6 BEAM AND PLATE MODELS

LEARNING OUTCOMES

Students are able to solve the lecture problems, home problems, and exercise problems on the topics of week 49:

- ☐ Virtual work densities of the Bernoulli and Timoshenko beam models
- ☐ Displacement analysis by beam elements
- ☐ Virtual work densities of the Reissner-Mindlin and Kirchhoff plate models
- ☐ Displacement analysis by plate models

6.1 CONTINUOUS APPROXIMATIONS

Virtual work density expressions can be used with various approximation types in line, rectangle, circular, etc. domains. Valid selections for a simply supported Kirchhoff plate in bending on a rectangle domain $\Omega = [0, L] \times [0, H]$ are, e.g.,

- ☐ Polynomial basis approximation $w(x, y) = a_0 xy(x - L)(y - H)$

- ☐ Double sine series approximation $w(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \sin(i\pi \frac{x}{L}) \sin(j\pi \frac{y}{H})$

Although parameters a_0, a_{ij} etc. of the continuous approximations on Ω may not be displacements of certain points, the recipe for finding their values is the same as for the nodal values and an approximation based on element interpolants on Ω^e ($\Omega = \cup \Omega^e$ and $\cap \Omega^e = \varnothing$).

CONTINUOUS SERIES SOLUTION

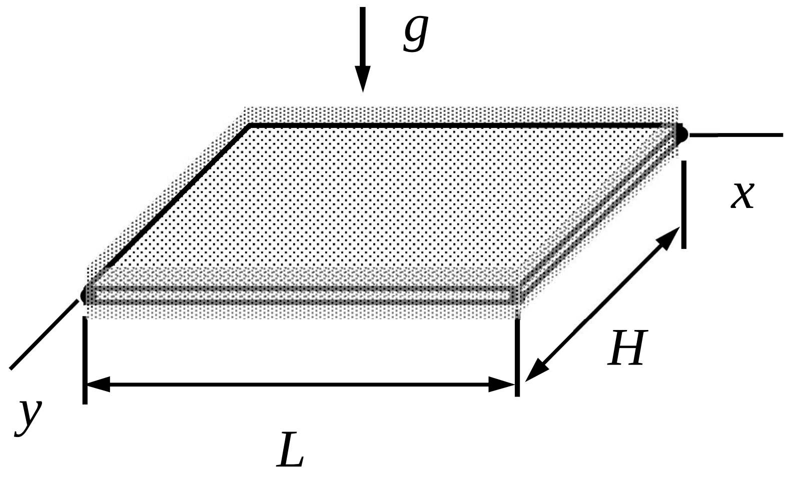
To find an approximate solution with a continuous series approximations for displacements/rotations and the virtual work density of a model

- Start with a linear combination of given functions with unknown coefficients (weights) $a_0, a_1, \ldots, a_n$. The series should satisfy the displacement/rotation boundary conditions no matter the coefficients.

- Substitute the series into the virtual work density expressions and continue with the recipe of the course to find the values of the coefficients.

Examples of useful function sets are polynomials of increasing order, harmonic functions of decreasing wavelength, etc. Mathematically, the function set used should be complete so that the interpolation error reduces in the number of terms.

EXAMPLE 6.1 Consider pure bending of a rectangle Kirchhoff plate $\Omega=(0,L)\times(0,H)$. Derive the series solution $w(x,y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}\sin(i\pi\frac{x}{L})\sin(j\pi\frac{y}{H})$ by considering the coefficients a_{ij} as the unknowns of the virtual work expression. Thickness t, Young's modulus E, and Poisson's ratio ν, and distributed load $f_z=\rho tg$ in direction of $z-$axis are constants.



Answer $a_{ij}=16\frac{f}{D}\frac{1}{ij\pi^2}/[(\frac{i\pi}{L})^2+(\frac{j\pi}{H})^2]^2 \quad i,j\in\{1,3,5,\ldots\},\ \ a_{ij}=0$ otherwise

- Shape functions need not to be polynomials. The well-known double sine-series solution to plate bending problem on a rectangle is an example of this theme. The solution uses the orthogonality properties of the sine and cosine functions (like)

$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx = \delta_{ij}\frac{L}{2} \quad \text{and} \quad \int_0^L \sin(i\pi\frac{x}{L})dx = \frac{L}{i\pi}[1-(-1)^i]$$

$$\int_0^H \sin(i\pi\frac{y}{H})\sin(j\pi\frac{y}{H})dy = \delta_{ij}\frac{H}{2} \quad \text{and} \quad \int_0^H \sin(i\pi\frac{y}{H})dy = \frac{H}{i\pi}[1-(-1)^i]$$

- When the series approximation is substituted there, virtual work expression becomes a variational expression for the unknown coefficients. Using then orthogonality of the sines and cosines on $\Omega=(0,L)\times(0,H)$, virtual work expressions of the internal and external forces boil down to

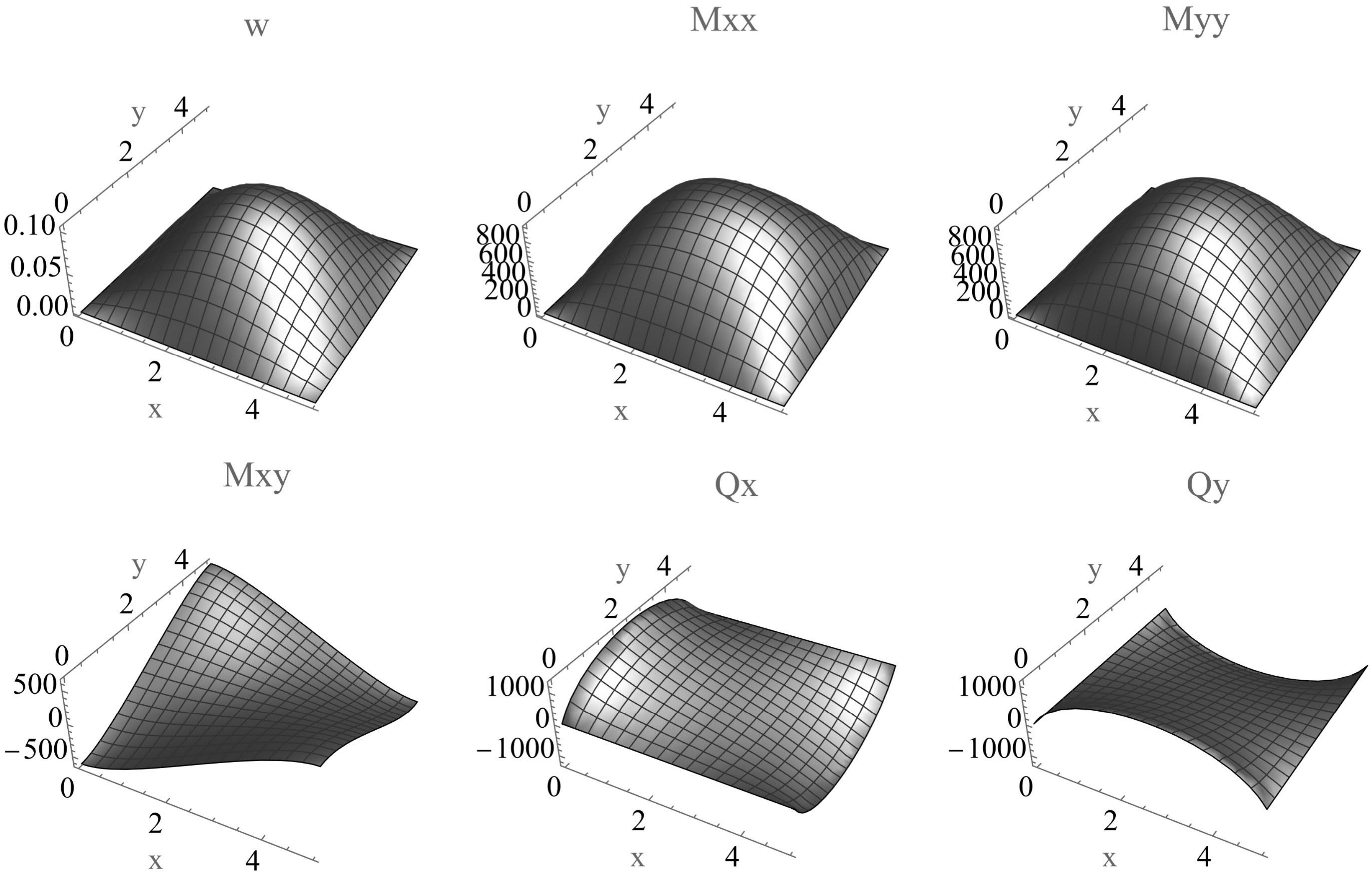
$$\delta W^{\text{int}} = -\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta a_{ij} \frac{Et^3}{12(1-\nu^2)} \frac{LH}{4} [(\frac{i\pi}{L})^2 + (\frac{j\pi}{H})^2]^2 a_{ij},$$

