PHYS-E0461 Introduction to plasma physics for fusion and space applications Exercise set 11

1. (**Just for fun**) (4p)

   (a) Why not produce fusion energy with today's tokamaks?
   Consider the medium-size, high-performance tokamak ASDEX Upgrade at Max Planck institute, Garching, Germany. Its size is specified by its major radius of 1.75 m and minor radius of 0.6 m. Its magnetic cage is specified by the toroidal magnetic field of 2.5 T and plasma current of 1 MA. If real DT fuel were used, we would produce alpha particles with energy of 3.5 MeV. Estimate the Larmor radius and banana width of these particles in a tokamak of the size of ASDEX Upgrade. How well do you think they would be confined? (Hints: For banana width you can use the *poloidal Larmor radius.* Make a sensible assumption about the velocity direction.) (2p)

   (b) Suppose that some electromagnetic instability in a D-D reactor limits the plasma beta β to $(m_e/M_i)^{1/2}$, where M_i is the mass of a deuteron. Let the magnetic field be limited to 20 T (by the strength of materials). If $T_e = T_i = 20\,\mathrm{keV}$, find the maximum plasma density that can be contained. (2p)

2. (**On ohmic heating**) (6p)

   Let's try to heat a tokamak plasma ohmically. Let the (constant) plasma current density along $\boldsymbol{B}$ be $j = 10^5\,\mathrm{A\,m^{-2}}$, and assume constant plasma density at $n = 10^{19}\,\mathrm{m^{-3}}$. The ohmic heating, $dw/dt = \eta \cdot j^2$, goes predominantly to electrons. Here $w = n_e T_e$ is the electron energy density. Assuming the so-called *Spitzer* resistivity, $\eta = 5.2 \times 10^{-5} \cdot Z \ln(\Lambda)/(T_{e,\mathrm{eV}}^{3/2})\ \Omega\,\mathrm{m}$ ($T_{e,\mathrm{eV}} = T_e/1\mathrm{eV}$), calculate the time dependence of the electron temperature! Assume a hydrogen plasma, so that $Z = 1$, and a constant Coulomb logarithm of $\ln\Lambda = 10$. If we start with $T_e = 10\,\mathrm{eV}$, how long does it take to reach 1 keV? How about the more fusion-relevant temperature of 10 keV?

3. (**Finding an instability 'our way', part 1**) (6p)

   In the lectures, fairly sophisticated methods for analyzing plasma instability were introduced. However, we can apply our existing tools from wave analysis to get some idea about instabilities called *two-stream instabilities.* An instability has to be fed to grow, and in this case it comes from the relative velocity of two species. Let us analyze the simplest possible case: an electron *stream* with the uniform velocity $\boldsymbol{V}_0$ streaming through a stationary bulk of ions.

   (a) Write down the relevant equations for electrons and ions assuming a cold ($T_e, T_i \approx 0$), initially homogeneous and steady-state plasma, and linearize them. There is no magnetic field and, thus, no preferred direction. (2p)

   (b) Now take the *real* plane wave trial solution for the electric field that arises, $E(x,t) = E_0 \exp(ikx - i\omega t)$, to find expressions for the perturbed electron and ion densities. (2p)

   (c) Using the expressions for the perturbed densities found in (b), use the Poisson equation to derive the dispersion relation (2p)
   $$1 = \omega_p^2 \left[\frac{m/M}{\omega^2} + \frac{1}{(\omega - kv_0)^2} \right].$$

4. (**Finding an instability 'our way', part 2**) (6p)

   To have an instability, the frequency ω has to be complex: $\omega = \omega_r + i\omega_i$. Thus, the electrostatic perturbation becomes $E(x,t) = E_0 e^{ikx - i\omega_r t} e^{\omega_i t}$, i.e., possibly growing exponentially. The dispersion relation is a 4th order equation for ω, so instead of trying to solve it, let's try to see when it gives an instability, i.e., is it possible that some of the roots are NOT real. Take shorthand notations $x \equiv \omega/\omega_p$ and $a \equiv kV_0/\omega_p$, and re-write the dispersion relation in problem 3 using these terms. Then call the function on the right-hand side $f(x)$, sketch its behaviour, and determine under what circumstances this function crosses the value 1. These are the *real* solutions to the dispersion relation. If there are less than four real solutions, the system can exhibit an instability. Can you find a condition for the parameter a that corresponds to the dispersion relation having four real solutions? (Hint: do not expect to find 'A Given Answer', but try to see how things depend on the parameter a, i.e., on the wave vector k.)

5. (**Food for thought**)

   During lectures, the DT reaction was always brought up as The Energy Producing Reaction due to its higher fusion cross section. However, for a more distant future, one considers using DD reactions. So, there must be pros and cons for both reactions. Analyze them. Return your short write-up in MyCourses before the next lecture.