Aalto University
School of Electrical
Engineering

ELEC-E4130 Electromagnetic fields, Autumn 2021

# Antennas and radiating systems

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December 8, 2021

# Schematic view of the big picture

**Antennas** are designed to efficiently radiate and receive EM waves, or equivalently, to convert between guided and unguided waves:

# Antennas and radiating systems

## Outline for lecture weeks 11–12

**Week 11:** Basics of antenna radiation

- Elemental dipoles [11–2]
- Antenna parameters [11–3]
- Thin linear antennas [11–4]
- Antenna arrays [11–5]

**Week 12:** Antenna systems and more

- Receiving antennas [11–6]
- Transmit-receive systems [11–7]
- Microstrip antennas [extra]
- Aperture radiators [11–9]

No lecture on Dec 6 (Indepence Day).

D.K. Cheng, *Field and Wave Electromagnetics*, 2nd Ed., Chapter 11

# Time-harmonic potentials

From lecture week 3, we have $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla V - j\omega\mathbf{A}$, and the time-harmonic **retarded potential solution**:

$$V = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho\, e^{-jkR}}{R} dv', \qquad \mathbf{A} = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J}\, e^{-jkR}}{R} dv', \qquad k = \omega\sqrt{\mu\varepsilon}$$

For a given source current $\mathbf{J}$ in free space, we'll calculate the radiated fields as

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}\, e^{-jkR}}{R} dv', \qquad \mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A}, \qquad \mathbf{E} = \frac{1}{j\omega\varepsilon_0}\nabla \times \mathbf{H}}$$

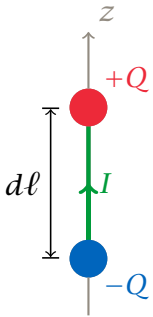
where $R$ is the distance between $dv'$ and the observation point.

The vector potential $\mathbf{A}$ is a useful intermediate step in the calculation, while the scalar potential $V$ can be ignored. (Equation of continuity: $\nabla \cdot \mathbf{J} = -j\omega\rho$)

# Radiation fields of elemental dipoles

Cheng 11-2

# Hertzian dipole




Short conducting wire ($d\ell \ll \lambda$) between conducting spheres, with uniform current

$$i(t) = I\cos(\omega t) = \mathcal{R}e\left[Ie^{+j\omega t}\right], \quad I > 0$$

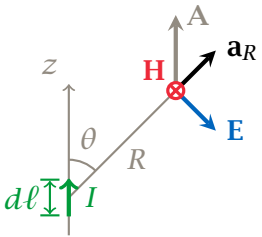
$\Rightarrow$ charge accumulation at the ends

$$i(t) = \pm\frac{dq(t)}{dt} \quad \Leftrightarrow \quad I = \pm j\omega Q$$

So this is an **oscillating electric dipole** with electric moment $\mathbf{p} = \mathbf{a}_z Q d\ell$.

However, for calculating the radiated fields, we can forget the charge and just assume a uniform current $I$ of length $d\ell$ in free space.

# EM fields of Hertzian dipole = elemental electric dipole




**Outline of the solution:**

1. Calculate the vector potential $\mathbf{A} = \mathbf{a}_z A_z$ due to $I$
2. Express $\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta$
3. Calculate the magnetic field $\mathbf{H} = \mathbf{a}_\phi H_\phi$ due to $\mathbf{A}$
4. Calculate the electric field $\mathbf{E} = \mathbf{a}_R E_R + \mathbf{a}_\theta E_\theta$ due to $\mathbf{H}$

Step 1 is a trivial integral while steps 3–4 are straightforward but somewhat tedious curls in spherical coordinates. Use the **formula sheet** to get steps 2–4 right!

In the far field, we get a spherical wave that locally is like a plane wave (like in the figure).

# Vector potential of Hertzian dipole

## 1. Vector potential

⇊ Note that $\beta = k = k_0$ in Ch. 11

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}e^{-jkR}}{R} dv' = \mathbf{a}_z \frac{\mu_0 I d\ell}{4\pi R} e^{-j\beta R}, \qquad \boxed{\beta = k_0 = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda}}$$

Since the dipole is infinitesimally small (and $d\ell \ll R$), we assume that $R$ is constant in the integral, that is, $R$ is just the spherical coordinate $R$.

## 2. In spherical coordinates using the formula sheet (subst. $A_x, A_y = 0$, $A_z = 1$)

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \quad \Rightarrow \quad \mathbf{a}_z = \mathbf{a}_R\cos\theta - \mathbf{a}_\theta\sin\theta$$

$$\Rightarrow \quad \boxed{\mathbf{A} = (\mathbf{a}_R\cos\theta - \mathbf{a}_\theta\sin\theta)\frac{\mu_0 I d\ell}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right) = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta}$$

Note: $\mathbf{A}$ in the formula sheet is not the same as the vector potential $\mathbf{A}$ in this case.

# 3. Magnetic field of Hertzian dipole

$$\mathbf{A} = \mathbf{a}_R \frac{\mu_0 I d\ell}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)\cos\theta - \mathbf{a}_\theta \frac{\mu_0 I d\ell}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)\sin\theta$$

$$\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} = \frac{1}{\mu_0 R^2\sin\theta}\begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial\theta} & 0 \\ A_R & RA_\theta & 0 \end{vmatrix}$$

$$= \mathbf{a}_\phi \frac{1}{\mu_0 R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial}{\partial\theta}(A_R)\right]$$

$$= \mathbf{a}_\phi \frac{I d\ell}{4\pi R}\left[\frac{\partial}{\partial R}\left(-e^{-j\beta R}\sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{e^{-j\beta R}}{R}\cos\theta\right)\right]$$

$$= \mathbf{a}_\phi \frac{I d\ell}{4\pi R}\left[j\beta\, e^{-j\beta R}\sin\theta + \frac{e^{-j\beta R}}{R}\sin\theta\right]$$

$$\Rightarrow \quad \boxed{\mathbf{H} = -\mathbf{a}_\phi \frac{I d\ell}{4\pi}\beta^2\sin\theta\left[\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2}\right]e^{-j\beta R} = \mathbf{a}_\phi H_\phi}$$

# 4. Electric field of Hertzian dipole

$$\mathbf{H} = -\mathbf{a}_\phi \frac{I d\ell}{4\pi} \beta^2 \sin\theta \left[ \frac{1}{j\beta R} + \frac{1}{(j\beta R)^2} \right] e^{-j\beta R}$$

$$\mathbf{E} = \frac{1}{j\omega\varepsilon_0} \nabla \times \mathbf{H} = \frac{1}{j\omega\varepsilon_0} \frac{1}{R^2 \sin\theta} \begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R \sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & 0 \\ 0 & 0 & (R\sin\theta) H_\phi \end{vmatrix}$$

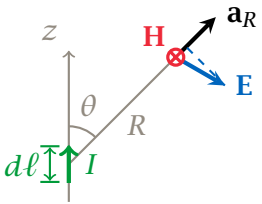
$$= \frac{\eta_0}{j\beta} \left[ \mathbf{a}_R \frac{1}{R\sin\theta} \frac{\partial}{\partial\theta} \left( H_\phi \sin\theta \right) - \mathbf{a}_\theta \frac{1}{R} \frac{\partial}{\partial R} \left( R H_\phi \right) \right] \qquad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$= -\frac{\eta_0}{j\beta} \frac{I d\ell \beta^2}{4\pi} \left\{ \mathbf{a}_R \frac{1}{R\sin\theta} \frac{\partial}{\partial\theta} \left( \sin^2\theta \right) \left[ \frac{1}{j\beta R} + \frac{1}{(j\beta R)^2} \right] e^{-j\beta R} \right.$$
$$\left. - \mathbf{a}_\theta \frac{1}{R} \frac{\partial}{\partial R} \left[ \left( \frac{1}{j\beta} + \frac{1}{(j\beta)^2 R} \right) e^{-j\beta R} \right] \sin\theta \right\}$$

$$\boxed{\begin{aligned} \mathbf{E} = -\frac{I d\ell}{4\pi} \eta_0 \beta^2 \Bigg\{ & \mathbf{a}_R 2\cos\theta \left[ \frac{1}{(j\beta R)^2} + \frac{1}{(j\beta R)^3} \right] \\ & + \mathbf{a}_\theta \sin\theta \left[ \frac{1}{j\beta R} + \frac{1}{(j\beta R)^2} + \frac{1}{(j\beta R)^3} \right] \Bigg\} e^{-j\beta R} \end{aligned}}$$

# EM fields of Hertzian dipole

**General solution** ($d\ell$ is infinitesimally small, but no other approximations)




$$E_R = -\frac{Id\ell}{4\pi}\eta_0\beta^2 2\cos\theta\left[\frac{1}{(j\beta R)^2} + \frac{1}{(j\beta R)^3}\right]e^{-j\beta R}$$

$$E_\theta = -\frac{Id\ell}{4\pi}\eta_0\beta^2 \sin\theta\left[\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2} + \frac{1}{(j\beta R)^3}\right]e^{-j\beta R}$$

$$H_\phi = -\frac{Id\ell}{4\pi}\beta^2 \sin\theta\left[\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2}\right]e^{-j\beta R}$$

**Important special cases**

- In the near field, $\beta R \ll 1$ and $e^{-j\beta R} \approx 1$ ($\approx$ statics)
- In the far field, $\beta R \gg 1$

# Near fields of Hertzian dipole

In the near field, when $\beta R \ll 1$ and $R \gg d\ell$, we get

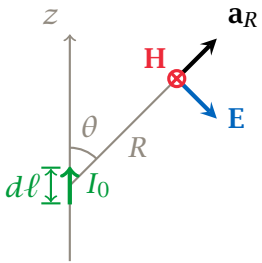
$$\mathbf{E} = \frac{p}{4\pi\varepsilon_0 R^3}\left(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta\right), \qquad p = \frac{I d\ell}{j\omega}$$

$$\mathbf{H} = \mathbf{a}_\phi \frac{I d\ell}{4\pi R^2}\sin\theta$$

These expressions are identical to the electrostatic field by a $z$-directed dipole with moment $p$ and the magnetostatic field by a $z$-directed current element $I d\ell$.