$$\delta W^{\text{ext}} = -\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta a_{ij} f_{ij}, \text{ where } \; f_{ij} = \int_0^L \int_0^H f(x,y) \sin(i\pi \frac{x}{L}) \sin(j\pi \frac{y}{H}) dx dy.$$

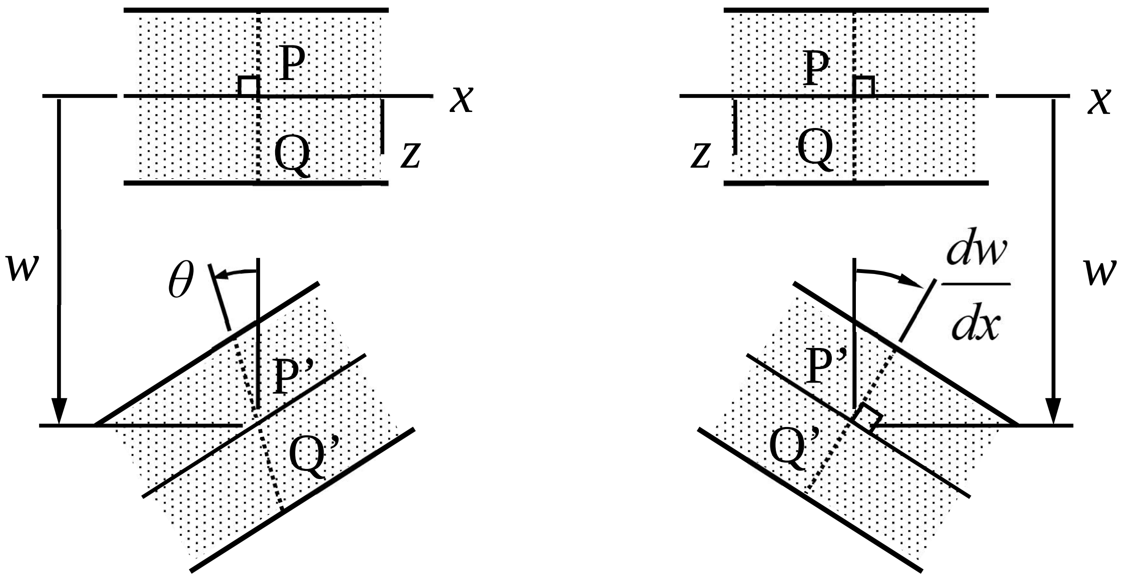
- As the terms are not connected in the virtual work expression (the matrix of the equation system implied by the principle of virtual work is diagonal), the fundamental lemma of variation calculus implies that

$$a_{ij} = 16 \frac{f}{D} \frac{1}{ij\pi^2} / [(\frac{i\pi}{L})^2 + (\frac{j\pi}{H})^2]^2 \quad i, j \in \{1,3,5,\ldots\}. \leftarrow$$

- With $L = H = 5\,\text{m}$, $t = 1\,\text{cm}$, $E = 210\,\text{GPa}$, $\nu = 0.3$, $\rho = 8000 \frac{\text{kg}}{\text{m}^3}$, $g = 9.81 \frac{\text{m}}{\text{s}^2}$, and 100 terms.


w
y
4
2
0
0.10
0.05
0.00
0
2
x
4
Mxx
y
4
2
0
800
600
400
200
0
0
2
x
4
Myy
y
4
2
0
800
600
400
200
0
0
2
x
4
Mxy
y
4
2
0
500
0
−500
0
2
x
4
Qx
y
4
2
0
1000
0
−1000
0
2
x
4
Qy
y
4
2
0
1000
0
−1000
0
2
x
4

6.2 BEAM MODEL



Timoshenko **Bernoulli**

Normal planes to the (material) axis of beam remain planes (Timoshenko) and normal to the axis (Bernoulli) in deformation. Mathematically $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta} \times \vec{\rho}_{\mathrm{PQ}}$ (rigid body motion with translation point P). In addition, normal stress in small dimensions vanishes.

- In terms of the displacement components $u(x)$, $v(x)$, $w(x)$ and rotation components $\phi(x)$, $\theta(x)$, $\psi(x)$ of the translation point, the Timoshenko model displacement components are ($\vec{u} = (u\vec{i} + v\vec{j} + w\vec{k}) + (\phi\vec{i} + \theta\vec{j} + \psi\vec{k}) \times (y\vec{j} + z\vec{k})$)

$$u_x(x, y, z) = u(x) + \theta(x)z - \psi(x)y,$$

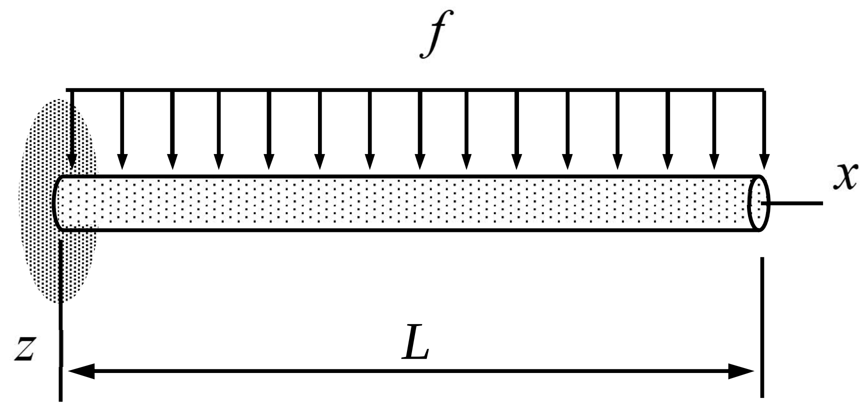
$$u_y(x, y, z) = v(x) - \phi(x)z,$$

$$u_z(x, y, z) = w(x) + \phi(x)y$$

In Bernoulli model, additionally $\theta = -dw/dx$ and $\psi = dv/dx$ so that normal planes remain normal to the axis.

- The kinematic assumption of the beam model means that $\sigma_{zz} = \sigma_{yy} = 0$.

EXAMPLE 6.2. Consider the beam of length L shown. Material properties E and G, cross-section properties A and I, and loading f are constants. Determine the deflection and rotation ($\theta = -dw/dx$) at the free end according to the Bernoulli beam model.



