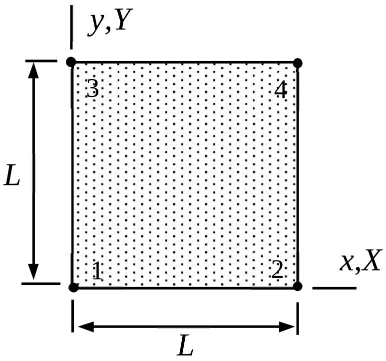
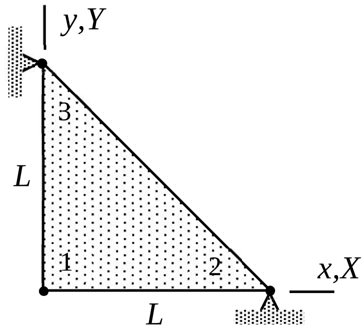
MEC-E1050 Finite Element Method in Solid, week 48/2021

1. Determine stress components at the midpoint of element shown if u_{Y2} is non-zero and the other nodal displacements are zeros. The approximations to the displacement components u, v are bi-linear. The material parameters E, ν and thickness t are constants. Use the strain-displacement and stress-strain relationship of linearly elastic isotropic material and assume plane stress conditions.



Answer $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \dfrac{u_{Y2}}{2L} \dfrac{E}{1-\nu^2} \begin{Bmatrix} -\nu \\ -1 \\ (1-\nu)/2 \end{Bmatrix}$

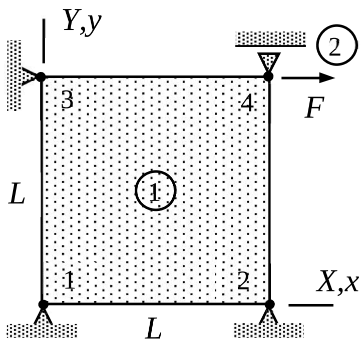
2. Determine the stress components σ_{xx}, σ_{yy}, and σ_{xy} of the triangle element shown in terms of the displacement components u_{X1}, u_{Y1} of node 1. Assume plane-strain conditions and use linear approximation to displacement components. The material parameters E, ν and thickness t are constants.



Answer

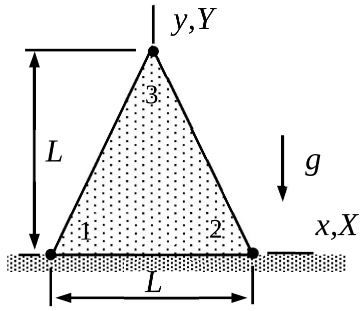
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = -\frac{E}{L(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \\ (1-2\nu)/2 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}$$

3. A thin slab (1) of square shape is loaded by a point force (2) as shown in the figure. Derive the relationship between the force F and the displacement u_{X4} of its point of action. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. External distributed forces vanish. Assume plane stress conditions and use a bilinear approximation.



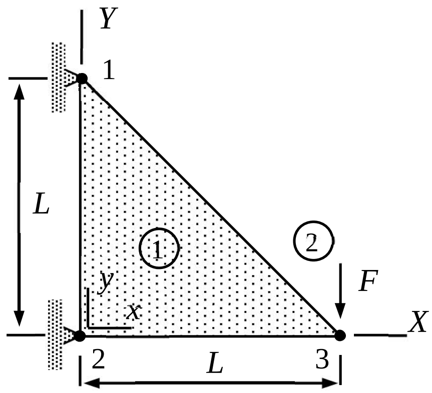
Answer $u_{X4} = \dfrac{6F}{Et} \dfrac{1-\nu^2}{3-\nu}$

4. A long wall having triangular cross-section, and made of homogeneous, isotropic, linearly elastic material, is subjected to its own weight. Material properties E, ν, ρ are constants. Determine the displacement components u_{X3} and u_{Y3} of node 3. Nodes 1 and 2 are fixed. Use just one three-node element and assume plane strain conditions.



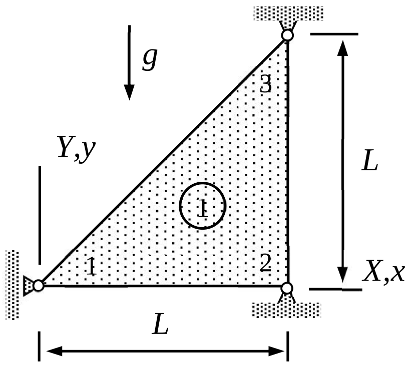
Answer $u_{X3} = 0$, $u_{Y3} = -\frac{1}{3}\frac{(1+\nu)(1-2\nu)}{1-\nu}\frac{\rho g L^2}{E}$

5. A thin triangular slab of thickness t is loaded by a point force at node 3. Nodes 1 and 2 are fixed. Derive the virtual work expression δW of the structure in terms of u_{X3} and u_{Y3}, and solve for the nodal displacements. Approximation is linear and material parameters E and ν are constants. Assume plane-stress conditions.