We didn't cover electrostatics and magnetostatics on this course, and for antenna applications these near field expressions are not that essential.

# Far fields of Hertzian dipole




In the far field, when $\beta R \gg 1$, we get

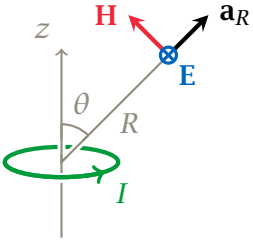
$$\boxed{\begin{aligned} \mathbf{E} &= \mathbf{a}_\theta V_0 \left( \frac{e^{-j\beta R}}{R} \right) \sin\theta \\ \mathbf{H} &= \mathbf{a}_\phi \frac{V_0}{\eta_0} \left( \frac{e^{-j\beta R}}{R} \right) \sin\theta \end{aligned}}$$

$$V_0 = j\beta\eta_0 \frac{I d\ell}{4\pi}, \quad [V_0] = \mathrm{V}$$

The far field is locally like a plane wave whose amplitude depends on the direction ($\sin\theta$) and distance from the antenna ($1/R$).

# EM fields of magnetic dipole

The small current loop in the figure (radius $b$, current phasor $I$) is a **magnetic dipole** with magnetic moment $\mathbf{m} = \mathbf{a}_z m = \mathbf{a}_z I \pi b^2$.




$$H_R = -\frac{j\omega\mu_0 m}{4\pi\eta_0}\beta^2 2\cos\theta\left[\frac{1}{(j\beta R)^2} + \frac{1}{(j\beta R)^3}\right]e^{-j\beta R}$$

$$H_\theta = -\frac{j\omega\mu_0 m}{4\pi\eta_0}\beta^2 \sin\theta\left[\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2} + \frac{1}{(j\beta R)^3}\right]e^{-j\beta R}$$

$$E_\phi = \frac{j\omega\mu_0 m}{4\pi}\beta^2 \sin\theta\left[\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2}\right]e^{-j\beta R}$$

The integral for calculating $\mathbf{A}$ is quite tedious [see the textbook for details], but the end result is very similar to the Hertzian (electric) dipole, due to **duality**.

$$Id\ell = j\beta m \quad \Leftrightarrow \quad \begin{cases} \mathbf{E}_e = \eta_0 \mathbf{H}_m \\ \mathbf{H}_e = -\mathbf{E}_m/\eta_0 \end{cases} \qquad \begin{cases} \text{subscript } e = \text{electric dipole} \\ \text{subscript } m = \text{magnetic dipole} \end{cases}$$

The figure shows the directions of the far fields where $\beta R \gg 1$.

# Antenna patterns and antenna parameters

Cheng 11-3

# Radiation fields and radiation patterns

We are often primarily interested in the far fields or **radiation fields** of an antenna. The electric radiation field can be written in the general form

$$\mathbf{E} = \mathbf{a}\, V_0 \left( \frac{e^{-j\beta R}}{R} \right) F(\theta, \phi)$$

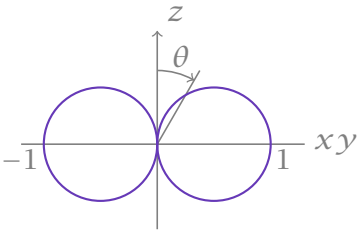
where $\mathbf{a}$ is a unit vector (linear combination of $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$), $V_0$ is a constant, and $F(\theta, \phi)$ is normalized so that $\max |F| = 1$.

The **radiation pattern** or **antenna pattern** $F(\theta, \phi)$ is the relative field strength as function of direction at a fixed distance $R$ in the far field.

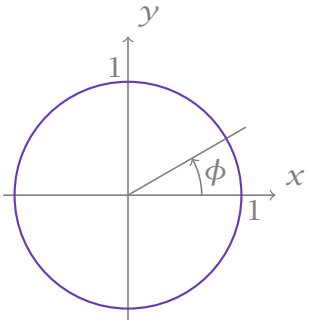
Instead of a three-dimensional plot of $F(\theta, \phi)$, we typically plot $|F|$ in two orthogonal planes that include the main radiation direction.

# Radiation patterns of a Hertzian dipole

$|F| = |\sin\theta|$




(a) $E$-plane pattern

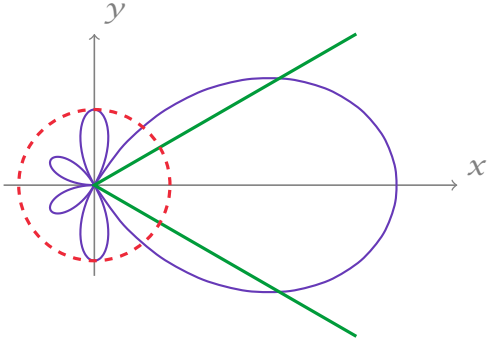


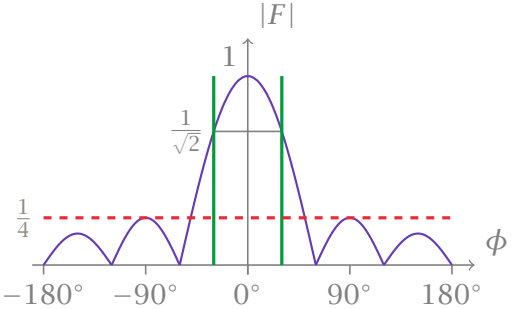

(b) $H$-plane pattern

Since $\mathbf{E} = \mathbf{a}_\theta E_\theta$ and $\mathbf{H} = \mathbf{a}_\phi H_\phi$ for the $z$-oriented Hertzian dipole, any plane containing the $z$ axis is an ***E*-plane** while the $xy$ plane is the ***H*-plane**.

# Beamwidth and sidelobe level

Radiation pattern with **main beam** in the $\phi = 0°$ direction and 4 **sidelobes**




The (3 dB) **beamwidth** or width of main beam is the angle between the points with amplitude $1/\sqrt{2}$ in the radiation pattern. In this case $60°$.

The **sidelobe level** is the amplitude of the largest sidelobes. Here $1/4$.

If the sidelobes are very small, it is convenient to plot the radiation pattern in decibels: $F_{\mathrm{dB}} = 10 \log |F|^2 = 20 \log |F|$. (Here the sidelobe level is $-12$ dB.)

# Directive gain and directivity

Radiation intensity $U = R^2 \mathcal{P}_{av}$ is the time-average radiated power per unit solid angle in the far field, and the total time-average power radiated is

$$P_r = \oint \boldsymbol{\mathcal{P}}_{av} \cdot d\mathbf{s} = \oint U \, d\Omega = \int_0^{2\pi} \int_0^{\pi} U(\theta, \phi) \, \sin\theta \, d\theta \, d\phi$$

## Directive gain

$$\boxed{G_D(\theta, \phi) = \frac{U(\theta, \phi)}{P_r/(4\pi)} = \frac{4\pi \, U(\theta, \phi)}{\oint U \, d\Omega} = \frac{\text{intensity}}{\text{avg. intensity}}}$$

## Directivity

$$\boxed{D = \frac{U_{\max}}{U_{\text{av}}} = \frac{4\pi \, U_{\max}}{P_r} = \frac{4\pi}{\int_0^{2\pi} \int_0^{\pi} |F(\theta, \phi)|^2 \sin\theta \, d\theta \, d\phi}}$$

Note that $U \sim R^2 \, |E|^2 \sim |F|^2$ in the far field, and $\max |F| = 1$.