Answer $w(L) = \dfrac{fL^4}{8EI}$ and $\theta(L) = -\dfrac{dw}{dx}(L) = -\dfrac{fL^3}{6EI}$

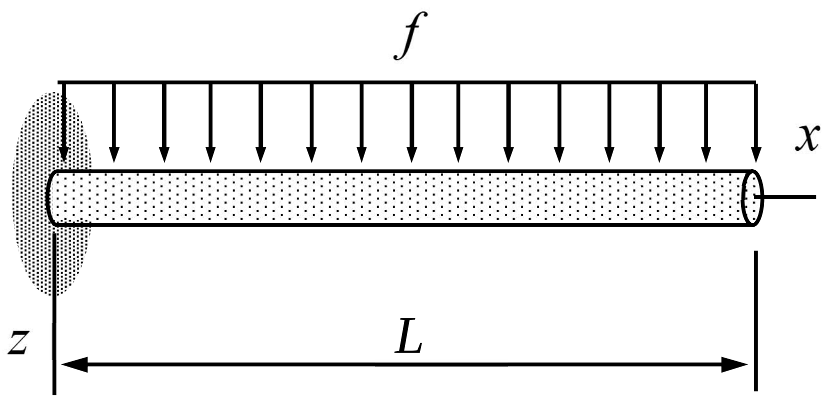
- Mathematica solution according to the Bernoulli model is obtained with the problem description:

	model	properties	geometry
1	BEAM	{{E, G}, {A, Iy, Iz}, {0, 0, f}}	Line[{1, 2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, θY[2], 0}

$$\left\{uZ[2] \to \frac{f L^4}{8 E Iy}, \theta Y[2] \to -\frac{f L^3}{6 E Iy}\right\}$$

EXAMPLE 6.3. Consider the beam of length L shown. Material properties E and G, cross-section properties A and I, and loading f are constants. Determine the deflection and rotation at the free end according to the Timoshenko beam model.



"Timoshenko effect"
$\sim 1+(t/L)^2$

Answer $w(L)=\frac{1}{8}\frac{fL^4}{EI}(1+\frac{4EI}{\kappa_z GAL^2})$ and $\theta(L)=-\frac{1}{6}\frac{fL^3}{EI}$

- Mathematica solution according to the Timoshenko model is obtained with the problem description ($\kappa_y = \kappa_z = 6/7$ are the shear correction factors for a circular cross-section):

	model	properties	geometry
1	BEAM	{{E, G, κy, κz}, {A, I, I}, {0, 0, f}}	Line[{1, 2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, θY[2], 0}

$$\left\{ uZ[2] \to \frac{1}{8} f L^2 \left(\frac{L^2}{E I} + \frac{4}{A G \kappa z} \right), \theta Y[2] \to -\frac{f L^3}{6 E I} \right\}$$

BEAM MODEL VIRTUAL WORK DENSITY

Beam element combines the bar, torsion, and the *xz*-plane and *xy*-plane bending modes

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d^2\delta v/dx^2 \\ d^2\delta w/dx^2 \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du/dx \\ d^2v/dx^2 \\ d^2w/dx^2 \end{Bmatrix} - \frac{d\delta\phi}{dx} GI_{rr} \frac{d\phi}{dx},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ -d\delta w/dx \\ d\delta v/dx \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}.$$

Bar and bending modes are connected unless the first moments S_z, S_y and the cross moment I_{zy} (off-diagonal terms of the matrix) of the cross-section vanish.

- If the loading modes are not connected, the simplest element interpolants (approximations) to u and ϕ are linear and those for v and w cubic ($\xi = x/h$):

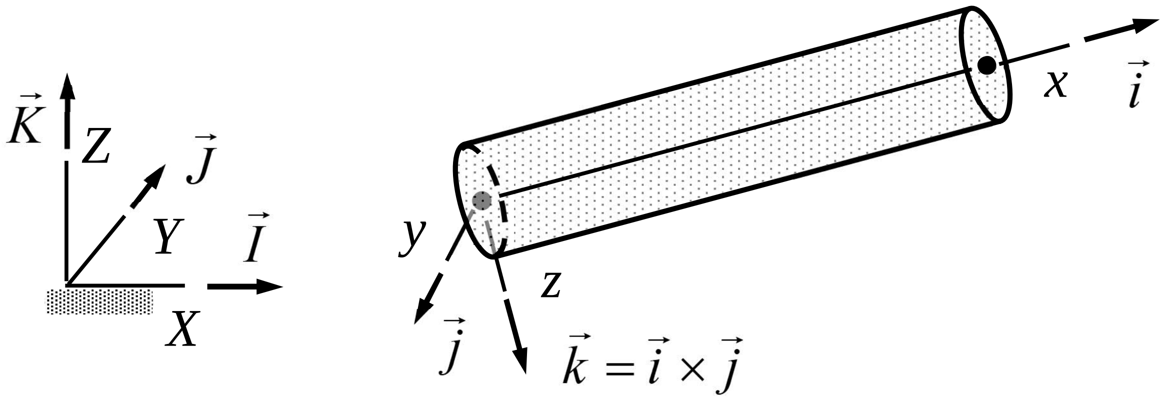
$$u(x) = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \text{ and } \phi(x) = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix},$$

$$v(x) = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ \hdashline u_{y2} \\ \theta_{z2} \end{Bmatrix} \text{ and } w(x) = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1} \\ -\theta_{y1} \\ \hdashline u_{z2} \\ -\theta_{y2} \end{Bmatrix}.$$

If the loading modes are connected, a quadratic three-node interpolant (approximation) to u is needed. Therefore, the clever selection $S_y = S_z = 0$ and $I_{yz} = 0$ of the material coordinate system simplifies calculations a lot.

BEAM COORDINATE SYSTEM

The *x*-axis of the material system is aligned with the axis of the body. The coordinates of end nodes define the components of $\vec{i}$. The orientation of $\vec{j}$ is one of the geometrical parameters of the element contribution and it has to be given in the same manner as the moments of area.



NOTICE: Mathematica code assumes that the *y* and *Y* axes are aligned, i.e., $\vec{j}=\vec{J}$ unless the direction of *y*-axis is specified explicitly in the beam element description.

MOMENTS OF CROSS-SECTION

Cross-section geometry of a beam have effect on the constitutive equation through moments of area (material is assumed to be homogeneous):

Zero moment: $A = \int dA$

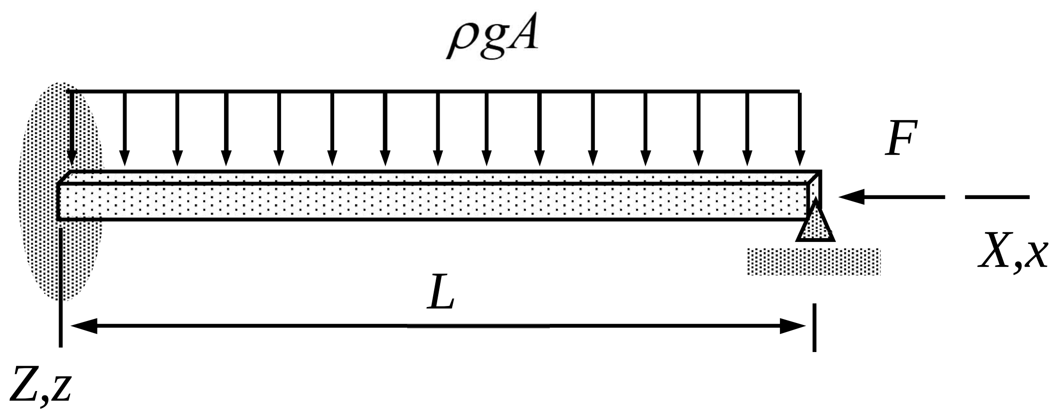
First moments: $S_z = \int ydA$ and $S_y = \int zdA$

Second moments: $I_{zz} = \int y^2 dA$, $I_{yy} = \int z^2 dA$, and $I_{yz} = \int yzdA$

Polar moment: $I_{rr} = \int y^2 + z^2 dA = I_{zz} + I_{yy}$

The moments depend on the selections of the material coordinate system. The origin and orientation can always be chosen so that $S_z = S_y = I_{yz} = 0$.