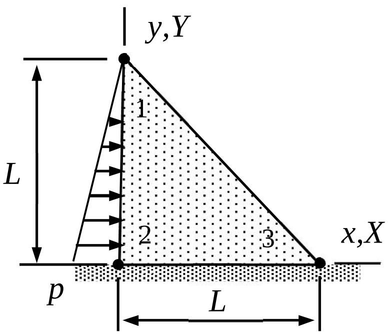
Answer $u_{X3} = 0$, $u_{Y3} = -4\frac{F}{tE}(1+\nu)$

6. A thin triangular slab (assume plane stress conditions) loaded by its own weight is allowed to move vertically at node 1 and nodes 2 and 3 are fixed. Find the displacement u_{Y1}. Material parameters E, ν , ρ and thickness t of the slab are constants.



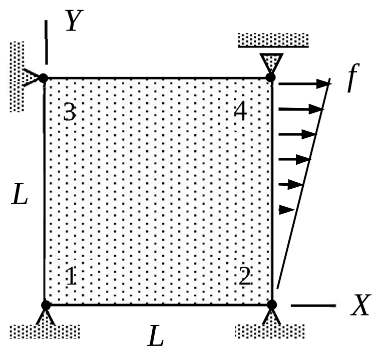
Answer $u_{Y1} = -\frac{2}{3}(1+\nu)\frac{\rho g L^2}{E}$

7. A long dam of homogeneous, isotropic, linearly elastic material, is subjected to water pressure on one side. Material properties E and ν are constants. Determine the displacement components u_{X1} and u_{Y1} of node 1. Nodes 2 and 3 are fixed. Use a three-node element and assume plane strain conditions. Consider a slab of thickness t in calculations. The peak value of the linearly varying pressure is p.



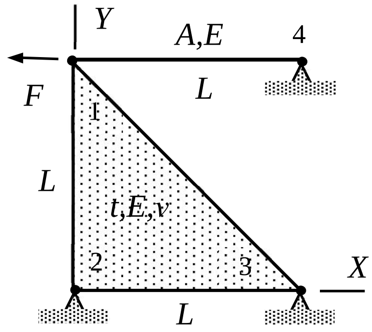
Answer $u_{X1} = \frac{2}{3}\frac{pL}{E}(1+\nu)$, $u_{Y1} = 0$

8. A thin slab is loaded by a distributed force as shown. Derive the relationship between the force peak value f and displacement u_{X4}. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. Assume plane stress conditions and use the virtual work density of the thin slab and a bilinear approximation.



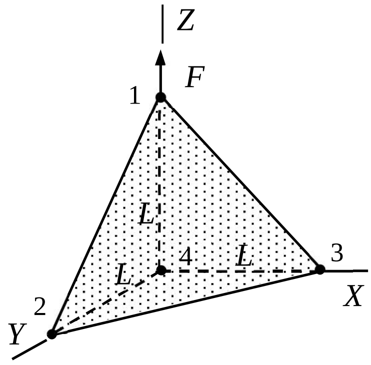
Answer $u_{X4} = 2\dfrac{fL}{Et}\dfrac{1-\nu^2}{3-\nu}$

9. A structure, consisting of a thin slab and a bar, is loaded by a horizontal force F acting on node 1. Material properties are E and ν, thickness of the slab is t and the cross-sectional area of the bar is A. Determine displacement of node 1 u_{X1} and u_{Y1} by using a linear bar element and a linear plane-stress element.



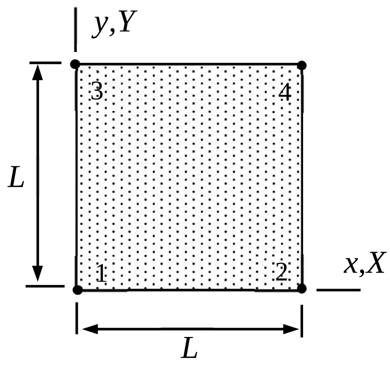
Answer $u_{X1} = -4\dfrac{L(1+\nu)}{Lt+4A(1+\nu)}\dfrac{F}{E}$ and $u_{Y1} = 0$

10. Point force F is acting on node 1 of the tetrahedron element of the figure. Nodes 2, 3 and 4 are fixed so that the displacement components are zeros. Determine displacement u_{Z1} of node 1 if $u_{X1} = u_{Y1} = 0$. Material properties E and ν are constants. Use linear approximation.



Answer $u_{Z1} = 6\dfrac{(1+\nu)(1-2\nu)}{1-\nu}\dfrac{F}{EL}$

Determine stress components at the midpoint of element shown if u_{Y2} and the other nodal displacements are zeros. The approximations to the displacement components u, v are bi-linear. The material parameters E, ν and thickness t are constants. Use the strain-displacement and stress-strain relationship of linearly elastic isotropic material and assume plane-stress conditions.



Solution

Under the plane-stress condition, the stress-strain and strain-displacement relationships of isotropic linearly elastic material are

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=[E]_\sigma\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} \text{ with } \begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial u/\partial y+\partial v/\partial x\end{Bmatrix} \text{and } [E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}.$$

The material parameters are Young's modulus E and Poisson's ratio ν. The relationships can be used to calculate stress out of the given displacement components.

