Problem Set 6

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26/11/2021

Exercise 1 - PS6

A consumer's utility function is

$$u(x, y) = 2 \ln x + 3 \ln y.$$

The consumer's budget constraint is

$$p_x x + p_y y \leq w,$$

where the parameters p_x, p_y and w are all strictly positive.

(a) Solve the consumer's utility maximization problem.

(b) Use the envelope theorem to estimate the change in the indirect utility function (i.e., the problem's value function) when the price p_y is changed to $p_y + \epsilon$, with $\epsilon > 0$.

(c) Use the envelope theorem to estimate the change in the indirect utility function (i.e., the problem's value function) when the utility function is changed to $\tilde{u}(x, y) = 2.2 \ln x + 3 \ln y$.

Exercise 1 - Solution

The Lagrangian is

$$\mathcal{L} = 2\ln x + 3\ln y - \lambda(p_x x + p_y y - w)$$

$$x^* = \frac{2w}{5p_x}, \quad y^* = \frac{3w}{5p_y}, \quad \lambda^* = \frac{5}{w}.$$

Let v be the indirect utility function of this problem. By the envelope theorem,

$$\frac{dv}{dp_y} = \frac{\partial \mathcal{L}}{\partial p_y}(x^*, y^*, \lambda^*) = -\lambda^* y^* = -\frac{3}{p_y}.$$

Since $dp_y = \epsilon$, we have $dv = -\frac{3}{p_y} dp_y = -\frac{3\epsilon}{p_y}$.

Similarly, let's rewrite the utility function as $u(x, y) = a\ln x + 3\ln y$. By the envelope theorem, and by evaluating all terms at $a = 2$,

$$\frac{dv}{da} = \frac{\partial \mathcal{L}}{\partial a}(x^*, y^*, \lambda^*) = \ln x^* = \ln(2w) - \ln(5p_x).$$

Since $da = 0.2$, we have

$dv = (\ln(2w) - \ln(5p_x))\, da = 0.2\,(\ln(2w) - \ln(5p_x)).$

Exercise 2

Consider the following maximization problem:

$$\begin{aligned} \max_{x,y} \quad & x + y \\ \text{s.t.} \quad & xy \geq 1. \end{aligned}$$

(a) Check if the non-degeneracy constraint qualification (NDCQ) is satisfied.

(b) Form the Lagrangian and find all the points that satisfy the first-order conditions for optimality.

(c) Is any point you found in (b) a solution to this problem? How can you reconcile your results with the Proposition at p. 12 in the slides from Lecture 14?

Exercise 2 - Solution

The NDCQ fails only at $(0, 0)$, which does not belong to the constraint set. Hence the NDCQ is always satisfied. The only point which satisfies the first-order conditions is $x = y = -1$ and $\lambda = 1$. However, this is not a solution. In fact, this problem does not have any solution at all. Notice that, when x and y are both positive, it is always feasible to increase either of them without bound, so leading to arbitrarily large values of the objective function. This is consistent with the Proposition at p. 12, which asserts that, if a solution exists, then there must be a Lagrange multiplier as well. However, the Proposition does not say that the existence of Lagrange multiplier (which is what happens in this problem) guarantees the existence of a solution.

Exercise 3

Consider the following constrained maximization problem:

$$\max_{x,y,z} \quad f(x,y,z) = ax + by + cz$$
$$\text{s.t.} \quad g(x,y,z) = \alpha x^2 + \beta y^2 + \gamma z^2 \leq L,$$

where $a, b, c, \alpha, \beta, \gamma, L$ are all positive parameters.

(a) Solve the maximization problem.

(b) Let V be the value function of this maximization problem, $\mathcal{L}$ the corresponding Lagrangian function and λ the Lagrange multiplier. Verify that:

$$\frac{dV}{dL} = \lambda^*,$$

$$\frac{dV}{da} = \frac{\partial \mathcal{L}}{\partial a}(x^*, y^*, z^*),$$

$$\frac{dV}{d\beta} = \frac{\partial \mathcal{L}}{\partial \beta}(x^*, y^*, z^*).$$

Exercise 3 - Solution

The Lagrangian is

$$\mathcal{L} = ax + by + cz - \lambda(\alpha x^2 + \beta y^2 + \gamma z^2 - L)$$

