 equally spaced (in frequency) spectral components
- Apply slight dispersion
  - $\beta = k^{1.25}$
- Apply window to taper edges and reduce ringing

Wave velocity in presence of dispersion

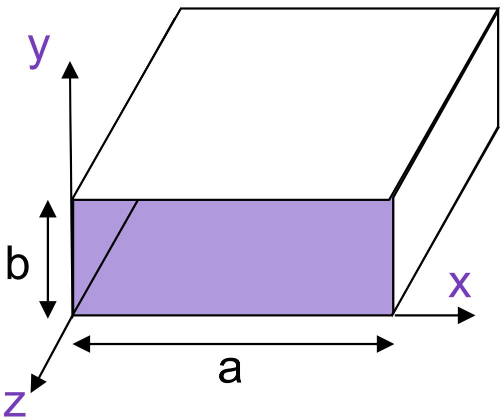


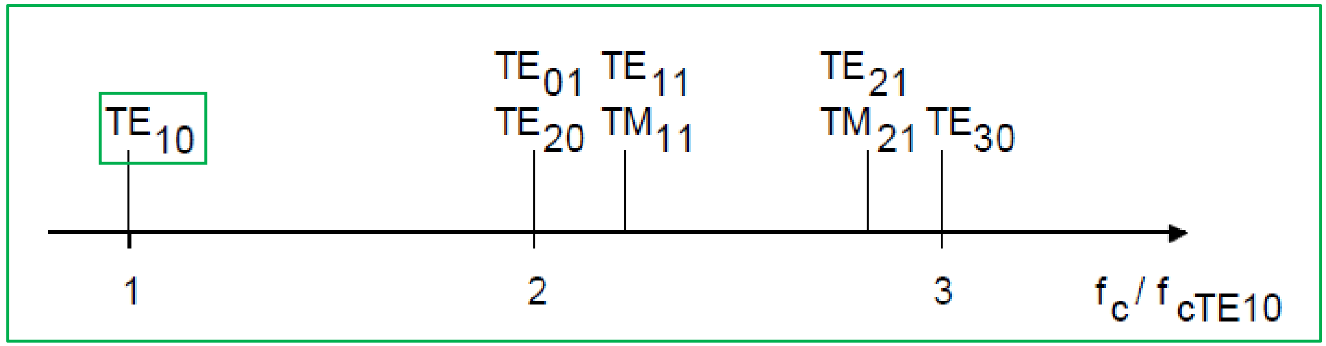
Propagation constant

$$\beta_m = (k_m)^{1.25}$$

- Waveform in **red**
- Envelope in **black**
- Phase velocity variation arises from beta not a linear function
- Pick a peak on the red curve. Note that the red curve peak moves faster than the black curve peak
  - $u_p > u_g$

In class exercise 1



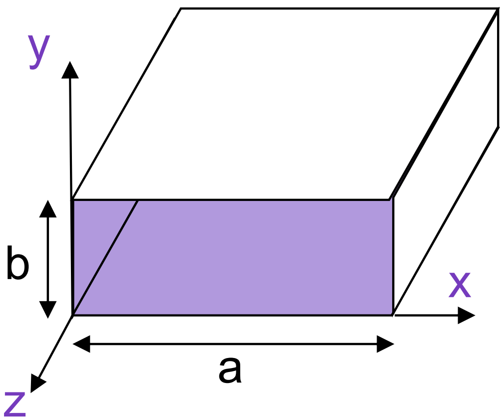


Cutoff frequency

$$f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{c}{2\sqrt{\varepsilon_r}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

- Consider the propagating waveguide modes above as a function of normalized fundamental mode frequencies
- Derive an expression for a and b such that the fundamental waveguide mode spans normalized bandwidth [1, 2]
- Assume a>b

In class exercise 1



Cutoff frequency

$$f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{c}{2\sqrt{\varepsilon_r}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

TE_{10} cutoff frequency

$$f_{c,TE,10} = \frac{c}{2\sqrt{\varepsilon_r}}\frac{1}{a}$$

Normalized cutoff frequency

$$\frac{f_c}{f_{c,TE,10}} = a\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Normalized cutoff frequency for TE_{20}

$$\frac{f_c}{f_{c,TE,10}}(m = 2, n = 0) = a\sqrt{\left(\frac{2}{a}\right)^2} = 2$$