Element approximation of the present case simplifies to (shape functions can be deduced from the figure with $\xi = x/L$ and $\eta = y/L$)

$$u=\begin{Bmatrix}(1-\xi)(1-\eta)\\ \xi(1-\eta)\\ (1-\xi)\eta\\ \xi\eta\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}=0 \quad \text{and} \quad v=\begin{Bmatrix}(1-\xi)(1-\eta)\\ \xi(1-\eta)\\ (1-\xi)\eta\\ \xi\eta\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ u_{Y2}\\ 0\\ 0\end{Bmatrix}=\frac{x}{L}(1-\frac{y}{L})u_{Y2} \quad \Rightarrow$$

$$\frac{\partial u}{\partial x}=0\,, \quad \frac{\partial u}{\partial y}=0\,, \quad \frac{\partial v}{\partial x}=\frac{1}{L}(1-\frac{y}{L})u_{Y2}\,, \text{ and } \frac{\partial v}{\partial y}=-\frac{x}{L}\frac{1}{L}u_{Y2}\,.$$

Strain components follow from the strain-displacement relationship

$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial u/\partial y+\partial v/\partial x\end{Bmatrix}=\frac{u_{Y2}}{L^2}\begin{Bmatrix}0\\ -x\\ L-y\end{Bmatrix}.$$

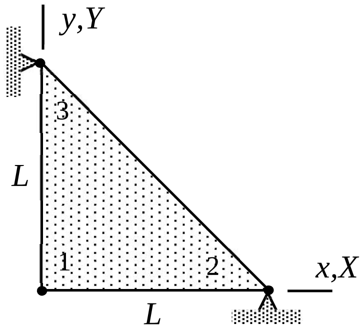
After that, stress components follow from the stress-strain relationship

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}\frac{u_{Y2}}{L^2}\begin{Bmatrix}0\\ -x\\ L-y\end{Bmatrix}=\frac{u_{Y2}}{L^2}\frac{E}{1-\nu^2}\begin{Bmatrix}-\nu x\\ -x\\ \frac{1-\nu}{2}(L-y)\end{Bmatrix}.$$

Evaluation at the midpoint $x = L/2$ and $y = L/2$ gives

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{u_{Y2}}{2L} \frac{E}{1-\nu^2} \begin{Bmatrix} -\nu \\ -1 \\ (1-\nu)/2 \end{Bmatrix}.$$

Determine the stress components σ_{xx}, σ_{yy}, and σ_{xy} of the triangle element shown in terms of the displacement components u_{X1}, u_{Y1} of node 1. Assume plane-strain conditions and use linear approximation to displacement components. The material parameters E, ν and thickness t are constants.



Solution

Under the plane-stress condition, the stress-strain and strain-displacement relationships of isotropic linearly elastic material and the matrix of elastic properties are

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=[E]_\varepsilon\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix},\quad \begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial u/\partial y+\partial v/\partial x\end{Bmatrix} \text{ and}$$

$$[E]_\varepsilon=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & (1-2\nu)/2\end{bmatrix} \text{ (from the formulae collection).}$$

The material parameters are Young's modulus E and Poisson's ratio ν. The relationships can be used to calculate stress out of the given displacement components.

Let us start with the approximation. Nodes 2 and 3 are fixed, u_{X1} and u_{Y1}. The shape function expressions can be deduced from the figure:

$$N_2=x/L,\quad N_3=y/L \quad\Rightarrow\quad N_1=1-N_2-N_3=1-x/L-y/L \quad\Rightarrow$$

$$u=\frac{1}{L}\begin{Bmatrix}L-x-y\\ x\\ y\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{X1}\\ 0\\ 0\end{Bmatrix}=(1-\frac{x}{L}-\frac{y}{L})u_{X1} \text{ and } v=\frac{1}{L}\begin{Bmatrix}L-x-y\\ x\\ y\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{Y1}\\ 0\\ 0\end{Bmatrix}=(1-\frac{x}{L}-\frac{y}{L})u_{Y1} \quad\Rightarrow$$

$$\frac{\partial u}{\partial x}=-\frac{1}{L}u_{X1},\quad \frac{\partial u}{\partial y}=-\frac{1}{L}u_{X1},\quad \frac{\partial v}{\partial x}=-\frac{1}{L}u_{Y1},\text{ and } \frac{\partial v}{\partial y}=-\frac{1}{L}u_{Y1}.$$

Strain components follow from the strain-displacement relationship

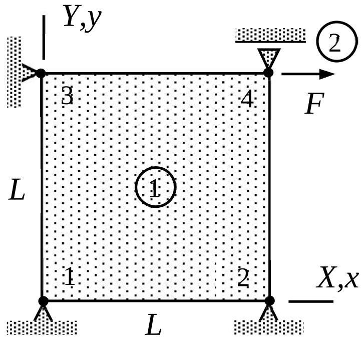
$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial u/\partial y+\partial v/\partial x\end{Bmatrix}=-\frac{1}{L}\begin{Bmatrix}u_{X1}\\ u_{Y1}\\ u_{X1}+u_{Y1}\end{Bmatrix}=-\frac{1}{L}\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix}.$$

After that, stress components follow from the stress-strain relationship

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_{\varepsilon} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \left(-\frac{1}{L} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}\right) \quad \Leftrightarrow$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = -\frac{E}{L(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \\ (1-2\nu)/2 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

A thin slab (1) of square shape is loaded by a point force (2) as shown in the figure. Derive the relationship between the force F and the displacement u_{X4} of its point of action. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. External distributed forces vanish. Assume plane stress conditions and use a bilinear approximation.