EXAMPLE 6.4. The beam of the figure is loaded by its own weight and a point force acting on the right end. Determine the displacement and rotation of the right end starting with the virtual density of the Bernoulli beam model. The x-axis of the material coordinate system is placed at the geometric centroid of the rectangle cross-section. Beam properties A, I_{yy}, I_{zz} and E of the planar problem are constants.



Answer: $u_{X2} = -\dfrac{FL}{EA}$ and $\theta_{Y2} = \dfrac{1}{48}\dfrac{\rho gAL^3}{EI_{yy}}$

- The left end of the beam is clamped and the right end simply supported. As the material and structural coordinate systems coincide $u_{x2}=u_{X2}$ and $\theta_{y2}=\theta_{Y2}$, the approximations of u and w simplify to

$$\begin{Bmatrix} u \\ w \end{Bmatrix} = \begin{Bmatrix} x/Lu_{X2} \\ L(x/L)^2(1-x/L)\theta_{Y2} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} du/dx \\ d^2w/dx^2 \end{Bmatrix} = \frac{1}{L}\begin{Bmatrix} u_{X2} \\ (2-6x/L)\theta_{Y2} \end{Bmatrix}.$$

- The moments of cross-section $S_y = S_z = 0$, I_{yy}, I_{zz} and $I_{yz} = 0$. As here $v=\phi=0$, $f_x = f_y = 0$ and $m_x = m_y = m_z = 0$, virtual work densities take the forms

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d^2\delta w/dx^2 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} EA & 0 \\ 0 & EI_{yy} \end{bmatrix} \begin{Bmatrix} du/dx \\ d^2w/dx^2 \end{Bmatrix} \text{ and } \delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ \rho g A \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{E}{L^2} \begin{bmatrix} A & 0 \\ 0 & I_{yy}(2-6x/L)^2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u_{X2} \\ \delta \theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ L(x/L)^2(1-x/L)\rho g A \end{Bmatrix}.$$

- Integrations over the domain $\Omega =]0, L[$ give the virtual work expressions

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} d\Omega = -\begin{Bmatrix} \delta u_{X2} \\ \delta \theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{E}{L} \begin{bmatrix} A & 0 \\ 0 & 4I_{yy} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega = \begin{Bmatrix} \delta u_{X2} \\ \delta \theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho g A L^2}{12} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad \Rightarrow$$

$$\delta W^{1} = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta \theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E}{L} \begin{bmatrix} A & 0 \\ 0 & 4I_{yy} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} - \frac{\rho g A L^2}{12} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \right).$$

- Virtual work expression of the point force follows from definition of work (or from the expression of formulae collection)

$$\delta W^2 = \begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -F \\ 0 \end{Bmatrix}.$$

- Principle of virtual work $\delta W = \delta W^1 + \delta W^2 = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E}{L}\begin{bmatrix} A & 0 \\ 0 & 4I_{yy} \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} -F \\ \rho g A L^2/12 \end{Bmatrix}\right) \quad \forall \begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix} \quad \Leftrightarrow$$

$$\frac{E}{L}\begin{bmatrix} A & 0 \\ 0 & 4I \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} -F \\ \rho g A L^2/12 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} = \begin{Bmatrix} -LF/EA \\ \rho g A L^3/(48 E I_{yy}) \end{Bmatrix}.$$

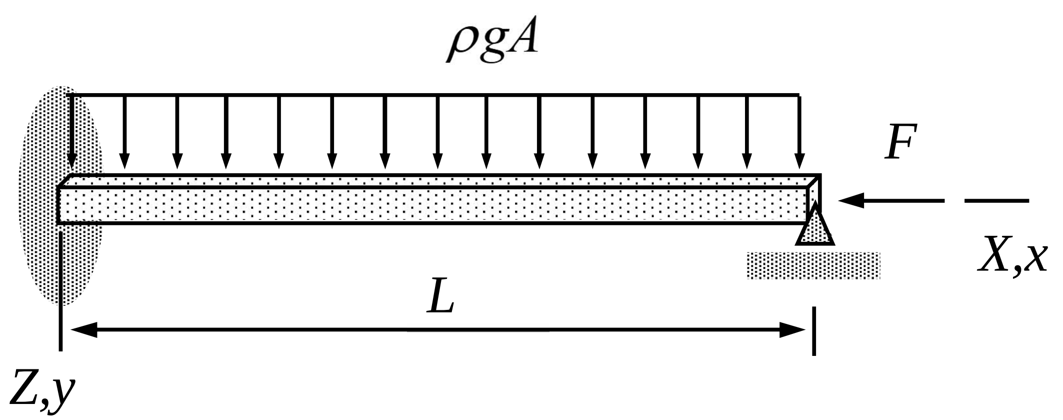
- Solution by the Mathematica code is obtained with the following problem description tables

	model	properties	geometry
1	BEAM	$\{\{E, G\}, \{A, Iyy, Izz\}, \{0, 0, A\,g\,\rho\}\}$	Line[{1, 2}]
2	FORCE	$\{-F, 0, 0\}$	Point[{2}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	$\{0, 0, 0\}$	$\{0, 0, 0\}$	$\{0, 0, 0\}$
2	$\{L, 0, 0\}$	$\{uX[2], 0, 0\}$	$\{0, \theta Y[2], 0\}$

$$\left\{uX[2] \to -\frac{F\,L}{A\,E},\ \theta Y[2] \to \frac{A\,g\,L^3\,\rho}{48\,E\,Iyy}\right\}$$

EXAMPLE 6.5. The Bernoulli beam of the figure is loaded by its own weight and a point force acting on the right end. Determine the displacement and rotation of the right end starting with the virtual density of the Bernoulli beam model. The x-axis of the material coordinate system is placed at the geometric centroid of the rectangle cross-section. Beam properties A, I_{yy}, I_{zz} and E are constants.



Answer: $u_{X2} = \dfrac{FL}{EA}$ and $\theta_{Y2} = \dfrac{1}{48}\dfrac{\rho g A L^3}{EI_{zz}}$

- Beam element definition of the Mathematica code requires the orientation of the $y-$axis unless $y-$ and $Y-$axes are aligned. Orientation is given by additional parameter defining the components of $\vec{j}$ in the structural coordinate system:

	model	properties	geometry
1	BEAM	{{E, G}, {A, Iyy, Izz, {0, 0, 1}}, {0, 0, A g ρ}}	Line[{1, 2}]
2	FORCE	{-F, 0, 0}	Point[{2}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, θY[2], 0}

$$\left\{uX[2] \to -\frac{F\,L}{A\,E},\ \theta Y[2] \to \frac{A\,g\,L^3\,\rho}{48\,Izz\,E}\right\}$$

TIMOSHENKO BEAM VIRTUAL WORK DENSITY

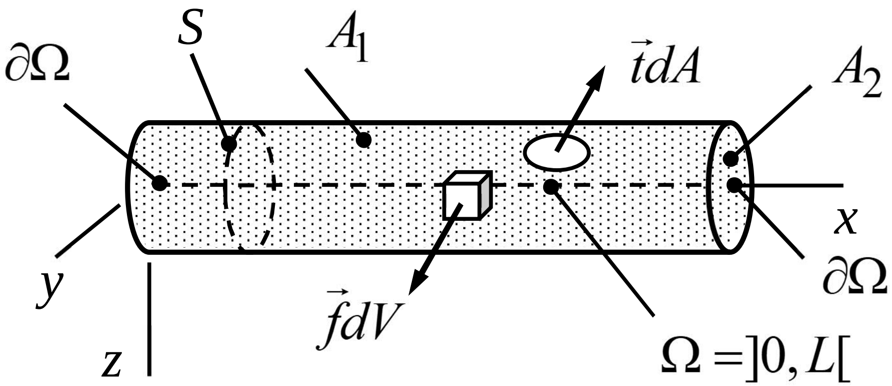
Timoshenko beam model takes into account transverse displacements due to shear. As the assumptions are less severe than those of the Bernoulli beam model, modelling error is smaller.