Normalized cutoff frequency for TE_{01}

$$\frac{f_c}{f_{c,TE,10}}(m = 0, n = 1) = a\sqrt{0 + \left(\frac{1}{b}\right)^2} = \frac{a}{b}$$

$$\boxed{\frac{a}{b} > 2 \quad \rightarrow \quad a > 2b}$$

- If a < 2b, the normalized bandwidth is less than 1
- If a > 2b, the normalized bandwidth = 1

Current density in the TE_{10} mode

TE boundary conditions

Generalized on top of PEC

$$\text{B.C.}\begin{cases} E_{t1} = 0 \\ H_{n1} = 0 \\ \mathbf{a_n} \times \mathbf{H_1} = \mathbf{J_S} \end{cases}$$

Tangential Components (u, v)

Normal Components (n)

The curl equation

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$

$$\begin{vmatrix} a_u & a_v & a_n \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial n} \\ H_u & H_v & \cancel{H_n} \end{vmatrix} = j\omega\varepsilon(\mathbf{a_u}\cancel{E_u} + \mathbf{a_v}\cancel{E_v} + \mathbf{a_n}E_n)$$

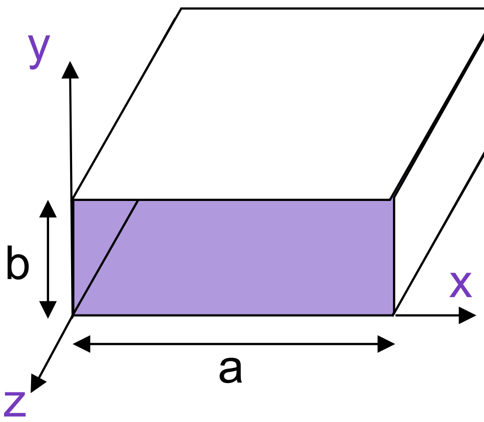
Compare the tangential components (in u, v)

It yields
$$\begin{cases} \dfrac{\partial}{\partial n}H_v = j\omega\varepsilon E_u = 0 \\ \dfrac{\partial}{\partial n}H_u = j\omega\varepsilon E_v = 0 \end{cases}$$

Therefore we have on a PEC
$$\begin{cases} \boxed{\dfrac{\partial H_{t1}}{\partial n} = 0} \\ E_{t1} = 0 \end{cases}$$

(implicit B.C.)

Dominant TE_{10} mode



The cutoff wave number is, $k_{c,10} = \dfrac{\pi}{a}$

The phase constant is thus given by,

$$\beta_{10} = \sqrt{k^2 - k_{c,10}^2} = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2} = k\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}$$

The cutoff wavelength is, $\lambda_{c,10} = \dfrac{2\pi}{k_c} = \dfrac{2\pi}{\pi/a} = 2a$

(the waveguide has to be at least greater than half of the wavelength in order for the wave to propagate)

The cutoff frequency is thus: $f_{c,10} = \dfrac{c/\sqrt{\varepsilon_r}}{\lambda_{c,10}} = \dfrac{c/\sqrt{\varepsilon_r}}{2a} = \dfrac{1}{2a\sqrt{\mu\varepsilon}}$

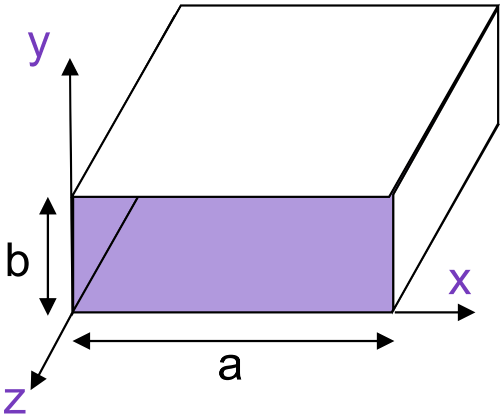
The guide wavelength:

$$\lambda_g = \frac{2\pi}{\beta_{10}} = \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}},$$

The phase velocity:

$$v_p = \frac{\omega}{\beta_{10}} = \frac{c/\sqrt{\varepsilon_r}}{\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}} = \frac{c/\sqrt{\varepsilon_r}}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}},$$