Solution

Let us start with the shape functions of element 1 and approximations. As nodes 1, 2, and 3 are fixed, it is enough to deduce the shape function of node 4

$$N_4 = \frac{xy}{L^2} .$$

Approximations to the displacement components and their derivatives with respect to x and y are

$$u = \frac{xy}{L^2} u_{X4}, \quad \frac{\partial u}{\partial x} = \frac{y}{L^2} u_{X4}, \text{ and } \frac{\partial u}{\partial y} = \frac{x}{L^2} u_{X4}$$

$$v = 0, \quad \frac{\partial v}{\partial x} = 0, \text{ and } \frac{\partial v}{\partial y} = 0 .$$

When the approximations are substituted there, the virtual work density of thin slab model simplifies to (plane stress conditions, only the internal part is needed)

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial \delta u / \partial x \\ \partial \delta v / \partial y \\ \partial \delta u / \partial y + \partial \delta v / \partial x \end{Bmatrix}^{\mathrm{T}} \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_\Omega^{\text{int}} = -\delta u_{X4} \frac{tE}{1-\nu^2} \frac{1}{L^4} (y^2 + \frac{1-\nu}{2} x^2) u_{X4} .$$

Integration over the domain occupied by the element gives the element contribution

$$\delta W^1 = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta u_{X4} \frac{Et}{6} \frac{3-\nu}{1-\nu^2} u_{X4} .$$

Virtual work expression of the point force (element 2) follows from the definition of work

$$\delta W^2 = \delta u_{X4} F .$$

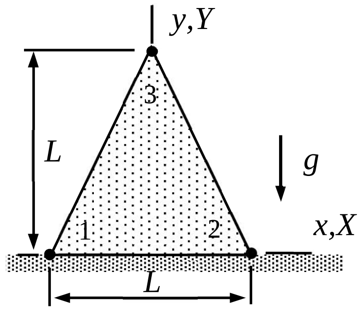
Virtual work expression of a structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^2 = \delta u_{X4} (-\frac{Et}{6} \frac{3-\nu}{1-\nu^2} u_{X4} + F) .$$

Finally, principle of virtual work in the form $\delta W = 0 \quad \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus imply that

$$u_{X4} = \frac{6F}{Et}\frac{1-\nu^2}{3-\nu}.$$

A long wall having triangular cross-section, and made of homogeneous, isotropic, linearly elastic material, is subjected to its own weight. Material properties E,ν,ρ are constants. Determine the displacement components u_{X3} and u_{Y3} of node 3. Nodes 1 and 2 are fixed. Use a three-node element and assume plane strain conditions.



Solution

Under the plane strain conditions, the virtual work densities (virtual works per unit area) of the thin slab model

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\varepsilon \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_\varepsilon = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress) and external forces acting on the element domain. Notice that the components f_x and f_y are external forces per unit area. Distributed forces on the boundaries and point forces are taken into account by separate force elements.

Shape function $N_3 = y/L$ of node 3 can be deduced from the figure. Linear approximations to the displacement components are

$$u = N_3 u_{X3} = \frac{y}{L} u_{X3} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{1}{L} u_{X3},$$

$$v = N_3 u_{Y3} = \frac{y}{L} u_{Y3} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{1}{L} u_{Y3} \ .$$

Virtual work of internal forces under the plane strain conditions with $G = E/(2+2\nu)$

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} \frac{tE}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \Rightarrow$$

$$\delta w_\Omega^{\text{int}} = -\frac{1}{L^2}\frac{Et}{1+\nu}\left(\frac{1-\nu}{1-2\nu} u_{Y3}\delta u_{Y3} + \frac{1}{2} u_{X3}\delta u_{X3}\right) \quad \Rightarrow$$

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega = \delta w_\Omega^{\text{int}} \frac{L^2}{2} = -\frac{1}{2}\frac{Et}{1+\nu}\left(\frac{1-\nu}{1-2\nu} u_{Y3}\delta u_{Y3} + \frac{1}{2} u_{X3}\delta u_{X3}\right).$$

Force density due to gravity is given by $f_x = 0$ and $f_y = -\rho g t$. Virtual work of external forces

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} = f_y \delta v = -\rho g t \frac{y}{L} \delta u_{Y3} \quad \Rightarrow$$

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega = -\int_0^L \int_{(y-L)/2}^{(L-y)/2} \rho g t \frac{y}{L} \delta u_{Y3}) dx dy = -\rho g t \delta u_{Y3} \int_0^L (L-y) \frac{y}{L} dy \quad \Rightarrow$$

$$\delta W^{\text{ext}} = -\rho g t \delta u_{Y3} \int_0^L (L-y) \frac{y}{L} dy = -\frac{1}{6} L^2 \rho g t \delta u_{Y3}.$$