# Directive gain and directivity of a Hertzian dipole

The electric far field or radiation field of a Hertzian dipole is

$$\mathbf{E} = \mathbf{a}_\theta V_0 \left( \frac{e^{-j\beta R}}{R} \right) \sin\theta \quad \Rightarrow \quad \left| F(\theta,\phi) \right|^2 = \sin^2\theta$$

Directive gain

$$G_D(\theta,\phi) = \frac{4\pi\, U(\theta,\phi)}{\oint U\, d\Omega} = \frac{4\pi \sin^2\theta}{\int_0^{2\pi}\int_0^{\pi} \sin^2\theta\, \sin\theta\, d\theta\, d\phi} = \frac{2\sin^2\theta}{\int_0^{\pi} \sin^3\theta\, d\theta} = \frac{3}{2}\sin^2\theta$$

Directivity $D = \max G_D = 1.5 \approx 1.76\,\mathrm{dB}$

Trigonometric integral using $\xi = \cos\theta$ substitution:

$$\int_0^{\pi} \sin^3\theta\, d\theta = \int_0^{\pi} \left(1 - \cos^2\theta\right) \sin\theta\, d\theta = \int_{-1}^{1} \left(1 - \xi^2\right) d\xi = \frac{4}{3}$$

# Example problem 11.1

The far field of an antenna is

$$\mathbf{E} = \mathbf{a}_\theta V_0 \left( \frac{e^{-j\beta R}}{R} \right) \sin\theta \left| \cos\frac{\phi}{2} \right|$$

Determine

**(a)** the direction of the main lobe,

**(b)** the (3 dB) beamwidth in the $E$-plane and $H$-plane, and

**(c)** the directivity of the antenna pattern.

# Solution

Radiation pattern $F(\theta,\phi) = \sin\theta \left|\cos\frac{\phi}{2}\right|$

**(a)** The main beam is in the $+x$ direction ($\theta = \pi/2, \phi = 0$) and there are no sidelobes.

**(b)** Beamwidth in $E$-plane:

$$|\sin\theta| = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \quad \Rightarrow \quad \Delta\theta = \frac{\pi}{2} = 90^\circ$$

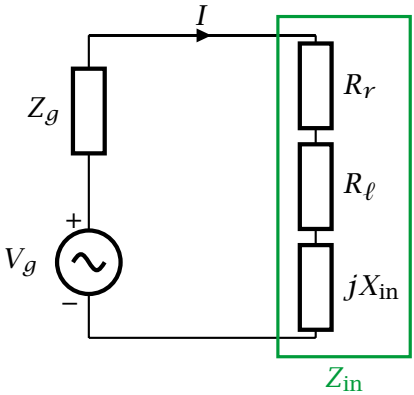
Beamwidth in $H$-plane:

$$\left|\cos\frac{\phi}{2}\right| = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \phi = \pm\frac{\pi}{2} \quad \Rightarrow \quad \Delta\phi = \pi = 180^\circ$$

**(c)** Directivity:

$$\oint |F|^2\, d\Omega = \underbrace{\int_0^{\pi} \sin^3\theta\, d\theta}_{4/3} \int_0^{2\pi} \underbrace{\cos^2(\phi/2)}_{\frac{1}{2}+\frac{1}{2}\cos\phi}\, d\phi = \frac{4\pi}{3} \quad \Rightarrow \quad D = \frac{4\pi}{\oint |F|^2\, d\Omega} = 3$$

# Input impedance, power gain and radiation efficiency




## Input impedance

$$Z_{\text{in}} = R_r + R_\ell + jX_{\text{in}}$$

$R_r$ = radiation resistance

$R_\ell$ = loss resistance

Total input power

$$P_i = P_r + P_\ell = \frac{1}{2}R_r I^2 + \frac{1}{2}R_\ell I^2$$

Antenna **power gain** or simply **gain**

$$\boxed{G_P = \frac{4\pi\, U_{\max}}{P_i}}$$

## Radiation efficiency

$$\boxed{\eta_r = \frac{G_P}{D} = \frac{P_r}{P_i} = \frac{R_r}{R_r + R_\ell}}$$

The antenna reactance $X_{\text{in}}$ is important for matching the antenna, but very difficult to calculate.

# Hertzian dipole, again

The radiation resistance of a Hertzian dipole (see the book for details)

$$R_r \approx 80\pi^2 \left(\frac{d\ell}{\lambda}\right)^2 \Omega, \qquad d\ell \ll \lambda$$

is very small and its loss resistance can be of the same size.

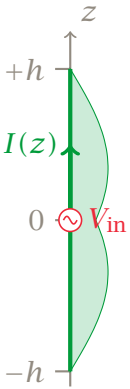
Moreover, it has a large capacitive input reactance.

So the Hertzian dipole is a poor radiator that is nearly impossible to match to a $50\,\Omega$ transmission line.

# Thin linear antennas

Cheng 11-4

# Center-fed linear dipole antenna




Thin straight conducting wire of length $2h$ comparable with the wavelength with a sinusoidal voltage source $V_{\text{in}}$ at the center.

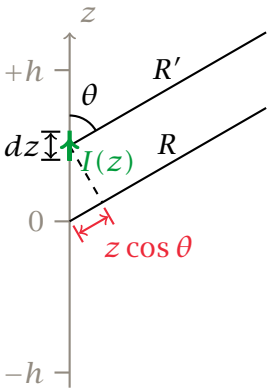
The current phasor $I(z)$ must be zero at the ends ($z = \pm h$), and we can approximate it with a sinusoidal:

$$I(z) = I_m \sin\left[\beta(h - |z|)\right], \quad -h < z < h,$$

where $I_m$ is a constant that depends on $V_{\text{in}}$.

**This current distribution is an approximation** that should be good enough for calculating the radiated fields with reasonable accuracy.

# Center-fed linear dipole antenna radiation




An infinitesimally short part of the dipole is a Hertzian dipole with length $dz$, current $I(z)$, and far field component

$$dE_\theta = j\beta\eta_0 \frac{I(z)dz}{4\pi}\left(\frac{e^{-j\beta R'}}{R'}\right)\sin\theta.$$

Using trigonometry we see that $R' = R - z\cos\theta$.

The difference between $R$ and $R'$ is important for the phase but irrelevant for the amplitude:

$$E_\theta = \frac{j\beta\eta_0 \sin\theta}{4\pi R}\int_{-h}^{h} I(z)e^{-j\beta(R-z\cos\theta)}dz$$

# Center-fed linear dipole antenna radiation (cont.)

The integral is a bit tedious but doable exploiting symmetry and some trig. formulas:

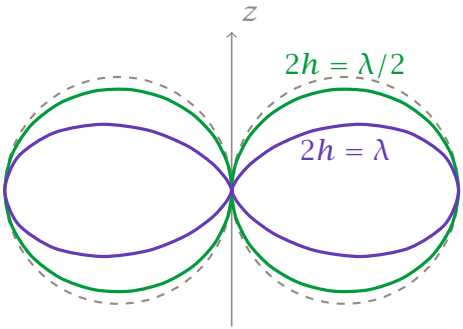
$$
\begin{aligned}
E_\theta &= \frac{j\beta\eta_0 I_m \sin\theta}{4\pi R} e^{-j\beta R} \int_{-h}^{h} \sin\left[\beta(h-|z|)\right] e^{j\beta z\cos\theta} dz \\
&= \frac{j\beta\eta_0 I_m \sin\theta}{4\pi R} e^{-j\beta R} 2 \int_{0}^{h} \sin\left[\beta(h-z)\right] \cos\left(\beta z\cos\theta\right) dz = \ldots \\
&= \frac{j\eta_0 I_m}{2\pi} \left(\frac{e^{-j\beta R}}{R}\right) F(\theta), \quad F(\theta) = \frac{\cos\left(\beta h\cos\theta\right) - \cos(\beta h)}{\sin\theta}
\end{aligned}
$$

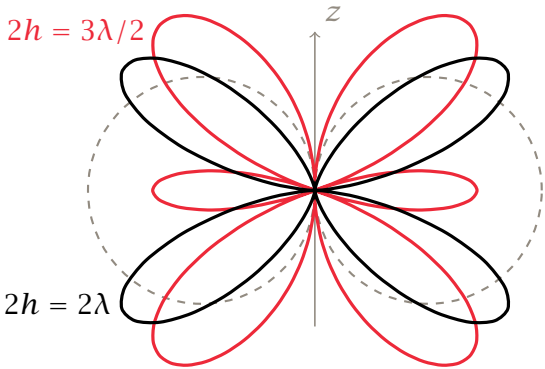
Here $F(\theta)$ looks like a **radiation pattern**, but it is unfortunately not normalized.

(The constant $\eta_0/(2\pi) \approx 60\,\Omega$.)

# E-plane radiation patterns for center-fed linear dipole antennas

Normalized $|F(\theta)|$ compared with the Hertzian dipole (dashed):




Longer dipole → narrower beam, but when $2h > \lambda$ several lobes appear.

Dipoles with length $2h = \lambda, 2\lambda, \ldots$ are difficult to feed since the current is approximately 0 at the center.

# Half-wave dipole (1/3)

For a half-wave dipole $\beta h = 2\pi h/\lambda = \pi/2$ and the pattern function

$$F(\theta) = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

happens to be normalized and non-negative for $0 \leq \theta \leq \pi$, and the current has its maximum at the feed $I(0) = I_m \sin(\beta h) = I_m$.