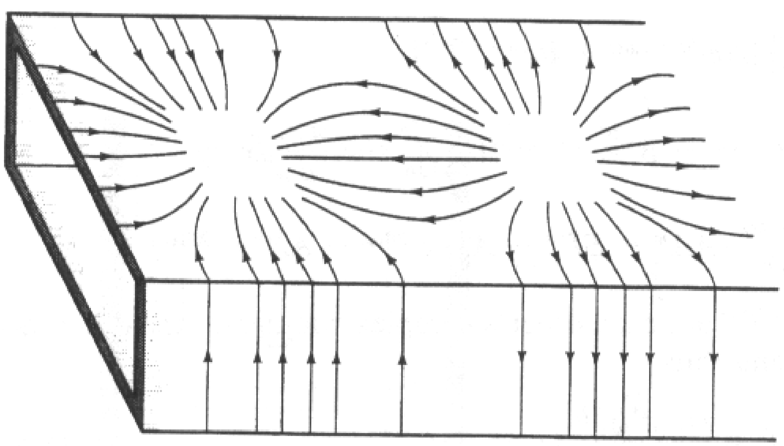
TE_{10} mode Surface Current density



TE_{10} fields

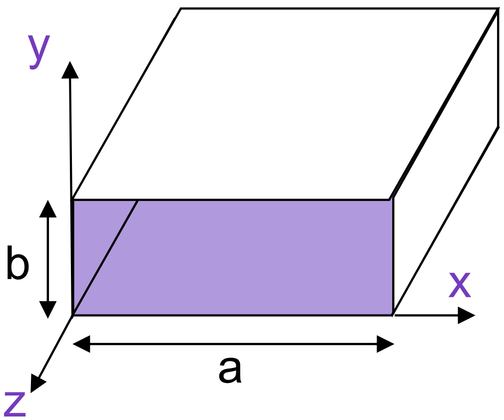
$$
\begin{cases}
E_y = \dfrac{-j\omega\mu a}{\pi} B_{10} \sin\left(\dfrac{\pi x}{a}\right) e^{-j\beta z} \\[2ex]
H_x = \dfrac{j\beta a}{\pi} B_{10} \sin\left(\dfrac{\pi x}{a}\right) e^{-j\beta z} \\[2ex]
H_z = B_{10} \cos\left(\dfrac{\pi x}{a}\right) e^{-j\beta z} \\[2ex]
E_z = E_x = H_y = 0
\end{cases}
$$

J_s plot for TE_{10} mode



- Lets use the fields and the boundary conditions to compute surface current densities (J_s) on the waveguide walls

TE_{10} mode Surface Current density



Boundary conditions: $\mathbf{J_s} = \mathbf{a_n} \times \mathbf{H_1}$

$$\begin{cases} H_z = A_{10}\cos\left(\frac{\pi x}{a}\right) e^{-j\beta z} \\ H_x = \frac{j\beta a}{\pi} A_{10}\sin\left(\frac{\pi x}{a}\right) e^{-j\beta z} \end{cases}$$

Side wall (x=0):

$H_x = 0, H_z \neq 0$

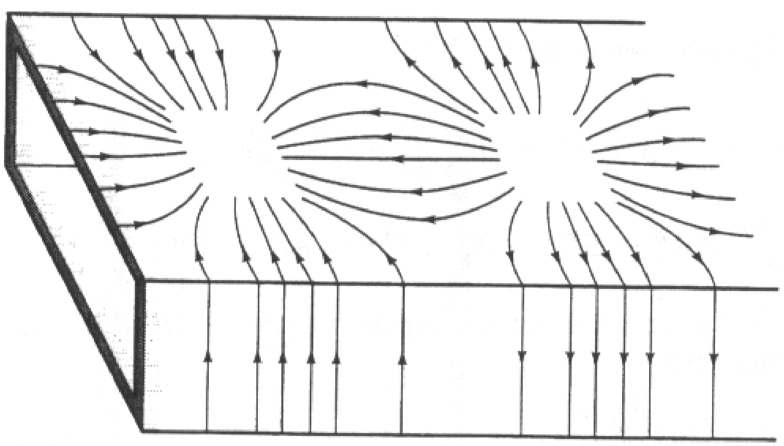
$\boxed{\mathbf{a_n} = \mathbf{a_x}}$

$$\mathbf{J_s} = \mathbf{a_x} \times \mathbf{a_z} H_z = -\mathbf{a_y}\cos(0)\, A_{10} e^{-j\beta z}$$

$$\boxed{\mathbf{J_s} = -\mathbf{a_y} A_{10} e^{-j\beta z}}$$

Mirror image

$\mathbf{J_s}$ plot for TE_{10} mode



Side wall (x=a):