The first order conditions are

$$a - 2\lambda\alpha x = 0 \tag{1}$$

$$b - 2\lambda\beta y = 0 \tag{2}$$

$$c - 2\lambda\gamma z = 0 \tag{3}$$

$$\lambda(\alpha x^2 + \beta y^2 + \gamma z^2 - L) = 0 \tag{4}$$

$$\lambda \geq 0 \tag{5}$$

Exercise 3 - Solution

a) $a, b, c, \alpha, \beta, \gamma, L$ are all positive. Therefore, from (1), (2), (3), λ cannot be 0. Then, x,y, z are also positive, and $\alpha x^2 + \beta y^2 + \gamma z^2 - L = 0$. From (1), (2), (3), $x = \frac{a}{2\alpha\lambda}$, $y = \frac{b}{2\beta\lambda}$, $z = \frac{c}{2\gamma\lambda}$. This is the solution of the maximization problem because L is concave.

Exercise 3 - Solution

b) From a,

$$\alpha x^2 + \beta y^2 + \gamma z^2 - L = 0 \tag{6}$$

$$\alpha {\frac{a}{2\alpha\lambda^*}}^2 + \beta {\frac{b}{2\beta\lambda^*}}^2 + \gamma {\frac{c}{2\gamma\lambda^*}}^2 - L = 0 \tag{7}$$

$$\lambda^* = \frac{1}{2} L^{\frac{-1}{2}} \sqrt{\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma}} \tag{8}$$

The value function is:

$$V = ax^* + by^* + cz^* = \frac{a^2}{2\alpha\lambda^*} + \frac{b^2}{2\beta\lambda^*} + \frac{c^2}{2\gamma\lambda^*} = \sqrt{L}\sqrt{\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma}} \tag{9}$$

Exercise 3 - Solution

$$\frac{dV}{dL} = \lambda^*$$

$$\frac{dV}{da} = \frac{1}{2}\sqrt{L}\frac{1}{\sqrt{\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma}}}\frac{2a}{\alpha} = \frac{a}{2\alpha\lambda^*} = \frac{\partial \mathcal{L}}{\partial a}$$

$$\frac{dV}{d\beta} = \frac{1}{2}\sqrt{L}\frac{1}{\sqrt{\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma}}}\frac{-b^2}{\beta} = \frac{-b^2}{4\beta\lambda^*} = -\lambda^* y^2 = \frac{\partial \mathcal{L}}{\partial \beta}$$

Exercise 4

Find the solutions of the following difference equations with the given initial values of x_0. In addition, determine whether the solutions are stable or not.

(a) $x_{t+1} = 3x_t - 5$, $x_0 = 3$;

(b) $4x_{t+1} = x_t + 2$, $x_0 = 2$;

(c) $-2x_{t+1} + 6x_t + 4 = 0$, $x_0 = 5$;

(d) $2x_{t+1} - 2x_t + 3 = 0$, $x_0 = 6$.

Exercise 4 - Solution

We can rewrite all the equations in the form $x_{t+1} = ax_t + b$. When $a \neq 1$, the solution is

$$x_t = a^t \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$$

(a) $x_t = \frac{1}{2} \cdot 3^t + \frac{5}{2}$; unstable because $a > 1$

(b) $x_t = \frac{4}{3} \left(\frac{1}{4} \right)^t + \frac{2}{3}$; stable because $|a| < 1$

(c) $x_t = 6 \cdot 3^t - 1$; unstable because $a > 1$

(d) Here $a = 1$ so we cannot use the formula above. The solution is $x_t = 6 - \frac{3}{2} t$, which is unstable

Exercise 5

Consider the following matrix:

$$A = \begin{pmatrix} 6 & -8 \\ 4 & -12 \end{pmatrix}.$$

(a) Find all the eigenvalues of A.

(b) Find an eigenvector for each eigenvalue of A.

(c) Use the eigenvalues and the eigenvectors you found to form a square matrix P and a diagonal matrix D such that

$$A = PDP^{-1}. \tag{10}$$

Verify that (10) is satisfied by the matrices you found.

(d) Use (10) to calculate A^n, where $n \geq 1$.

Exercise 5 - Solution

(a) The two eigenvalues are $\lambda_1 = -10$ and $\lambda_2 = 4$.

(b) Two possible eigenvectors are

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

(c) The decomposition is

$$A = PDP^{-1} = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -10 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -\frac{1}{7} & \frac{4}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix}$$

(d)

$$A^n = PD^nP^{-1} = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} (-10)^n & 0 \\ 0 & 4^n \end{pmatrix} \begin{pmatrix} -\frac{1}{7} & \frac{4}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix}$$