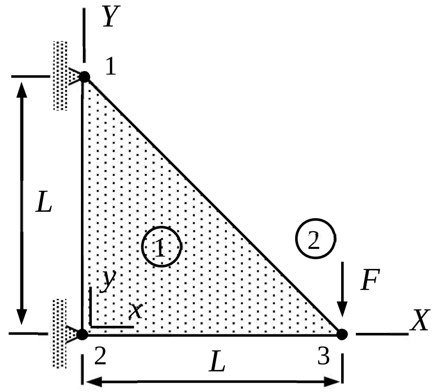
Virtual work expression in the sum of the internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\frac{1}{2} \frac{Et}{1+\nu} (\frac{1-\nu}{1-2\nu} u_{Y3} \delta u_{Y3} + \frac{1}{2} u_{X3} \delta u_{X3}) - \frac{1}{6} L^2 \rho g t \delta u_{Y3} .$$

Principle of virtual work $\delta W = 0$ and the basic lemma of variational calculus imply

$$u_{X3} = 0 \quad \text{and} \quad u_{Y3} = -\frac{1}{3}(1+\nu)\frac{1-2\nu}{1-\nu} L^2 \frac{\rho g}{E}.$$

A thin triangular slab of thickness t is loaded by a point force at node 3. Nodes 1 and 2 are fixed. Derive the virtual work expression δW of the structure in terms of u_{X3} and u_{Y3}, and solve for the nodal displacements. Approximation is linear and material parameters E and ν are constants. Assume plane stress conditions.



Solution

The virtual work densities (virtual works per unit area) of the thin slab model under the plane stress conditions

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{where}$$

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress), external forces acting on the element domain, and external forces acting on the edges. Notice that the components f_x and f_y are external forces per unit area. The forces acting on the element edges are taken into account by separate force elements.

Expressions of linear shape functions in material $xy-$coordinates can be deduced from the figure. Only the shape function of node 3 is actually needed:

$$N_3 = \frac{x}{L},\ \ N_1 = \frac{y}{L}\text{ , and } N_2 = 1 - N_1 - N_3 = 1 - \frac{x}{L} - \frac{y}{L} \qquad \Rightarrow$$

$$u = N_1 0 + N_2 0 + N_3 u_{X3} = \frac{x}{L}u_{X3} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1}{L}u_{X3} \quad \text{and} \quad \frac{\partial u}{\partial y} = 0\,,$$

$$v = N_1 0 + N_2 0 + N_3 u_{Y3} = \frac{x}{L}u_{Y3} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = \frac{1}{L}u_{Y3} \quad \text{and} \quad \frac{\partial v}{\partial y} = 0\,.$$

When the approximation is substituted there, virtual work expression of internal forces per unit area simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \delta u_{X3} \\ 0 \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L}\frac{tE}{2(1-\nu^2)}\begin{bmatrix} 2 & 2\nu & 0 \\ 2\nu & 2 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}\frac{1}{L}\begin{Bmatrix} u_{X3} \\ 0 \\ u_{Y3} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{2L^2(1-\nu^2)}\begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix}\begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}$$

As the integrand is constant, integration over the triangular domain gives

$$\delta W^{\mathrm{int}} = \int_A \delta w_\Omega^{\mathrm{int}} \mathrm{d}A = \delta w_\Omega^{\mathrm{int}} \frac{L^2}{2} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}.$$

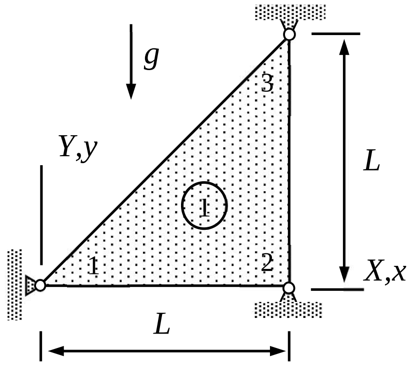
If also the point force is accounted for, the virtual work expression of the structure takes the form

$$\delta W = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix} \right).$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} = -4(1+\nu) \frac{F}{tE} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

A thin triangular slab (assume plane stress conditions) loaded by its own weight is allowed to move vertically at node 1 and nodes 2 and 3 are fixed. Find the displacement u_{Y1}. Material parameters E, ν, ρ and thickness t of the slab are constants.