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d\delta\psi/dx \\ -d\delta\theta/dx \end{Bmatrix}^{\mathrm{T}} E\begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix}\begin{Bmatrix} du/dx \\ d\psi/dx \\ -d\theta/dx \end{Bmatrix} - \begin{Bmatrix} -\delta\psi + d\delta v/dx \\ \delta\theta + d\delta w/dx \\ \delta d\phi/dx \end{Bmatrix}^{\mathrm{T}} \times$$

$$\times G\begin{bmatrix} A & 0 & -S_y \\ 0 & A & S_z \\ -S_y & S_z & I_{rr} \end{bmatrix}\begin{Bmatrix} -\psi + dv/dx \\ \theta + dw/dx \\ d\phi/dx \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}.$$

If $S_z = S_y = 0$ and $I_{yz} = 0$, bar, torsion, and bending modes contribute to the virtual work expression as if they were separate bar, torsion and bending elements.

- Beam is a thin body in two dimensions



- The kinematic assumption of the Timoshenko beam model and definition of strain give the displacement and the non-zero strain components ($R_\psi = -\psi + dv/dx$, $R_\theta = \theta + dw/dx$)

$$\begin{Bmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{Bmatrix} = \begin{Bmatrix} u(x) + z\theta(x) - y\psi(x) \\ v(x) - z\phi(x) \\ w(x) + y\phi(x) \end{Bmatrix} \Rightarrow$$

$$\varepsilon_{xx} = \frac{du}{dx} + z\frac{d\theta}{dx} - y\frac{d\psi}{dx} \quad \text{and} \quad \begin{Bmatrix} \gamma_{xy} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} -\psi + dv/dx - zd\phi/dx \\ \theta + dw/dx + yd\phi/dx \end{Bmatrix}.$$

- The kinetic assumptions $\sigma_{zz} = \sigma_{yy} = 0$ and the generalized Hooke's law give the non-zero stress components

$$\sigma_{xx} = E\varepsilon_{xx} = E(\frac{du}{dx} + z\frac{d\theta}{dx} - y\frac{d\psi}{dx}) \quad \text{and} \quad \begin{Bmatrix} \sigma_{xy} \\ \sigma_{zx} \end{Bmatrix} = G\begin{Bmatrix} -\psi + dv/dx - zd\phi/dx \\ \theta + dw/dx + yd\phi/dx \end{Bmatrix}.$$

- With notation $r^2 = y^2 + z^2$ the generic expressions for the virtual work densities per unit volume simplify to (some manipulations are needed here)

$$\delta w_V^{\mathrm{int}} = -\begin{Bmatrix} d\delta u/dx \\ d\delta\psi/dx \\ -d\delta\theta/dx \end{Bmatrix}^{\mathrm{T}} E\begin{bmatrix} 1 & -y & -z \\ -y & yy & yz \\ -z & zy & zz \end{bmatrix}\begin{Bmatrix} du/dx \\ d\psi/dx \\ -d\theta/dx \end{Bmatrix} - \begin{Bmatrix} -\delta\psi + d\delta v/dx \\ \delta\theta + d\delta w/dx \\ \delta d\phi/dx \end{Bmatrix}^{\mathrm{T}} \times$$

$$\times G\begin{bmatrix} 1 & 0 & -z \\ 0 & 1 & y \\ -z & y & r^2 \end{bmatrix}\begin{Bmatrix} -\psi + dv/dx \\ \theta + dw/dx \\ d\phi/dx \end{Bmatrix},$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -zf_y + yf_z \\ zf_x \\ -yf_x \end{Bmatrix},$$

$$\delta w_A^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -zt_y + yt_z \\ zt_x \\ -yt_x \end{Bmatrix}.$$

- Virtual work density of the internal forces is obtained as an integral over the small dimensions which is the cross-section (the volume element $dV = dAd\Omega$).

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d\delta\psi/dx \\ -d\delta\theta/dx \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du/dx \\ d\psi/dx \\ -d\theta/dx \end{Bmatrix} - \begin{Bmatrix} -\delta\psi + d\delta v/dx \\ \delta\theta + d\delta w/dx \\ d\delta\phi/dx \end{Bmatrix}^{\mathrm{T}} \times$$

$$\times G \begin{bmatrix} A & 0 & -S_y \\ 0 & A & S_z \\ -S_y & S_z & I_{rr} \end{bmatrix} \begin{Bmatrix} -\psi + dv/dx \\ \theta + dw/dx \\ d\phi/dx \end{Bmatrix}.$$

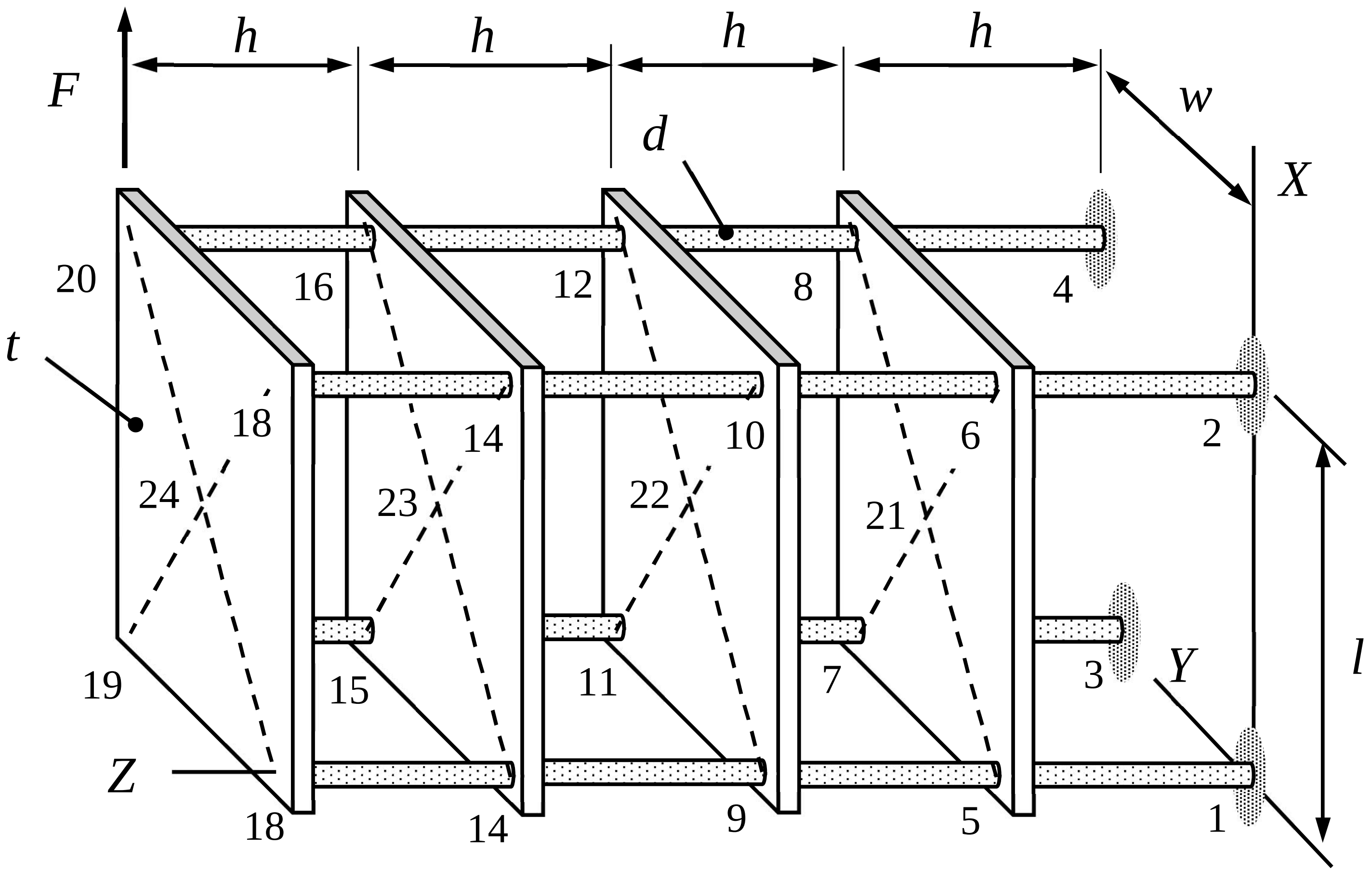
- The contributions coming from the external forces follow in the same manner. Assuming that the volume force is constant (in an element) and that the surface forces are acting on the end surfaces only, the expressions become