The far fields and time-average Poynting vector are

$$E_\theta = \frac{j\eta_0 I_m}{2\pi}\left(\frac{e^{-j\beta R}}{R}\right)F(\theta), \qquad H_\phi = \frac{jI_m}{2\pi}\left(\frac{e^{-j\beta R}}{R}\right)F(\theta),$$

$$\mathcal{P}_{\text{av}} = \frac{1}{2}\mathcal{R}e\left[\mathbf{E}\times\mathbf{H}^*\right] = \mathbf{a}_R\frac{\eta_0\left|I_m\right|^2}{8\pi^2R^2}F^2(\theta)$$

# Half-wave dipole (2/3)

Total radiated power

$$P_r = \int_0^{2\pi} \int_0^{\pi} \mathscr{P}_{\text{av}} R^2 \sin\theta \, d\theta \, d\phi = \frac{\eta_0 \, |I_m|^2}{4\pi} \int_0^{\pi} F^2(\theta) \sin\theta \, d\theta$$

$$= \frac{\eta_0 \, |I_m|^2}{4\pi} \underbrace{\int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \, d\theta}_{\approx 1.21883} \approx (36.54\,\Omega) \, |I_m|^2 = \frac{1}{2} R_r \, |I_m|^2$$

## Radiation resistance

$$\boxed{R_r \approx 73\,\Omega}$$

Moreover, the antenna reactance $X_{\text{in}} \approx 0$ if the dipole is slightly shorter ($2h \approx 0.48\lambda$), so a half-wave dipole is easy to feed.

# Half-wave dipole (3/3)

## Directivity

$$D = \frac{4\pi U_{\max}}{P_r} = \frac{4\pi R^2 \mathcal{P}_{\text{av}}(\theta = \pi/2)}{P_r} \approx 1.64 \approx 2.15\,\text{dB}$$

## (3 dB) Beamwidth

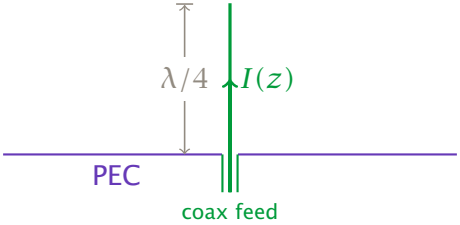
$$F(\theta) = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = 50.96^\circ \text{ or } 129.04^\circ \quad \Rightarrow \quad \Delta\theta \approx 78^\circ.$$

As was also evident from the plot a few slides back, the half-wave dipole is slightly more directive than the Hertzian dipole.

Electrically small antennas will almost always radiate like short dipoles. To get more directivity we need large antennas or antenna arrays.

# Example problem 11.2

Figure out the radiation resistance and directivity of a quarter-wave monopole over a PEC ground-plane.



# Solution

Using image theory, we get the same EM fields in the upper half-space if we remove the PEC plane and add the appropriate mirror image of the quarter wave monopole ⇒ half-wave dipole in free space.

The total radiated power $P_r = \frac{1}{2} R_r \, |I_m|^2$ is halved compared with the dipole, so the radiation resistance must also be halved:

$$R_r \approx 36.5\,\Omega$$

The directivity remains the same as for the half-wave dipole

$$D = \frac{U_{\max}}{U_{\text{av}}} = 1.64 = 2.15\,\text{dB}$$

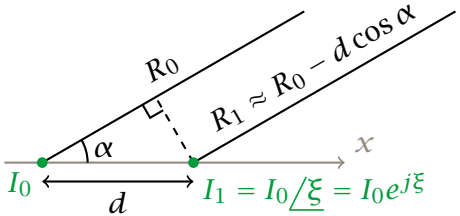
since both the maximum and average radiation intensity remains the same.

(One half of the total power distributed over one half of the solid angle gives the same average.)

# Antenna arrays

Cheng 11-5

# Two-element array




Two identical antennas 0 and 1 are fed with the same current amplitude but different phase. The observation point is in the far field.

The electric far field amplitude can be written in the form

$$E = E_m \left(\frac{e^{-j\beta R_0}}{R_0}\right) F(\theta,\phi) + E_m e^{j\xi} \left(\frac{e^{-j\beta R_1}}{R_1}\right) F(\theta,\phi)$$
$$\approx \frac{E_m F(\theta,\phi)}{R_0}\left[e^{-j\beta R_0} + e^{j\xi} e^{-j\beta(R_0 - d\cos\alpha)}\right]$$

where $E_m$ is a constant and $F(\theta,\phi)$ is the (normalized) radiation pattern. The difference between $R_0$ and $R_1$ is important for the phase and irrelevant for the amplitude. Using the formula sheet, we get $\cos\alpha = \mathbf{a}_x \cdot \mathbf{a}_R = \sin\theta\cos\phi$.

# Element factor and array factor

The far field amplitude can be put in the form

$$E = E_m \left( \frac{e^{-j\beta R_0}}{R_0} \right) F(\theta, \phi) \left[ 1 + e^{j\psi} \right] = E_m \left( \frac{e^{-j\beta R_0}}{R_0} \right) F(\theta, \phi) e^{j\psi/2} \underbrace{\left[ e^{-j\psi/2} + e^{j\psi/2} \right]}_{2\cos(\psi/2)}$$

where

$$\psi = \beta d \sin\theta \cos\phi + \xi$$

Thus we have

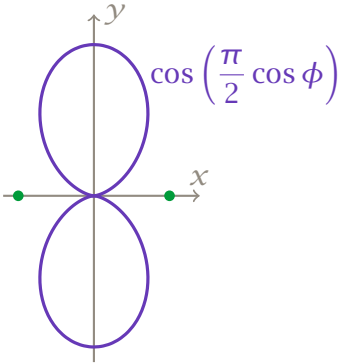
$$|E| = \frac{2\,|E_m|}{R_0} \underbrace{|F(\theta, \phi)|}_{\text{element}} \underbrace{\left| \cos\frac{\psi}{2} \right|}_{\text{array}}$$

Array radiation pattern = **element factor** × **array factor**

assuming that the coupling between the antenna elements can be neglected.

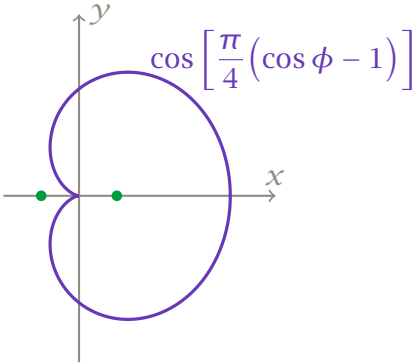
# H-plane patterns for two-element arrays of $z$-directed dipoles

## Broadside array




$$d = \lambda/2, \quad \xi = 0$$

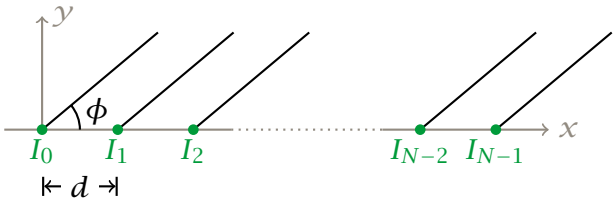
## Endfire array




$$d = \lambda/4, \quad \xi = -\pi/2$$

# General uniform linear array

$N$ uniformly spaced identical antennas along the $x$-axis, fed with equal amplitude and progressive phase shift $\xi$ between adjacent elements ($I_n = I_0 e^{jn\xi}$):




The normalized array factor in the $xy$-plane is

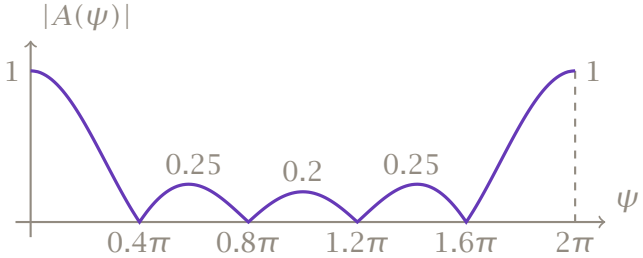
$$|A(\psi)| = \frac{1}{N}\left|1 + e^{j\psi} + e^{j2\psi} + \cdots + e^{j(N-1)\psi}\right| = \frac{1}{N}\left|\frac{1 - e^{jN\psi}}{1 - e^{j\psi}}\right|$$

or

$$\boxed{|A(\psi)| = \frac{1}{N}\left|\frac{\sin(N\psi/2)}{\sin(\psi/2)}\right|, \quad \psi = \beta d \cos\phi + \xi}$$

Choosing appropriate $\beta d$ and $\xi$ (and $N$) we can design some useful arrays.

# Normalized array factor of five-element array




As $\phi$ varies from $0$ to $2\pi$, $\psi = \beta d \cos\phi + \xi$ varies from $\beta d + \xi$ to $-\beta d + \xi$ covering the range $2\beta d$. This defines the **visible range** of the pattern.

# Array factor of the uniform linear array

$$|A(\psi)| = \frac{1}{N}\left|\frac{\sin(N\psi/2)}{\sin(\psi/2)}\right|, \quad \psi = \beta d\cos\phi + \xi$$

## Main beam direction

$$\psi = 0 = \beta d\cos\phi_0 + \xi \quad \Leftrightarrow \quad \cos\phi_0 = -\frac{\xi}{\beta d}$$

Broadside array: $\phi_0 = \pm\pi/2$ and $\xi = 0$
Endfire array: $\phi = 0$ and $\xi = -\beta d$

## Null locations

$$\sin(N\psi/2) = 0 \quad \Rightarrow \quad \psi = \pm\frac{2\pi k}{N}, \quad k = 1, 2, 3, \ldots$$

The 3 dB beamwidth needs to be solved numerically, but the width of the main beam between the first nulls ($k = 1$) is straightforward to calculate.