Solution

For the plane stress conditions and the thin-slab model, virtual work density of internal and external volume forces are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}, \text{ where } [E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}.$$

Let us start with the approximations. Only the shape function of node 1 is needed as the other nodes are fixed. By using linearity and conditions $N_1(0,0)=1$, $N_1(L,0)=N_1(0,L)=0$

$$N_1(x,y) = 1 - \frac{x}{L}.$$

Displacement components simplify to

$$u = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0,$$

$$v = (1-\frac{x}{L})u_{Y1} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = -\frac{u_{Y1}}{L} \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

When approximations are substituted there, virtual work density simplifies to

$$\delta w_{\Omega}^{\text{int}} = -\delta u_{Y1} \frac{1}{L^2} \frac{tE}{2+2\nu} u_{Y1}.$$

Integration over the domain gives the virtual work expression. As the integrand is constant

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dA = \frac{L^2}{2} \delta w_{\Omega}^{\text{int}} = -\delta u_{Y1} \frac{1}{4} \frac{tE}{1+\nu} u_{Y1}.$$

Virtual work expression of the external volume force due to gravity takes the form

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} dA = -\int_0^L \int_0^x (1-\frac{x}{L}) \delta u_{Y1} t\rho g dy dx = -\delta u_{Y1} \frac{1}{6} t\rho g L^2.$$

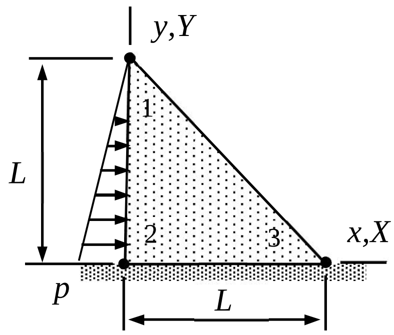
Virtual work expression of the thin slab is sum of the internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta u_{Y1}\left(\frac{1}{4}\frac{tE}{1+\nu}u_{Y1} + \frac{1}{6}t\rho g L^2\right).$$

Principle of virtual work $\delta W = 0 \;\forall \delta\mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{1}{4}\frac{tE}{1+\nu}u_{Y1} + \frac{1}{6}t\rho g L^2 = 0 \quad \Leftrightarrow \quad u_{Y1} = -\frac{2}{3}(1+\nu)\frac{\rho g L^2}{E}. \quad \leftarrow$$

A long dam of homogeneous, isotropic, linearly elastic material, is subjected to water pressure on one side. Material properties E and ν are constants. Determine the displacement components u_{X1} and u_{Y1} of node 1. Nodes 2 and 3 are fixed. Use a three-node element and assume plane strain conditions. Consider a slab of thickness t in calculations. The peak value of the linearly varying pressure is p.



Solution

Under the plane strain conditions, the virtual work densities of thin slab are

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\varepsilon} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_{\varepsilon} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}.$$

The external forces t_x and t_y (force per unit length in this case) acting on the element edges can be taken into account by a separate force element with the density expression (per unit length)

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}$$

although the expression is actually part of the thin slab model. The approximation on the boundary is just the restriction of the element approximation to the boundary.

Only the shape function for node 1 is needed as the other nodes are fixed (displacement vanishes). In terms of the displacement components u_{X1} and u_{Y1} of node 1, element approximations of the displacement components and their derivatives are

$$u = \frac{y}{L} u_{X1} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = \frac{1}{L} u_{X1},$$

$$v = \frac{y}{L} u_{Y1} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = \frac{1}{L} u_{Y1}.$$

When the approximation is substituted there, the virtual work densities simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ \delta u_{Y1}/L \\ \delta u_{X1}/L \end{Bmatrix}^{\mathrm{T}} \frac{Et}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{Y1}/L \\ u_{X1}/L \end{Bmatrix} \Rightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \dfrac{Et}{2(1+\nu)L^2} & 0 \\ 0 & \dfrac{Et(1-\nu)}{(1+\nu)(1-2\nu)L^2} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix},$$

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} = \begin{Bmatrix} \delta u_{X1} y/L \\ \delta u_{Y1} y/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} pt(1-y/L) \\ 0 \end{Bmatrix} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} pt(1-y/L)y/L \\ 0 \end{Bmatrix}.$$

Integrations over the element and edge 2-1 give the virtual work expressions (notice that the virtual work density of internal forces is constant)

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} d\Omega = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \dfrac{Et}{4(1+\nu)} & 0 \\ 0 & \dfrac{Et(1-\nu)}{2(1+\nu)(1-2\nu)} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_0^L \delta w_{\partial\Omega}^{\text{ext}} dy = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} ptL/6 \\ 0 \end{Bmatrix}.$$

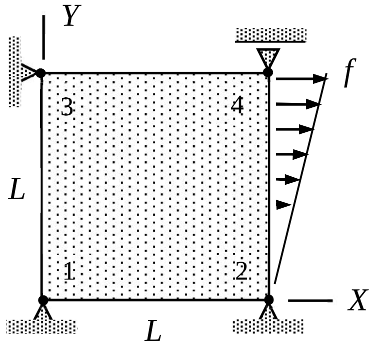
Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} (\begin{bmatrix} \dfrac{Et}{4(1+\nu)} & 0 \\ 0 & \dfrac{Et(1-\nu)}{2(1+\nu)(1-2\nu)} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} - \begin{Bmatrix} ptL/6 \\ 0 \end{Bmatrix}) = 0 \qquad \Rightarrow$$

$$\begin{bmatrix} \dfrac{Et}{4(1+\nu)} & 0 \\ 0 & \dfrac{Et(1-\nu)}{2(1+\nu)(1-2\nu)} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} - \begin{Bmatrix} ptL/6 \\ 0 \end{Bmatrix} = 0 \qquad \Leftrightarrow$$

$$u_{X1} = \frac{2}{3}\frac{pL}{E}(1+\nu) \quad \text{and} \quad u_{Y1} = 0.$$ ←

A thin slab is loaded by a distributed force as shown. Derive the relationship between the force peak value f and displacement u_{X4}. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. Assume plane-stress conditions and use the virtual work density of the thin slab and a bilinear approximation.