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix} \text{ and } \delta W^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix}.$$

The last contribution is taken care of by a point force element in the Mathematica code.

IMPORTANT. The simplest possible linear approximation to the displacement and rotation components does not give a good numerical method unless numerical tricks like under-integration etc. are applied. To avoid numerical problems, approximations should be chosen cubic even if the exact solution is a simple polynomial! The Mathematica code uses a cubic approximation to all the unknowns and static condensation to end up with a two-node element.

EXAMPLE 6.6. A structure is modeled by using 16 beams and 4 rigid bodies. Assuming that a point force with the magnitude *F* is acting as shown in the figure, determine the displacement of the point of action in the direction of the force.



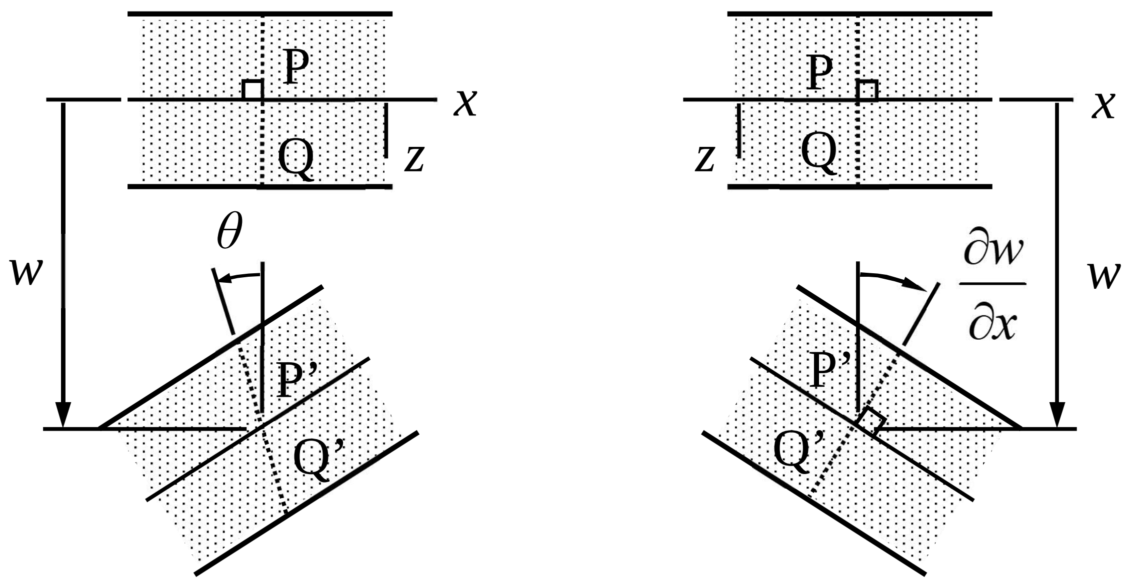
- Rigid bodies can be modelled by using a one node force element and rigid links with the other nodes. The problem description tables are given in the examples section of the Mathematica solver. The displacement of node 20 in the direction of X −axis as given by the solver is

$$u_{X20} = \frac{16Fh^3}{3\pi d^4 E}\left(\frac{64d^2+4l^2}{d^2+4l^2} + \frac{3w^2}{2h^2G/E+3l^2+3w^2}\right)$$

- If $E = 210\cdot 10^3\,\text{N/mm}^2$, $G = 80\cdot 10^3\,\text{N/mm}^2$, $d = 6.9\,\text{mm}$, $l = 408\,\text{mm}$, $w = 263\,\text{mm}$, $h = 170\,\text{mm}$, $F = 69\,\text{N}$, the displacement

$$u_{X20} = 1.56\,\text{mm}.$$

6.3 PLATE MODEL



Reissner-Mindlin **Kirchhoff**

Straight line segments perpendicular to the reference-plane remain straight in deformation (Reissner-Mindlin) and perpendicular to the reference-plane (Kirchhoff). In addition, transverse normal stress component is negligible.

- Normal line segments to the reference-plane move as rigid bodies. In terms of the displacement components $u(x,y)$, $v(x,y)$, $w(x,y)$ and rotation components $\phi(x,y)$, $\theta(x,y)$ of the translation point at the reference-plane, the displacement components are given by $\vec{u} = (u\vec{i} + v\vec{j} + w\vec{k}) + (\phi\vec{i} + \theta\vec{j}) \times z\vec{k}$). In component form

$$u_x(x,y,z) = u(x,y) + \theta(x,y)z,$$

$$u_y(x,y,z) = v(x,y) - \phi(x,y)z,$$

$$u_z(x,y,z) = w(x,y).$$

Rotation component in the z-direction is missing!

In the Kirchhoff model $u(x,y)$, $v(x,y)$, $w(x,y)$ and $\phi = \partial w / \partial y$ and $\theta = -\partial w / \partial x$ define the displacement field.