# Sidelobe locations and sidelobe level

If $N$ is large, the sidelobes (minor maximas) occur approximately when

$$|\sin(N\psi/2)| = 1 \quad \Rightarrow \quad \frac{N\psi}{2} = \pm(2m+1)\frac{\pi}{2}, \quad m = 1, 2, 3, \ldots$$

($m = 0$ is inside the main beam)

## First sidelobe locations

$$\psi = \pm\frac{3\pi}{N}$$

## Sidelobe level

$$\frac{1}{N}\left|\frac{1}{\sin\frac{3\pi}{2N}}\right| \approx \frac{1}{N}\left|\frac{2N}{3\pi}\right| = \frac{2}{3\pi} \approx 0.212$$

By increasing $N$, we can get a very narrow main beam, but the sidelobe level stays fairly large.

# Smaller sidelobes?

Five-element broadside array ($\xi = 0$) with tapered excitation with amplitude ratios $1:2:3:2:1$

$$
\begin{aligned}
|A(\psi)| &= \frac{1}{9}\left|1 + 2e^{j\psi} + 3e^{j2\psi} + 2e^{j3\psi} + e^{j4\psi}\right| \\
&= \frac{1}{9}\left|e^{j2\psi}\right|\left|3 + 2(e^{j\psi} + e^{-j\psi}) + (e^{j2\psi} + e^{-j2\psi})\right| \\
&= \frac{1}{9}\left|3 + 4\cos\psi + 2\cos(2\psi)\right|
\end{aligned}
$$

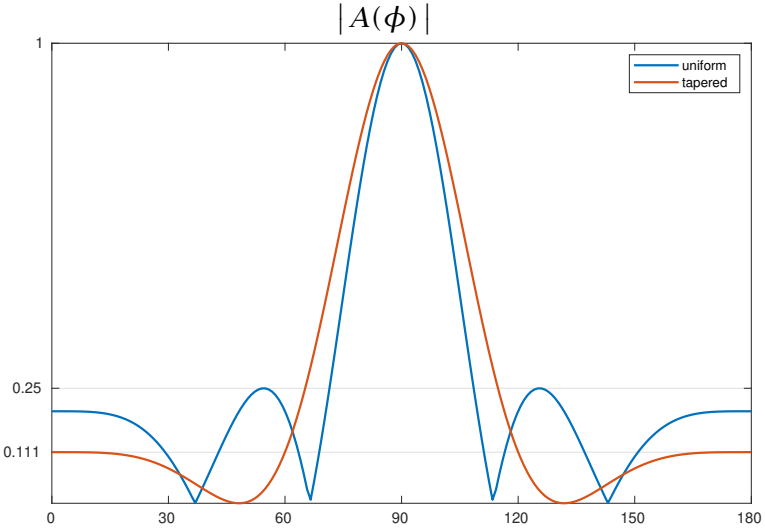
If the element spacing $d = \lambda/2$, we get $\psi = \pi\cos\phi$ and

$$
\left|A(\phi)\right| = \frac{1}{9}\left|3 + 4\cos(\pi\cos\phi) + 2\cos(2\pi\cos\phi)\right|
$$

The same array with uniform excitation has

$$
\left|A_u(\phi)\right| = \left|\frac{\sin\left(\frac{5\pi}{2}\cos\phi\right)}{5\sin\left(\frac{\pi}{2}\cos\phi\right)}\right|
$$

# Array factors for the five-element broadside arrays

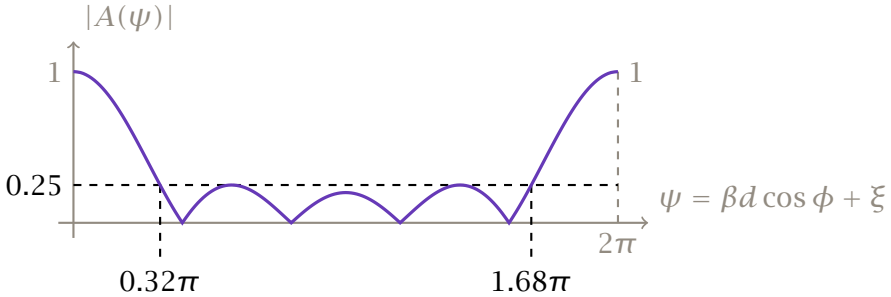



Tapered excitation gives smaller sidelobe level at the expense of a broader main beam.

(This holds for any electrically large antenna.)

# Example problem 11.3

Design an endfire array with five elements, as narrow main beam as possible, and sidelobe level 0.25. (Determine $\xi$ and the optimal $d$.)



# Solution

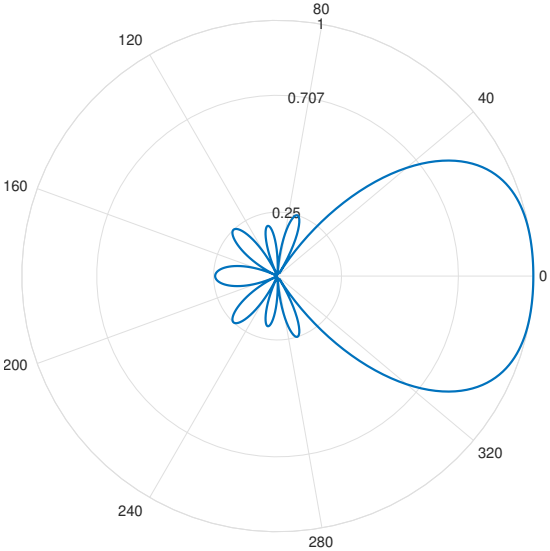
Endfire array:

$$\boxed{\xi = -\beta d} \quad \Rightarrow \quad \psi = \beta d \cos\phi - \beta d$$

Since $-2\beta d \leq \psi \leq 0$ in this case, the optimal choice is (from the figure):

$$-2\beta d = -1.68\pi \quad \Rightarrow \quad \boxed{d = 0.42\lambda}$$

The plot on the right is the array factor $|A(\phi)|$.



# Next week

This is the end of lecture week 11. We'll continue with transmit-receive systems and some other antenna types next week.

Next week we only have a lecture on Thursday (hall AS6 or online), since we celebrate Finnish Independence Day on Monday.

# Receiving antennas

Cheng 11-6

# Effective area

If the incident electromagnetic wave has time-average power density $\mathcal{P}_{\text{av}}$, the receiving antenna deliver the average power

$$\boxed{P_L = A_e \mathcal{P}_{\text{av}}}$$

to a matched load when the antenna is optimally oriented with respect to the polarization of the incident wave.

The quantity $A_e$ is called **effective area**, effective aperture, or receiving cross section.

The effective area is the receiving counterpart of the **directivity**

$$D = \frac{U_{\text{max}}}{U_{\text{av}}}$$

# Radio link between two lossless optimally oriented antennas


Antenna A radiates power $P_t$. At antenna B

$$\mathcal{P}_{\text{av}} = \frac{P_t}{4\pi r^2} D_{\text{A}}$$

and the power delivered to a matched load at antenna B is

$$P_L = A_{e\text{B}} \mathcal{P}_{\text{av}} \quad \Rightarrow \quad \frac{P_L}{P_t} = \frac{A_{e\text{B}} D_{\text{A}}}{4\pi r^2}$$

For the same link in the opposite direction, we get

$$\frac{P_L}{P_t} = \frac{A_{e\text{A}} D_{\text{B}}}{4\pi r^2}$$

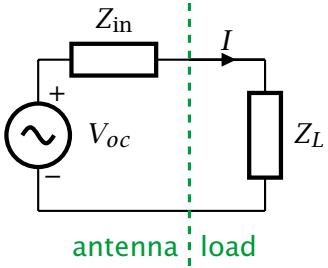
Due to **reciprocity** the power transfer ratio is the same both ways, so

$$\Rightarrow A_{e\text{B}} D_{\text{A}} = A_{e\text{A}} D_{\text{B}} \quad \Rightarrow \quad \frac{D_{\text{A}}}{A_{e\text{A}}} = \frac{D_{\text{B}}}{A_{e\text{B}}}$$

$\Rightarrow$ the ratio $D/A_e$ is the same constant for any (lossless) antenna.

# Receiving Hertzian dipole

The antenna impedance $Z_{\text{in}}$ is the same at transmission and reception.




For a lossless Hertzian dipole

$$Z_{\text{in}} = R_r + jX_{\text{in}}, \quad R_r = \underbrace{\frac{2\pi\eta_0}{3}}_{\approx 80\pi^2\,\Omega}\left(\frac{d\ell}{\lambda}\right)^2$$

The open circuit voltage $V_{oc} = E_i d\ell$ if $\mathbf{E}_i \parallel$ Hertzian dipole.