Solution

Under the plane stress conditions, the virtual work densities (virtual works per unit area) of the thin slab model

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress) and the external area forces acting on the element domain. The external forces t_x and t_y (tractions per unit length in this case) acting on the element edges can be taken into account by a separate force element with the density expression (per unit length)

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}$$

although the expression is actually part of the thin slab model. The approximation on the boundary is just the restriction of the element approximation to the boundary.

Only the shape function associated with node 4 is needed as the other nodes are fixed (displacement vanishes). In terms of the displacement component u_{X4} of node 4, approximations to the displacement components and their derivatives are

$$u = \mathbf{N}^{\mathrm{T}}\mathbf{a} = u_{X4}\frac{x}{L}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = u_{X4}\frac{1}{L}\frac{y}{L} \quad \text{and} \quad \frac{\partial u}{\partial y} = u_{X4}\frac{x}{L}\frac{1}{L},$$

$$v = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

Virtual work density of the internal forces simplifies to (when the approximations are substituted there)

$$\delta w_{\Omega}^{\text{int}} = -\delta u_{X4} \begin{Bmatrix} y \\ 0 \\ x \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^4} \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} y \\ 0 \\ x \end{Bmatrix} u_{X4} = -\delta u_{X4} \frac{1}{L^4} \frac{Et}{1-\nu^2} (y^2 + x^2 \frac{1-\nu}{2}) u_{X4} .$$

Integration over the element gives the virtual work expression of internal forces

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_{\Omega}^{\text{int}} dxdy = -\delta u_{X4} \frac{1}{L^4} \frac{Et}{1-\nu^2} (L^4 \frac{1}{3} + L^4 \frac{1}{3} \frac{1-\nu}{2}) u_{X4} \quad \Rightarrow$$

$$\delta W^{\text{int}} = -\delta u_{X4} \frac{Et}{1-\nu^2} \frac{3-\nu}{6} u_{X4} .$$

Virtual work expression of external forces t_x and t_y is obtained as an integral over the edge defined by $x = L$. The restriction of approximation to $x = L$ and the linear distribution $t_x = fy / L$ give

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} = \delta u_{X4} \frac{y}{L} (f \frac{y}{L}) \quad \Rightarrow \quad \delta W^{\text{ext}} = \int_0^L w_{\partial\Omega}^{\text{ext}} dy = \frac{fL}{3} \delta u_{X4} .$$

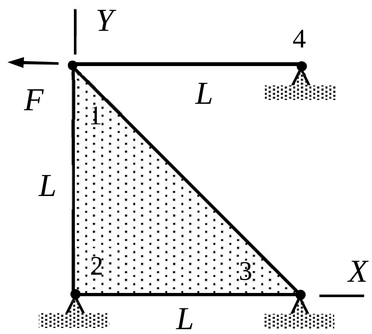
Virtual work expression is the sum of internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta u_{X4} (\frac{Et}{1-\nu^2} \frac{3-\nu}{6} u_{X4} - \frac{fL}{3}) .$$

Principle of virtual work $\delta W = 0 \ \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathrm{a} R = 0 \ \ \forall \delta \mathrm{a} \ \Leftrightarrow \ R = 0$ give

$$\frac{Et}{1-\nu^2} \frac{3-\nu}{6} u_{X4} - \frac{fL}{3} = 0 \quad \Leftrightarrow \quad u_{X4} = 2 \frac{fL}{Et} \frac{1-\nu^2}{3-\nu} .$$

A structure, consisting of a thin slab and a bar, is loaded by a horizontal force F acting on node 1. Material properties are E and ν, thickness of the slab is t, and the cross-sectional area of the bar A are constants. Determine displacement components u_{X1} and u_{Y1} of node 1 by using a linear bar element and a linear plane-stress element.



Solution

Under the plane stress conditions, the virtual work densities (virtual works per unit area) of the thin slab model

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\text{T}} t[E]_\sigma \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{where}$$

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress) and external forces acting on the element domain. Notice that the components f_x and f_y are external forces per unit area. Forces acting on the element edges can be taken into account by separate force elements.