$H_x = 0, H_z \neq 0$

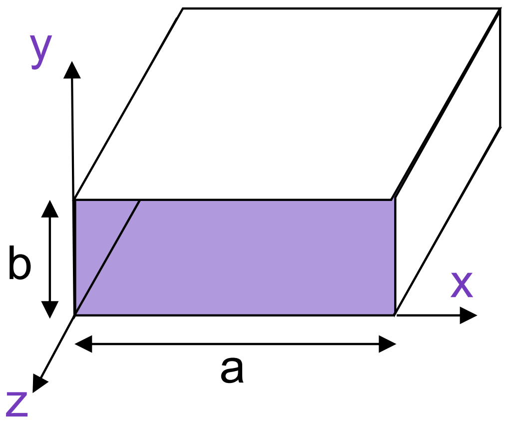
$\boxed{\mathbf{a_n} = -\mathbf{a_x}}$

Pointing into the dielectric

$$\mathbf{J_s} = -\mathbf{a_x} \times \mathbf{a_z} H_z = \mathbf{a_y}\cos(\pi)\, A_{10} e^{-j\beta z}$$

$$\boxed{\mathbf{J_s} = -\mathbf{a_y} A_{10} e^{-j\beta z}}$$

TE_{10} mode Surface Current density



Mirror image
180º out of phase

Bottom wall (y=0):

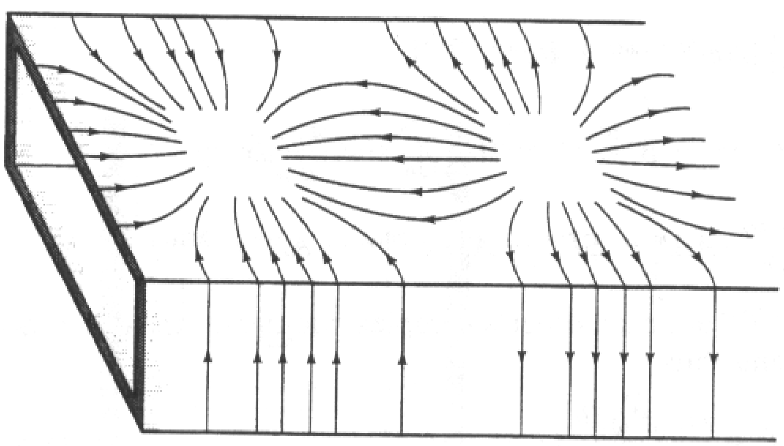
$\boxed{\mathbf{a_n} = \mathbf{a_y}}$

Pointing into the dielectric

$$\mathbf{J}_s = \mathbf{a_y} \times (\mathbf{a_z} H_z + \mathbf{a_x} H_x)$$

$$\boxed{\mathbf{J}_s = \left(\mathbf{a_x} A_{10} \cos \frac{\pi x}{a} - \mathbf{a_z} A_{10} \frac{j\beta_{10} a}{\pi} \sin \frac{\pi x}{a} \right) e^{-j\beta z}}$$

$\mathbf{J_s}$ plot for TE_{10} mode



Top wall (y=b):