Matched load $Z_L = Z_{\text{in}}^* = R_r - jX_{\text{in}}$

$$P_L = \frac{1}{2}\left|\frac{V_{oc}}{Z_{\text{in}} + Z_L}\right|^2 R_r = \frac{|E_i|^2\, d\ell^2}{8R_r}$$

Since $\mathscr{P}_{\text{av}} = \frac{|E_i|^2}{2\eta_0}$, the effective area is

$$A_e = \frac{P_L}{\mathscr{P}_{\text{av}}} = \frac{\eta_0 d\ell^2}{4R_r} = \frac{3\lambda^2}{8\pi} = \frac{\lambda^2}{4\pi}D$$

where $D = 3/2$ for a Hertzian dipole.

# Effective area $A_e$ and directivity $D$

For any lossless optimally oriented antenna, we have

$$\boxed{D = 4\pi \frac{A_e}{\lambda^2}}$$

The textbook uses the somewhat more general definition with angle dependent effective area and directive gain

$$G_D(\theta, \phi) = \frac{4\pi}{\lambda^2} A_e(\theta, \phi)$$

If we assume optimal orientation but allow losses, we get the same relation between antenna (power) gain and effective area

$$G_P = 4\pi \frac{A_e}{\lambda^2}$$

# Example problem 12.1

At frequency $10\,\mathrm{GHz}$, what is the effective area (in $\mathrm{cm}^2$) of the following antennas:

**(a)** A Hertzian dipole of length $\lambda/100$.

**(b)** A half-wave dipole.

**(c)** A horn-antenna with 20 dB directivity.

**Solution:**

**(a)** $A_e = \dfrac{3\lambda^2}{8\pi} \approx 1.1\,\mathrm{cm}^2$

**(b)** $A_e = \dfrac{\lambda^2}{4\pi} D \approx \dfrac{\lambda^2}{4\pi} 1.64 \approx 1.2\,\mathrm{cm}^2$

**(c)** $A_e = \dfrac{\lambda^2}{4\pi} D \approx \dfrac{\lambda^2}{4\pi} 100 \approx 72\,\mathrm{cm}^2$

The geometrical aperture of the horn antenna $\approx A_e$ (slightly larger), while the lengths of the dipoles ($0.03\,\mathrm{cm}$ and $1.5\,\mathrm{cm}$) doesn't seem to correlate with $A_e$.

# Effective area $A_e$ and effective length $\ell_e$

The effective area $A_e$ is convenient for calculating received power, and for an electrically large antenna (such as a parabolic dish) this area can be about the same as the physical size of the antenna.

For thin linear antennas $A_e$ is conceptually a bit odd and the (vector) **effective length $\boldsymbol{\ell}_e$** appears more related to the geometric size:

$$|V_{oc}| = |\boldsymbol{\ell}_e \cdot \mathbf{E}_i|$$

See [11–4.2] in the textbook for more details.

For a Hertzian dipole $\ell_e = d\ell$. For a half-wave dipole $\ell_e = \lambda/\pi < 2h$.

# Backscatter cross section

If a object intercepts all power that hits the area $\sigma_{bs}$ and scatter that energy uniformly in all directions, we get

$$\mathcal{P}_s = \frac{\sigma_{bs}\mathcal{P}_i}{4\pi r^2}$$

where

$$\begin{aligned} \mathcal{P}_i &= \text{time-average incident power density at the object} \\ \mathcal{P}_s &= \text{time-average scattered power density at the receiver} \\ r &= \text{distance between scatterer and receiver} \end{aligned}$$

The area $\sigma_{bs}$ is called **backscatter cross section** or **radar cross section**

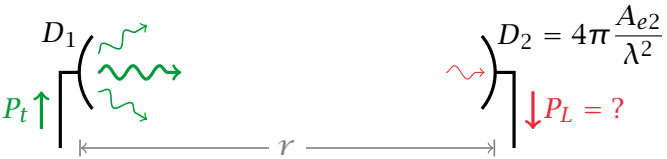
$$\boxed{\sigma_{bs} = 4\pi r^2 \frac{\mathcal{P}_s}{\mathcal{P}_i}}$$

Since $\mathcal{P}_s \sim 1/r^2$, this area is independent of $r$.

# Transmit-receive systems

Cheng 11-7

# Radio link from antenna 1 to antenna 2




Time-average power density at antenna 2

$$\mathscr{P}_{\text{av}} = \frac{P_t}{4\pi r^2} D_1$$

The time-average power delivered to a matched load at antenna 2 is

$$P_L = A_{e2}\mathscr{P}_{\text{av}} = \left(\frac{\lambda^2}{4\pi} D_2\right)\left(\frac{P_t D_1}{4\pi r^2}\right) = P_t D_1 D_2 \left(\frac{\lambda}{4\pi r}\right)^2$$

# Friis transmission formula

The **Friis transmission formula** can be written in several forms for lossless optimally oriented antennas

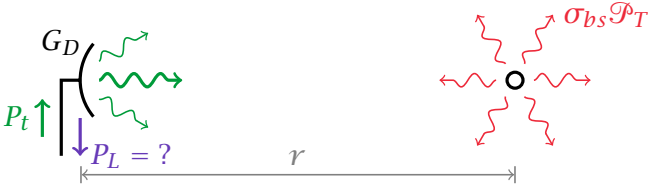
$$\boxed{\frac{P_L}{P_t} = D_1 D_2 \left(\frac{\lambda}{4\pi r}\right)^2 = \frac{A_{e1} A_{e2}}{r^2 \lambda^2}}$$

It is also common to express Friis transmission formula using antenna gain instead of directivity

$$\frac{P_L}{P_t} = G_{P1} G_{P2} \left(\frac{\lambda}{4\pi r}\right)^2$$

This version takes antenna losses into account, but still assume optimal orientation and matched load (and feed). Any mismatch at some point will decrease the power transmission.

# Radar equation




Time-average power density at the target

$$\mathscr{P}_T = \frac{P_t}{4\pi r^2} G_D \qquad G_D = \text{directive gain in target direction}$$

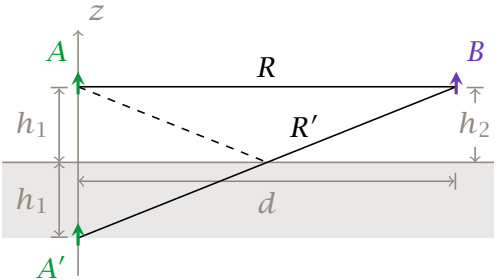
Using the radar cross section and the effective area we get the **radar equation**

$$P_L = A_e \frac{\sigma_{bs}\mathscr{P}_T}{4\pi r^2} = \left(\frac{\lambda^2}{4\pi} G_D\right) \frac{\sigma_{bs} P_t G_D}{(4\pi r^2)^2} \quad \Leftrightarrow \quad \boxed{\frac{P_L}{P_t} = \frac{\sigma_{bs}\lambda^2}{(4\pi)^3 r^4} G_D}$$

Note the $1/r^4$ dependence. Radar systems need high power to get good range?

# Antenna link above a PEC ground

Link from Hertzian dipole $A$ to $B$ above a PEC ground




The PEC plane can be replaced with the image dipole $A'$.

If $d \gg h_1, h_2$ we have $R' \approx R$ for the amplitude, but the difference in path length $R' - R$ can make the received signals in-phase or opposite phase or something in between.

Due to this multi-path transmission, we get the **path-gain factor** $|F|$ compared to the free-space link between the antennas. In this case

$$0 \leq |F| \leq 2.$$

In real life situations $|F|$ is difficult to estimate.

# Example problem 12.2

For the $S$-band data downlink of the Aalto-1 satellite, the following estimates can be found: Transmit power 1.3 W, frequency 2.4 GHz, satellite antenna gain 9 dB, ground station antenna gain 30 dB. The satellite orbit is 900 km above the ground and the maximum distance for the link is about 3000 km.

What is the received power at min and max distance, if we use this data and ignore atmospheric attenuation and any other loss or mismatch in the system?

**Solution:** Friis transmission formula gives

$$P_L \approx 1.3\,\mathrm{pW} \approx -89\,\mathrm{dBm} \qquad \text{and} \qquad P_L \approx 0.11\,\mathrm{pW} \approx -99\,\mathrm{dBm}$$

Be careful with the unit conversions and decibels!

# Outlook

This was the last lecture on the course. This week we have 2 homework problems instead of the usual 4.

**Midterm 2** on Monday will have the same format as midterm 1. It's nominally 2 h, with 15 min extra at the end to account for scanning and uploading the answers.

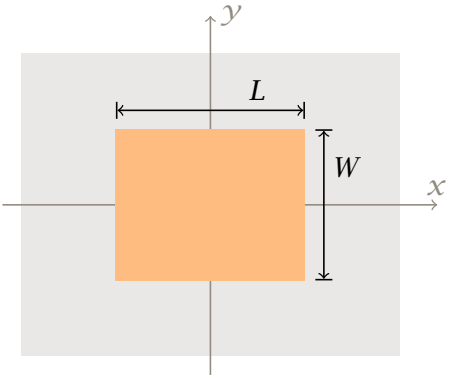
The **retake exam on January 10** is a possibility to retake one or both of the midterms. Since the teaching in the Spring term is supposed to be on campus, also the retake exam will be held on campus.