Element contribution for the thin slab needs to be derived from approximation and virtual work densities. Approximations to the displacement components depend only on the shape function associated with node 1 as the other nodes are fixed (displacement vanishes). In terms of the displacement components u_{X1} and u_{Y1}

$$u = u_{X1}\frac{y}{L} \quad\Rightarrow\quad \frac{\partial u}{\partial x} = 0 \quad\text{and}\quad \frac{\partial u}{\partial y} = u_{X1}\frac{1}{L},$$

$$v = u_{Y1}\frac{y}{L} \quad\Rightarrow\quad \frac{\partial v}{\partial x} = 0 \quad\text{and}\quad \frac{\partial v}{\partial y} = u_{Y1}\frac{1}{L}.$$

Virtual work density of the internal forces simplifies to (when the approximations are substituted there)

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} 0 \\ \delta u_{Y1} \\ \delta u_{X1} \end{Bmatrix}^{\text{T}} \frac{1}{L^2}\frac{Et}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{Y1} \\ u_{X1} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\text{T}} \frac{1}{L^2}\frac{Et}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}$$

Virtual work expression is the integral of density over the domain occupied by the element (note that the virtual work density is constant in this case). Therefore

$$\delta W^1 = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{2}\frac{Et}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

Virtual work expression of the bar element is given in the formula collection with $u_{x1} = u_{X1}$ and $u_{x2} = 0$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X1} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

Virtual work expression of the point force follows e.g. directly from the definition (force multiplied by the virtual displacement in its direction)

$$\delta W^3 = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F \\ 0 \end{Bmatrix}.$$

Virtual work expression of the structure is the sum of element contributions $\delta W = \delta W^1 + \delta W^2 + \delta W^3$

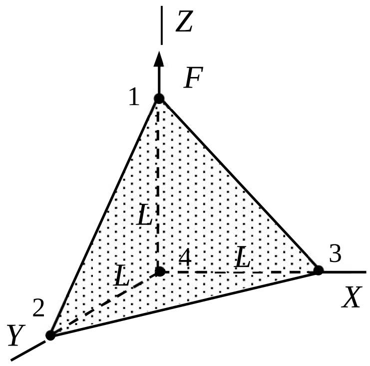
$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} (\frac{1}{2}\frac{Et}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \end{Bmatrix}) \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} (\begin{bmatrix} \frac{1}{4}\frac{Et}{1+\nu} + \frac{EA}{L} & 0 \\ 0 & \frac{1}{2}\frac{Et}{1-\nu^2} \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \end{Bmatrix}).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\begin{bmatrix} \frac{1}{4}\frac{Et}{1+\nu} + \frac{EA}{L} & 0 \\ 0 & \frac{1}{2}\frac{Et}{1-\nu^2} \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad u_{X1} = -\frac{4(1+\nu)L}{tL + 4(1+\nu)A}\frac{F}{E} \text{ and } u_{Y1} = 0.$$

Point force F is acting on node 1 of the tetrahedron element of the figure. Nodes 2, 3 and 4 are fixed so that the displacement components are zeros. Determine displacement u_{Z1} of node 1 if $u_{X1} = u_{Y1} = 0$. Material properties E and ν are constants. Use linear approximation.



Solution

Virtual work density of the solid model is (only internal forces in this problem)

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta w / \partial z \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial w / \partial z \end{Bmatrix} - \begin{Bmatrix} \partial\delta u / \partial y + \partial\delta v / \partial x \\ \partial\delta v / \partial z + \partial\delta w / \partial y \\ \partial\delta w / \partial x + \partial\delta u / \partial z \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} \partial u / \partial y + \partial v / \partial x \\ \partial v / \partial z + \partial w / \partial y \\ \partial w / \partial x + \partial u / \partial z \end{Bmatrix} \text{ where}$$

$$[E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \text{ and } G = \frac{E}{2+2\nu} .$$

Approximations to the displacement components depend only on the shape function associated with node 1 as the other nodes are fixed (displacement vanishes). In addition, the only non-zero displacement component is u_{Z1}. Here, $u = v = 0$ and

$$w = \mathbf{N}^{\mathrm{T}}\mathbf{a} = u_{Z1}\frac{z}{L} \quad \Rightarrow \quad \frac{\partial w}{\partial x} = 0 , \; \frac{\partial w}{\partial y} = 0 , \text{ and } \frac{\partial w}{\partial z} = u_{Z1}\frac{1}{L} .$$

Virtual work density of the internal forces simplifies to (when the approximations are substituted there)

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z1} / L \end{Bmatrix}^{\mathrm{T}} \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z1} / L \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{E}{2+2\nu} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta w_V^{\text{int}} = -\delta u_{Z1}\frac{1}{L^2}\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}u_{Z1} .$$

Virtual work expression of the body element is integral of the density over the domain occupied by the element (note that the virtual work density is constant and volume $V = L^3 / 6$)

$$\delta W^1 = -\delta u_{Z1}\frac{L}{6}\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}u_{Z1} .$$

Virtual work expression of the given force follows, e.g., directly from the definition: force multiplied by the virtual displacement in its direction.

$$\delta W^2 = \delta u_{Z1} F \; .$$

Virtual work expression of the structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^2 = -\delta u_{Z1} \left(\frac{L}{6} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} u_{Z1} - F \right) .$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta u_{Z1} \left(\frac{L}{6} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} u_{Z1} - F \right) = 0 \quad \forall \delta u_{Z1} \quad \Leftrightarrow \quad \frac{L}{6} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} u_{Z1} - F = 0 \quad \Leftrightarrow$$

$$u_{Z1} = 6 \frac{(1+\nu)(1-2\nu)}{1-\nu} \frac{F}{EL} .$$