KIRCHHOFF PLATE VIRTUAL WORK DENSITY

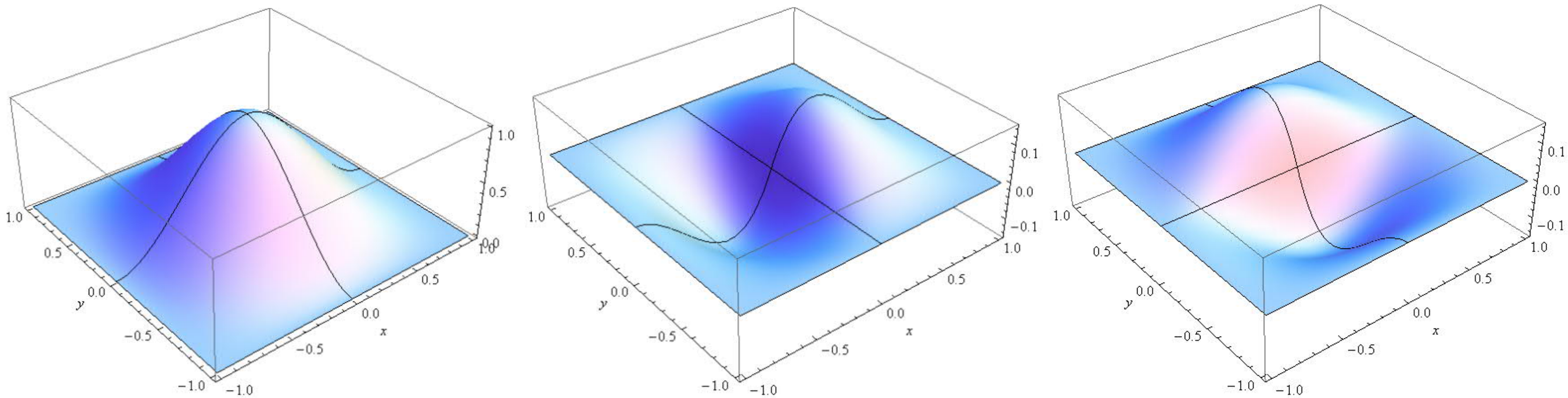
Virtual work densities combine the plane-stress thin slab and the plate bending modes which disconnect if the first moment of thickness vanishes. Virtual work densities of the bending mode of the Kirchhoff plate are

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} [E]_{\sigma} \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix}, \quad \left(\begin{Bmatrix} \phi \\ \theta \end{Bmatrix} = \begin{Bmatrix} \partial w / \partial y \\ -\partial w / \partial x \end{Bmatrix} \right)$$

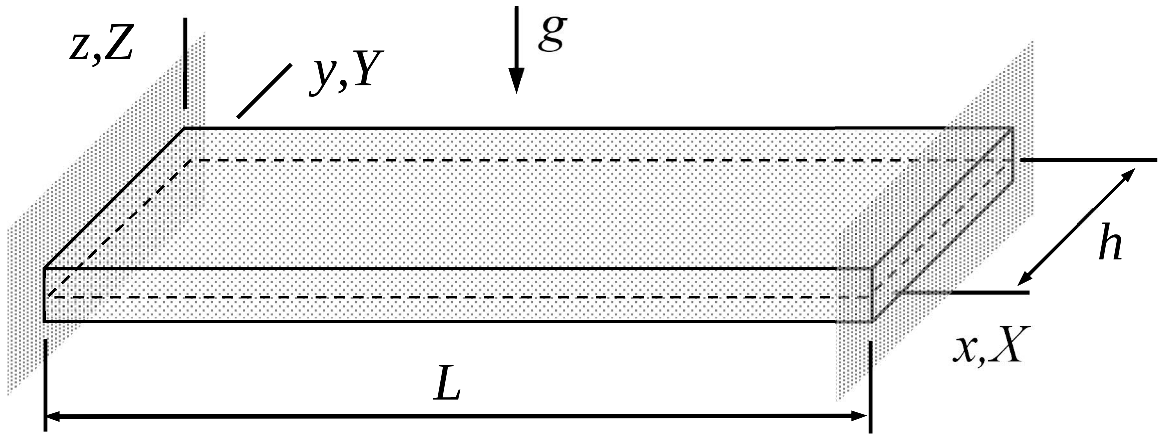
$$\delta w_{\Omega}^{\text{ext}} = \delta w f_z .$$

The planar solution domain (reference-plane) can be represented by triangular or rectangular elements. Interpolation of displacement components $w(x, y)$ should be continuous and have also continuous derivatives at the element interfaces.

- The severe continuity requirement of the approximation at the element interfaces is problematic in practice and cannot be satisfied with a simple interpolation of the nodal values. The figure illustrates the shape functions corresponding to displacement and rotation at a typical node in a patch of 4 square elements. The shape functions vanish outside the patch. In the course, Kirchhoff model is used only in calculations with domains of one element (no interfaces - no problems).



EXAMPLE 6.7. Consider a plate strip loaded by its own weight. Determine the deflection w according to the Kirchhoff model. Thickness, length of the plate are t, L, and h, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Use the one parameter approximation $w(x) = a_0(1-x/L)^2(x/L)^2$.



Answer: $$w = -\frac{\rho g L^4}{2Et^2}(1-\nu^2)(1-\frac{x}{L})^2(\frac{x}{L})^2$$

- Approximation satisfies the boundary conditions ‘a priori’ and contains a free parameter a_0 (not associated with a node) to be solved by using the principle of virtual work:

$$w = a_0(1-\frac{x}{L})^2(\frac{x}{L})^2 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = a_0 \frac{2}{L^2}[1-6\frac{x}{L}+6(\frac{x}{L})^2] \text{ and } \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x \partial y} = 0 .$$

- When the approximation is substituted there, virtual work densities simplify to

$$\delta w_\Omega^{\text{int}} = -a_0 \delta a_0 \frac{Et^3}{3(1-\nu^2)} \frac{1}{L^4}[1-6\frac{x}{L}+6(\frac{x}{L})^2]^2,$$

$$\delta w_\Omega^{\text{ext}} = -\delta a_0 (1-\frac{x}{L})^2(\frac{x}{L})^2 \rho g t .$$

- Integrations over the domain $\Omega =]0,L[\times]0,h[$ give the virtual works of internal and external forces

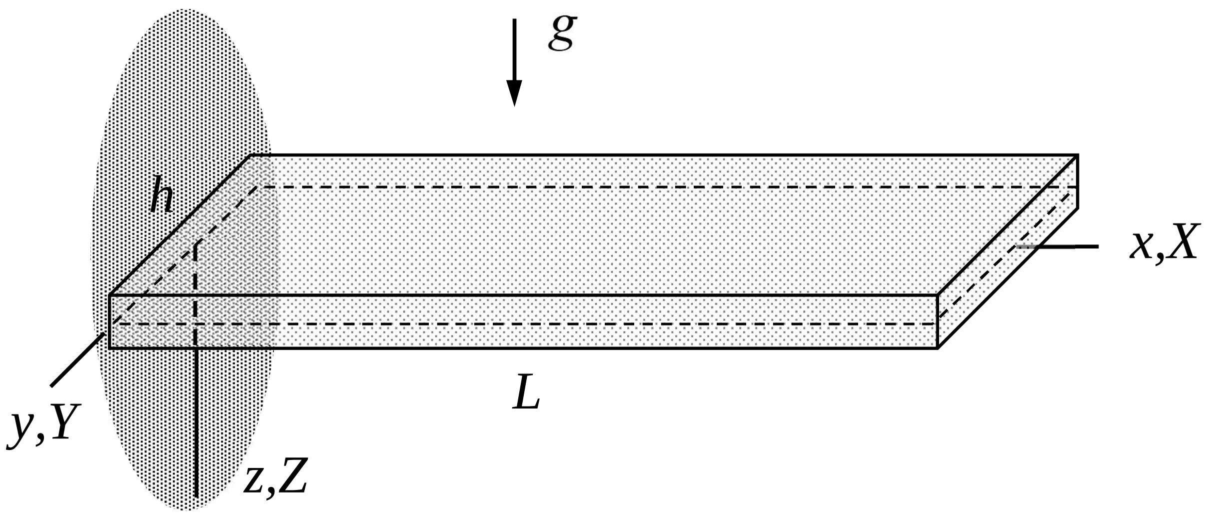
$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega = -a_0 \delta a_0 \frac{1}{15} \frac{hEt^3}{L^3(1-\nu^2)},$$

$$\delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega = -\delta a_0 \frac{1}{30} \rho g t L h.$$

- Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give finally $\forall \delta a_0$

$$\delta W = -\delta a_0 \left(\frac{1}{15} \frac{hEt^3}{L^3(1-\nu^2)} a_0 + \frac{1}{30} \rho g t L h\right) = 0 \quad \Leftrightarrow \quad a_0 = -\frac{1}{2} \frac{\rho g t L^4}{Et^2}(1-\nu^2). \quad \Leftarrow$$

EXAMPLE 6.8. A rectangular plate is loaded by its own weight. Determine the deflection of the plate at the free end by using the Kirchhoff plate model and one element. Thickness, width, and length of the plate are t, h and L, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν of the material are constants. Assume that deflection w depends only on x.