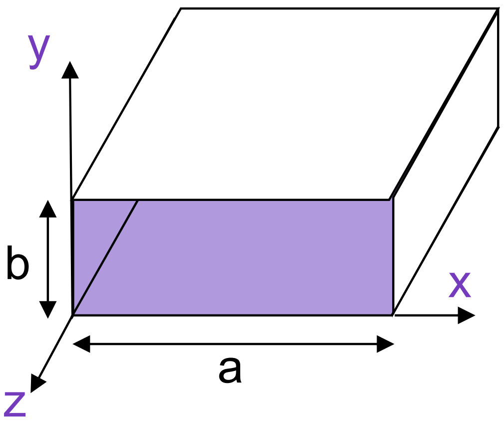
$\boxed{\mathbf{a_n} = -\mathbf{a_y}}$

Pointing into the dielectric

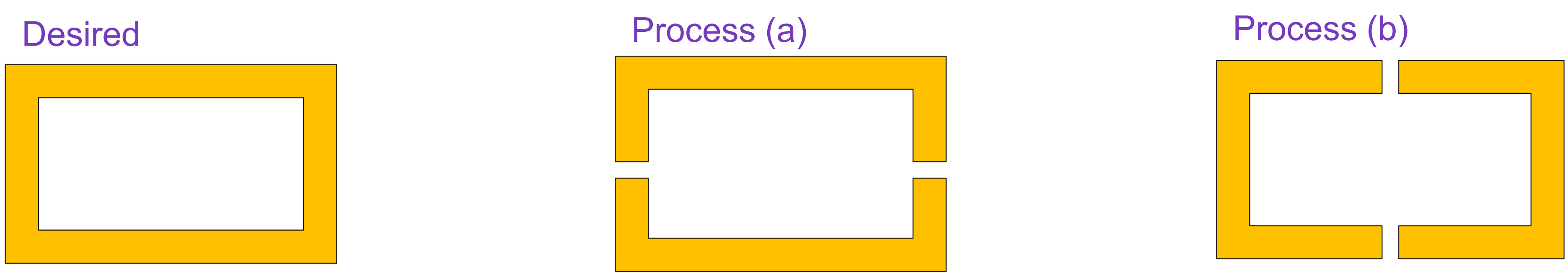
$$\mathbf{J}_s = -\mathbf{a_y} \times (\mathbf{a_z} H_z + \mathbf{a_x} H_x)$$

$$\boxed{\mathbf{J}_s = \left(-\mathbf{a_x} A_{10} \cos \frac{\pi x}{a} + \mathbf{a_z} A_{10} \frac{j\beta_{10} a}{\pi} \sin \frac{\pi x}{a} \right) e^{-j\beta z}}$$

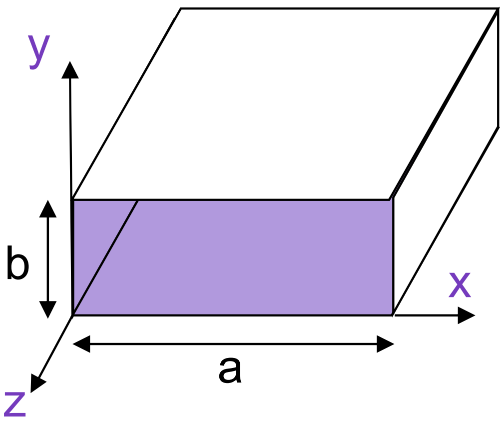
In class exercise 2



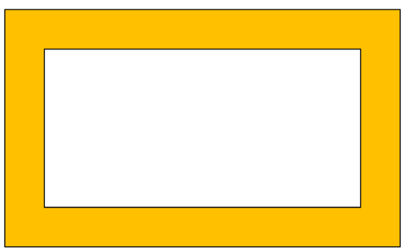
- Its difficult to cut a deep hole with a rectangular cross section
- Rectangular cross section pipes can be extruded but not with sufficient tolerance for high frequency operation
- High frequency waveguides are often manufactured in split blocks where two rectangular trenches are cut and then the two halves are pieced together
- Which process would you use to create a high frequency rectangular waveguide (a) or (b) and why? Consider only the TE_{10} mode

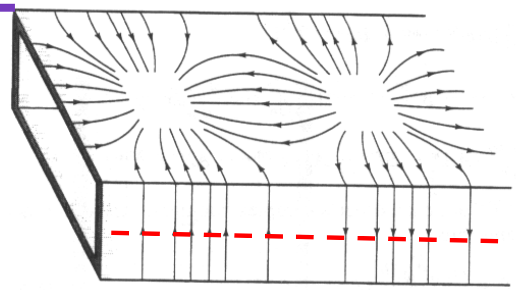


In class exercise 2



Desired

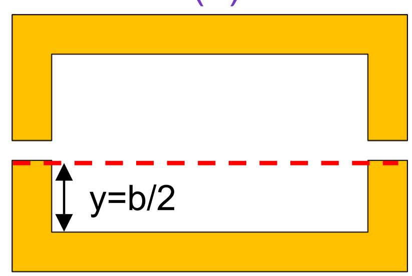


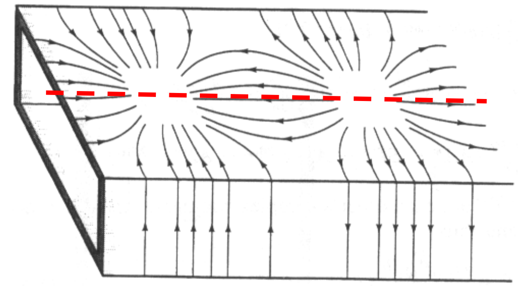


$$\mathbf{J_s}(\boldsymbol{x} = \mathbf{0}, \boldsymbol{y} = \boldsymbol{b}/\mathbf{2}) = \mathbf{a_y} \mathrm{A_{10}} \mathrm{e}^{-\mathrm{j}\beta\mathrm{z}}$$

$$\mathbf{J_s}(\boldsymbol{x} = \boldsymbol{a}, \boldsymbol{y} = \boldsymbol{b}/\mathbf{2}) = \mathbf{a_y} \mathrm{A_{10}} \mathrm{e}^{-\mathrm{j}\beta\mathrm{z}}$$