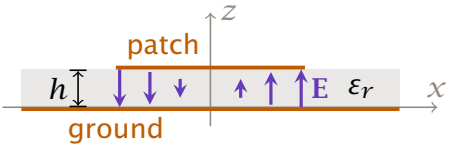
Please look at the remaining slides about Microstrip antennas and Aperture radiators on your own. Those topics will, however, not be included in Midterm 2 (or the corresponding retake).

# Microstrip antennas

(Extra material)

# Rectangular patch antenna




Rectangular metallic patch on top of a dielectric substrate with metal ground plane below.

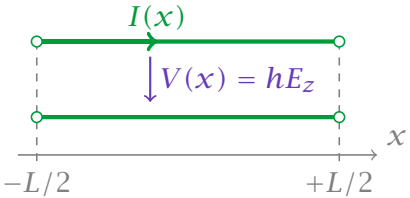
Antenna = open resonator

The electric field is mainly between the patch and the ground

$$\mathbf{E} = \mathbf{a}_z E_z(x, y).$$

Let's assume that $W < L$ (and $h \ll L, W$) and study the lowest resonant mode.

# Transmission line model and EM fields




Standing wave in a $L = \lambda/2$ transmission line with open ends:

$$V(x) = V_0 \sin(\beta x)$$
$$I(x) = -j\frac{V_0}{Z_0}\cos(\beta x)$$

with $\beta = 2\pi/\lambda = \pi/L$

Length $L = \lambda/2$ in the substrate gives the resonant frequency $f$ and electric field is

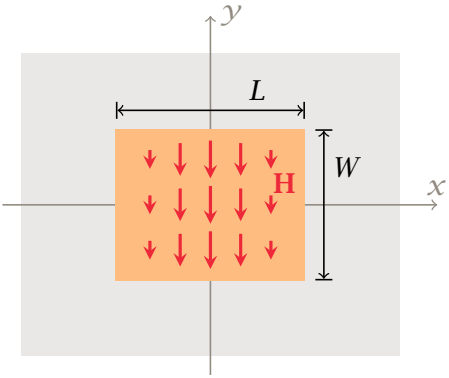
$$\mathbf{E} = \mathbf{a}_z E_0 \sin(kx), \quad \beta = k = \frac{\pi}{L}$$

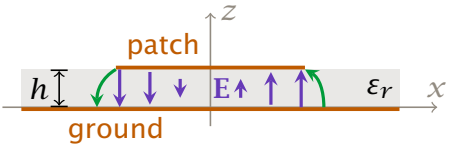
Magnetic field

$$\mathbf{H} = \frac{j}{\omega\mu}\nabla \times \mathbf{E} = -\mathbf{a}_y \frac{j}{k\eta}\frac{\partial E_z}{\partial x}$$
$$= -\mathbf{a}_y \frac{jE_0}{\eta}\cos(kx)$$

Note that $k$, $\eta$, $\lambda$ are the parameters in the dielectric substrate, not in free space.

# Lowest resonant mode




Ideal resonator model gives

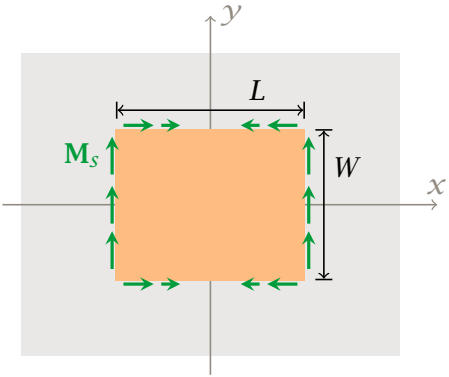
$$\mathbf{E} = \mathbf{a}_z E_0 \sin\left(\frac{\pi x}{L}\right)$$

$$\mathbf{H} = -\mathbf{a}_y \frac{jE_0}{\eta} \cos\left(\frac{\pi x}{L}\right)$$

under the patch and zero elsewhere.

The fringing E-field makes the patch effectively a bit longer and the fields leaking out from the resonator also cause radiation.

# Patch antenna radiation




Equivalent magnetic surface current at the edges of the resonator

$$\mathbf{M}_s = -\mathbf{a}_n \times \mathbf{E}$$

We can approximate the radiation using two $y$-oriented magnetic currents (or dipoles) of length $W$ spaced $L$ apart close to a PEC-plane.

This gives a main lobe in the $+z$ direction with $x$-directed E-field and $y$-directed H-field.

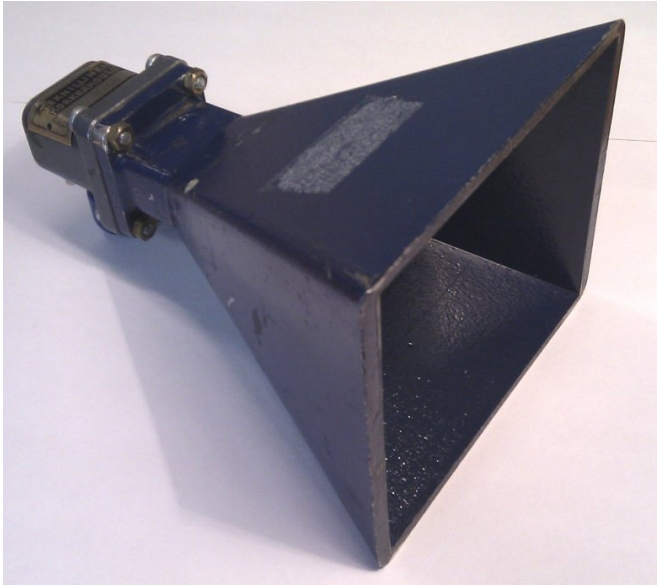
Typical directivity could be 7 to 8 dB.

[C.A. Balanis, *Antenna theory: analysis and design*, 3rd Ed., Wiley 2005, Ch. 14]

# Aperture radiators

Cheng 11-9

# Example: X-band horn antenna



A horn antenna borrowed from the lab in 2012.

How does it work and how well can we calculate its directivity?

# Coaxial to waveguide adapter


## Adapter HP X281A

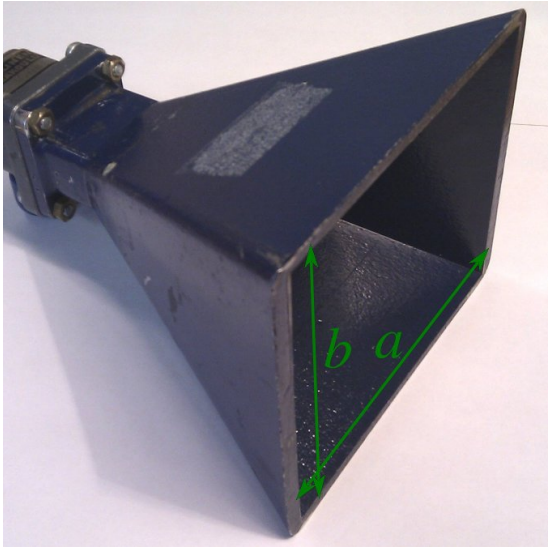
- $50\,\Omega$ coaxial type-N connector
- WR90-waveguide (X-band: $8.2\ldots12.4\,\mathrm{GHz}$, $0.90'' \times 0.40'' = 22.9\,\mathrm{mm} \times 10.2\,\mathrm{mm}$)
- Maximum VSWR $1.25$

Let's assume that the operation frequency is

$$\boxed{f = 10\,\mathrm{GHz}}$$

and that the adapter works perfectly near the center of the X-band.

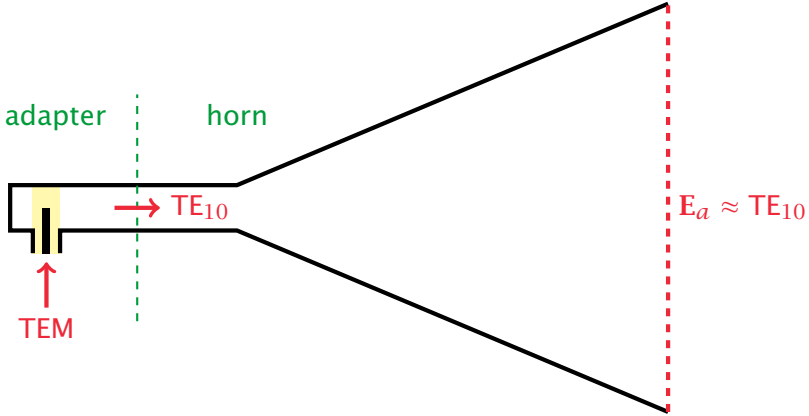
# Waveguide horn




## Rectangular aluminium horn

- $24\,\mathrm{mm} \times 10\,\mathrm{mm}$ waveguide of length $22\,\mathrm{mm}$
- Horn aperture $a = 105\,\mathrm{mm}$, $b = 89\,\mathrm{mm}$, horn length $95\,\mathrm{mm}$
- Manufacturer and model unknown. (No datasheet found for this part.)

# Cross section of the horn



# Aperture field radiation

## Assumptions

- The aperture field $\mathbf{E}_a(x, y)$ is known and has approximately uniform phase and linear polarization (e.g. $\mathbf{a}_y$) in the aperture.
- The fields are approximately zero in the rest of the plane containing the aperture.

