Problem Set 7

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Exercise 1 - PS7

Consider the following matrix:

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}.$$

(a) Find all the eigenvalues of A and determine their multiplicity.

(b) Show that A is diagonalizable. Specifically, form a matrix P such that $D = P^{-1}AP$ is diagonal and verify that $D = P^{-1}AP$ holds.

Exercise 1 - Solution

The eigenvalues are $r_1 = 3$ (multiplicity 2) and $r_2 = 5$ (multiplicity 1).

An eigenvector for r_2 is $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. We can also find two linearly independent eigenvectors for r_1, e.g. $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\boldsymbol{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Therefore, we can form

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix},$$

and the diagonal matrix D is

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Exercise 2

Consider the following system of difference equations:

$$x_{t+1} = -5x_t + 2y_t$$
$$y_{t+1} = -2x_t - y_t,$$

with $t = 0, 1, 2, \dots$.

(a) Find the general solution.

(b) Find the solution for the initial conditions $x_0 = 2$ and $y_0 = 5$.

(c) Is the steady state $(x^*, y^*) = (0, 0)$ globally asymptotically stable? Why or why not?

Exercise 2 - Solution

The system's coefficient matrix has one eigenvalue $r = -3$ with multiplicity 2 and only one linearly independent eigenvector. An eigenvector is $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and a generalized eigenvector is $\boldsymbol{w} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$.
The general solution is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \left(c_0(-3)^t + tc_1(-3)^{t-1}\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_1(-3)^t \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}.$$

In the solution for the initial conditions $x_0 = 2$ and $y_0 = 5$, we have $c_0 = 2$ and $c_1 = 6$. The steady state is not stable because $|r| = 3 \geq 1$.

Exercise 3

Consider the following system of difference equations:

$$x_{t+1} = -2x_t - 15y_t$$
$$y_{t+1} = \frac{1}{2}x_t + ay_t$$

with $t = 0, 1, 2, \ldots$ and where $a \in \mathbb{R}$ is a parameter.

(a) Find all the values of a such that the system has a unique steady state.

(b) From now on, assume $a = \frac{7}{2}$. Find all the steady states.

(c) Find the general solution.

(d) Find the solution for the initial conditions $x_0 = -5$ and $y_0 = 1$.

(e) Is any steady state globally asymptotically stable? Why or why not?

Exercise 3 - Solution

The system has a unique steady state when det(I-A) is not zero, which is when a is not 7/2.
Any (x^*, y^*) such that $x^* + 5y^* = 0$ is a steady state.
The system's coefficient matrix has eigenvalues $r_1 = \frac{1}{2}$ and $r_2 = 1$, and the corresponding eigenvectors are $\boldsymbol{v}_1 = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$. The general solution is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = c_1 \left(\frac{1}{2}\right)^t \begin{pmatrix} -6 \\ 1 \end{pmatrix} + c_2 1^t \begin{pmatrix} -5 \\ 1 \end{pmatrix}.$$

When the initial conditions are $x_0 = -5$ and $y_0 = 1$, the solution is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \end{pmatrix},$$

so that $c_1 = 0$ and $c_2 = 1$. Notice that the solution is constant because the initial condition is a steady state. However, every steady state is unstable because $|r_2| = 1 \geq 1$.

Exercise 4

Consider the following system of difference equations:

$$x_{t+1} = 4x_t - 8y_t - 8$$
$$y_{t+1} = \frac{5}{8}x_t - 2y_t + 1$$

with $t = 0, 1, 2, \ldots$

(a) Find the steady state.

(b) Is the steady state unique? Why or why not?

(c) Find the general solution.

(d) Find the solution for the initial conditions $x_0 = 1$ and $y_0 = 1$.

(e) Is the steady state globally asymptotically stable? Why or why not?

Exercise 4 - Solution

We have that $\det(I - A) = -4 \neq 0$, so there is a unique steady state which is $(x^*, y^*) = (8, 2)$. The system's coefficient matrix has eigenvalues $r_1 = -1$ and $r_2 = 3$, and the corresponding eigenvectors are $\boldsymbol{v}_1 = \begin{pmatrix} \frac{8}{5} \\ 1 \end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$. The general solution is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = c_1(-1)^t \begin{pmatrix} \frac{8}{5} \\ 1 \end{pmatrix} + c_2(3)^t \begin{pmatrix} 8 \\ 1 \end{pmatrix} + \begin{pmatrix} 8 \\ 2 \end{pmatrix}.$$

In the solution for the initial conditions $x_0 = 1$ and $y_0 = 1$, we have that $c_1 = -\frac{5}{32}$ and $c_2 = -\frac{27}{32}$. The steady state is not stable because $|r_i| \geq 1$ for $i = 1, 2$.

Exercise 5

Solve the following initial value problems:

(a) $y\dot{y} = t$, $y(\sqrt{2}) = 1$;

(b) $y^2\dot{y} = t + 1$, $y(1) = 1$;

(c) $\dot{y} = \frac{y^3}{t^3}$, $y(1) = 1$;

(d) $\dot{y} = \frac{t^3}{y^3}$, $y(1) = 1$.

Exercise 5 - Solution

(a) By separating variables,

$$ydy = tdt.$$

Integrating both sides

$$\int ydy = \int tdt$$

and then evaluating the integrals yields

$$y^2 = t^2 + 2C.$$

At the initial condition, it must hold that

$$1 = 2 + 2C,$$

from which we easily get $C = -\frac{1}{2}$. Thus the unique solution of the IVP is

$$y(t) = \sqrt{t^2 - 1}.$$

Exercise 5 - Solution

(b) By separating variables,

$$y^2 dy = (t+1)dt.$$

Integrating both sides

$$\int y^2 dy = \int (t+1)dt$$

and then evaluating the integrals yields

$$y^3 = \frac{3}{2}t^2 + 3t + 3C.$$

At the initial condition, it must hold that

$$1 = \frac{3}{2} + 3 + 3C,$$

from which we obtain $C = -\frac{7}{6}$. Thus the unique solution of the IVP is

$$y(t) = \sqrt[3]{\frac{3}{2}t^2 + 3t - \frac{7}{2}}.$$

Exercise 5 - Solution

(c) By separating variables,

$$\frac{1}{y^3}dy = \frac{1}{t^3}dt.$$

Integrating both sides

$$\int \frac{1}{y^3}dy = \frac{1}{t^3}dt$$

and then evaluating the integrals yields

$$\frac{1}{y^2} = \frac{1}{t^2} + (-2)c.$$

At the initial condition, it must hold that

$$1 = 1 + (-2)c,$$

from which we obtain $C = 0$. Thus the unique solution of the IVP is

$$y(t) = t.$$

Exercise 5 - Solution

(d) By separating variables,

$$y^3 dy = t^3 dt.$$

Integrating both sides

$$\int y^3 dy = t^3 dt$$

and then evaluating the integrals yields

$$y^4 = t^4 + 4c.$$

At the initial condition, it must hold that

$$1 = 1 + 4c,$$

from which we obtain $C = 0$. Thus the unique solution of the IVP is

$$y(t) = t.$$