Process (a)

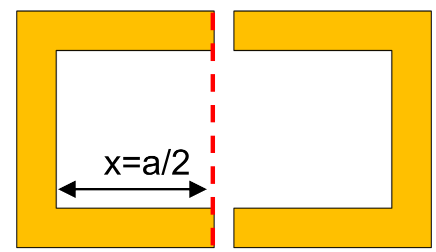




$$\mathbf{J_s}\left(\boldsymbol{x} = \frac{\boldsymbol{a}}{\mathbf{2}}, \boldsymbol{y} = \mathbf{0}\right) = \mathbf{a_z} A_{10} \frac{j\beta_{10} a}{\pi} e^{-j\beta z}$$

$$\mathbf{J_s}\left(\boldsymbol{x} = \frac{\boldsymbol{a}}{\mathbf{2}}, \boldsymbol{y} = \mathbf{0}\right) = -\mathbf{a_z} A_{10} \frac{j\beta_{10} a}{\pi} e^{-j\beta z}$$

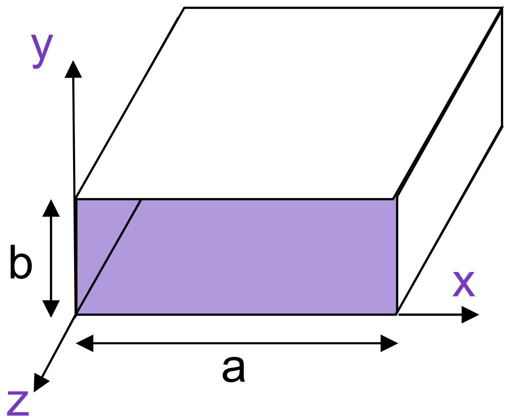
Process (b)



- The break in the metal creates resistance.
- In process (a) the current is <u>orthogonal</u> to the break and therefore must cross it
- In process (b) the current is <u>parallel</u> to the break and therefore never traverses the break

Conclusions

Real WG



Some standard waveguide specs.

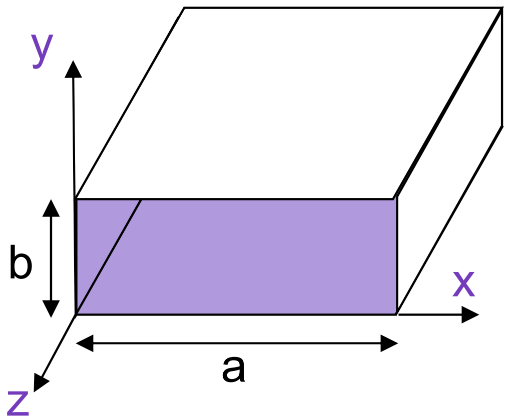
Guide	Size (inch)	Rec. (GHz)	f_c (GHz)	Band	(GHz)
WR650	6.500×3.250	1.12 - 1.70	0.91	L	(1.0 - 2.0)
WR284	2.840×1.340	2.60 - 3.95	2.08	S	(2.0 - 4.0)
WR187	1.872×0.872	3.95 - 5.85	3.15	C	(4.0 - 8.0)
WR90	0.900×0.400	8.20 - 12.40	6.56	X	(8.0 - 12.0)
WR62	0.622×0.311	12.40 - 18.00	9.49	Ku	(12.0 - 18.0)
WR42	0.420×0.170	18.00 - 26.50	14.05	K	(18.0 - 27.0)
WR28	0.280×0.140	26.50 - 40.00	21.08	Ka	(27.0 - 40.0)

a b $f_c(TE_{10})$

$<f_c(TE_{20})$

- All the above waveguides follow the convention **a >= 2b**
- They are all intended to operate in the TE_{10} (lowest order mode)
- Recommended operating range provides buffer above the TE10 cutoff frequency and below the TE_{20} and TE_{01} cutoff frequencies

Propagation



$$\beta \neq k$$

- The one conductor geometry supports TE/TM operation
- The longitudinal phase variation of a TE/TM is not equal to the free space (plane wave) TEM phase variation
  - $\beta \rightarrow$ rectangular waveguide longitudinal phase variation
  - $k \rightarrow$ free space longitudinal phase variation
- β is a strong function of frequency and geometry

Next time

- Use surface current densities to compute loss and realistic performance of waveguide