## Radiation pattern

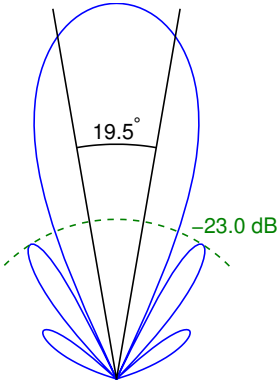
$$F(\theta, \phi) = \iint\limits_{\text{aperture}} E_a(x', y') e^{j\beta \sin\theta (x' \cos\phi + y' \sin\phi)} \, dx' \, dy'$$

- The (unnormalized) radiation pattern is essentially the 2D Fourier transform of the aperture distribution. [See the textbook for more details.]
- Fourier transform tables help to analyze aperture radiators, and inverse transforms can also be used for synthesis problems.

# Radiation pattern, beamwidth, sidelobe level, and directivity

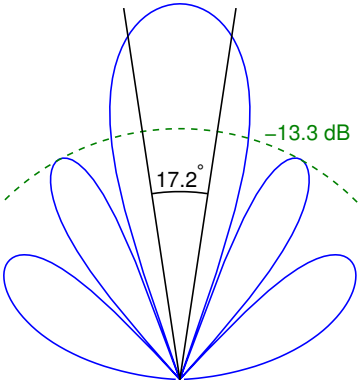
## X-band horn assuming $TE_{10}$ aperture field

$xz$ plane ($H$-plane)




aperture distribution $\cos(\pi x/a)$,
$-a/2 < x < a/2$, $a = 105\,\mathrm{mm}$

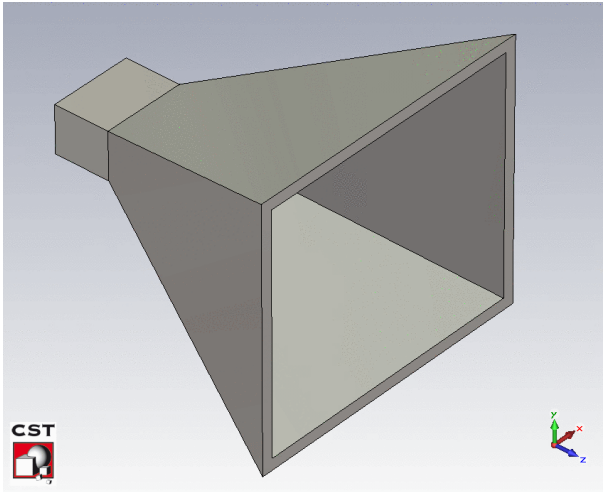
$yz$ plane ($E$-plane)




constant aperture distribution
$b = 89\,\mathrm{mm}$

$$\boxed{D = 20.2\,\mathrm{dB}}$$

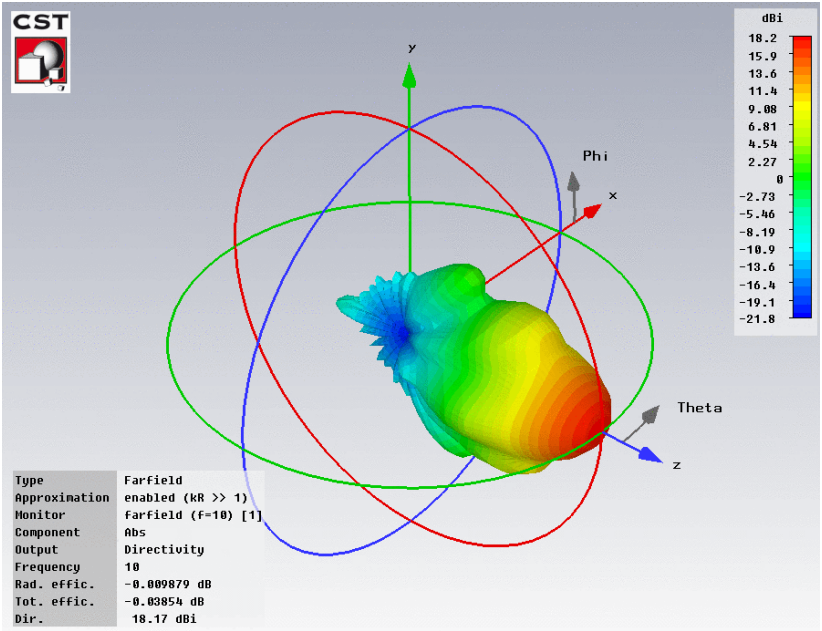
# Simulation geometry



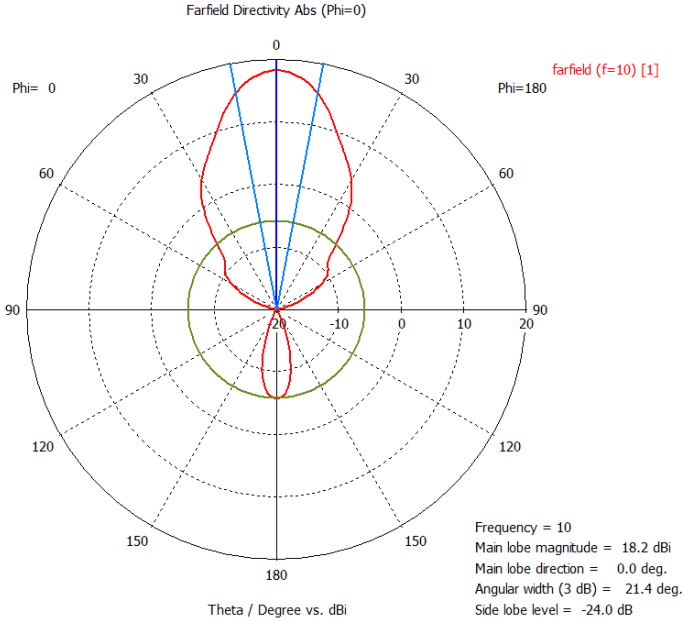

Geometrically accurate model of the horn + ideal waveguide port

*CST Microwave Studio 2011*

# 3D radiation pattern

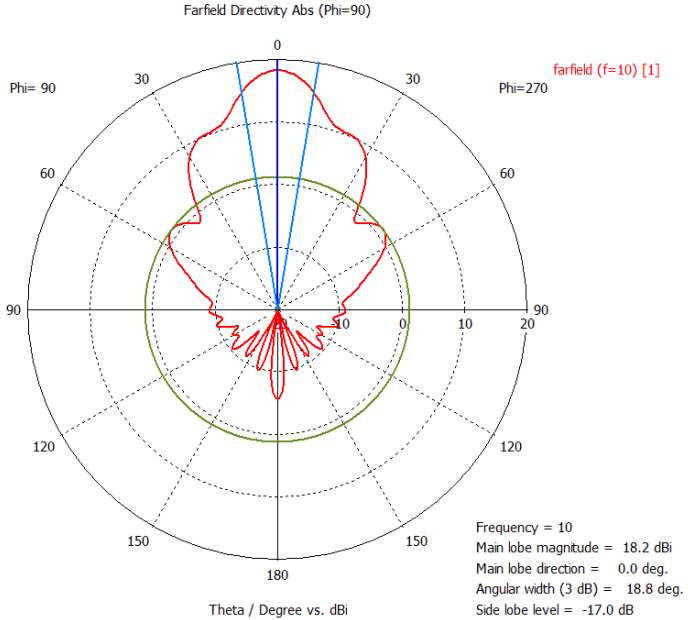


# Directive gain in $xz$ plane




Frequency = 10
Main lobe magnitude = 18.2 dBi
Main lobe direction = 0.0 deg.
Angular width (3 dB) = 21.4 deg.
Side lobe level = -24.0 dB

# Directive gain in $yz$ plane



# Summary of the results

| | Formulas | CST |
|---|---|---|
| Directivity | $20.2\,\mathrm{dB}$ | $18.2\,\mathrm{dB}$ |
| H-plane beamwidth | $19.5^\circ$ | $21.4^\circ$ |
| H-plane sidelobe level | $-23.0\,\mathrm{dB}$ | $-24.0\,\mathrm{dB}$ |
| E-plane beamwidth | $17.2^\circ$ | $18.8^\circ$ |
| E-plane sidelobe level | $-13.3\,\mathrm{dB}$ | $-17.0\,\mathrm{dB}$ |

There might be some small inaccuracy in the CST simulation, but the main discrepancy is that the actual aperture field deviates from $\mathrm{TE}_{10}$ significantly.

# Aperture efficiency

For aperture antennas it makes sense to define the **aperture efficiency** as:

$$\eta_a = \frac{A_e}{A} = \frac{\text{effective area}}{\text{physical area}}$$

For the previous horn antenna we get

$$A_e = \frac{\lambda^2}{4\pi} D \approx 74.9\,\text{cm}^2 \qquad \eta_a \approx 80\% \qquad \text{(calulated)}$$

$$A_e \approx 47.3\,\text{cm}^2 \qquad \eta_a \approx 51\% \qquad \text{(simulated)}$$

$$A = 93.5\,\text{cm}^2$$