Answer: $w(L) = u_{Z2} = \dfrac{3}{2}\dfrac{g\rho L^4}{Et^2}(1-\nu^2)$ (Bernoulli beam $w(L) = \dfrac{3}{2}\dfrac{g\rho L^4}{Et^2}$)

- As the solution is assumed to depend on x only and the material and structural coordinate systems coincide, one may use the cubic approximation of the Bernoulli beam model (bending in $xz-$plane and $\xi = x/L$). Let us denote the displacement and rotation at the free end by $u_{z2} = u_{Z2}$ and $\theta_{y2} = \theta_{Y2}$ to get

$$w = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ L(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ L\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ \hdashline u_{Z2} \\ -\theta_{Y2} \end{Bmatrix} = \begin{Bmatrix} (3-2x/L)(x/L)^2 \\ -L(x/L)^2(x/L-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} \Rightarrow$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{L^2}\begin{Bmatrix} 6(L-2x)/L \\ -2(L-3x) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} \quad \text{and} \quad \frac{\partial^2 \delta w}{\partial x^2} = \begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2}\begin{Bmatrix} 6(L-2x)/L \\ -2(L-3x) \end{Bmatrix}.$$

- When the approximation is substituted there, virtual work densities simplify to

$$\delta w_\Omega^{\text{int}} = -\frac{Et^3}{12(1-\nu^2)L^4}\begin{Bmatrix}\delta u_{Z2}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\frac{1}{L^2}(6L-12x)^2 & \frac{1}{L}(2L-6x)(6L-12x)\\ \frac{1}{L}(2L-6x)(6L-12x) & (2L-6x)^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix}$$

$$\delta w_\Omega^{\text{ext}} = \begin{Bmatrix}\delta u_{Z2}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\rho g t\begin{Bmatrix}(3-2x/L)(x/L)^2\\ -L(x/L)^2(x/L-1)\end{Bmatrix}.$$

- Integrations over the domain $\Omega =]0,L[\times]0,h[$ give the virtual works of internal and external forces

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega = -\frac{Et^3 h}{12(1-\nu^2)L^3}\begin{Bmatrix}\delta u_{Z2}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}12 & 6L\\ 6L & 4L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega = \begin{Bmatrix}\delta u_{Z2}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{\rho g t h L}{12}\begin{Bmatrix}6\\ L\end{Bmatrix}.$$

- Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give finally

$$\delta W = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{Et^3 h}{12(1-\nu^2)L^3} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \frac{\rho g t h L}{12} \begin{Bmatrix} 6 \\ L \end{Bmatrix} \right) = 0 \quad \forall \begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix} \quad \Leftrightarrow$$

$$\frac{Et^3 h}{12(1-\nu^2)L^3} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \frac{\rho g t h L}{12} \begin{Bmatrix} 6 \\ L \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = \frac{\rho g t (1-\nu^2) L^4}{Et^3} \begin{Bmatrix} 3/2 \\ -2/L \end{Bmatrix}.$$

A more detailed analysis may give dependence on y − coordinate which was excluded by the displacement assumption of the simplified analysis.

REISSNER-MINDLIN PLATE VIRTUAL WORK DENSITY

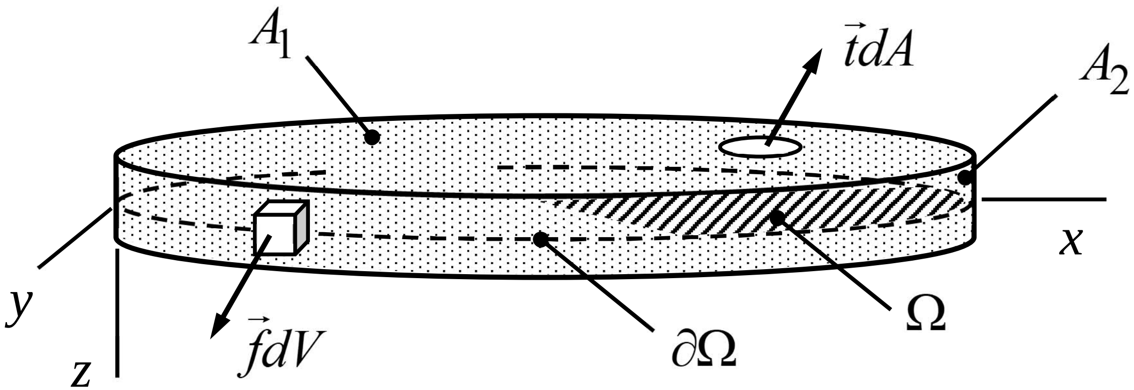
Virtual work densities combine the thin slab and plate bending modes which disconnect if the first moment of thickness vanishes. Virtual work densities of the Reissner-Mindlin plate bending mode are

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} -\partial\delta\theta / \partial x \\ \partial\delta\phi / \partial y \\ \partial\delta\phi / \partial x - \partial\delta\theta / \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_{\sigma} \begin{Bmatrix} -\partial\theta / \partial x \\ \partial\phi / \partial y \\ \partial\phi / \partial x - \partial\theta / \partial y \end{Bmatrix} - \begin{Bmatrix} \partial\delta w / \partial y - \delta\phi \\ \partial\delta w / \partial x + \delta\theta \end{Bmatrix}^{\mathrm{T}} tG \times,$$

$$\times \begin{Bmatrix} \partial w / \partial y - \phi \\ \partial w / \partial x + \theta \end{Bmatrix}, \qquad \delta w_{\Omega}^{\text{ext}} = \delta w f_z .$$

The planar solution domain can be represented by triangular or rectangular elements. Interpolation of displacement and rotation components $w(x, y)$, $\phi(x, y)$, and $\theta(x, y)$ should be continuous at the element interfaces.

- Plate is a thin body in one-dimension



- Normals to the reference plane (not necessarily the symmetry or mid-plane) remain straight in deformation. Kinematic assumption $\vec{u} = (u\vec{i} + v\vec{j} + w\vec{k}) + (\phi\vec{i} + \theta\vec{j}) \times z\vec{k}$ gives the displacement components and strains

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{Bmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \end{Bmatrix} + z \begin{Bmatrix} \theta(x,y) \\ -\phi(x,y) \\ 0 \end{Bmatrix} \quad \Rightarrow$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} + z \begin{Bmatrix} \partial \theta / \partial x \\ -\partial \phi / \partial y \\ \partial \theta / \partial y - \partial \phi / \partial x \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial w / \partial y - \phi \\ \partial w / \partial x + \theta \end{Bmatrix}.$$

- The constitutive equations of a linearly elastic isotropic material and kinetic assumption $\sigma_{zz} = 0$ give the non-zero stress components

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \text{ and } \begin{Bmatrix} \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = G \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}.$$

- The generic expression of δw_V^{int} simplifies to a sum of thin slab, bending, transverse shear and interaction parts of which the last vanishes if the material is homogeneous and the reference plane coincides with the symmetry plane. With that assumption

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} [E]_\sigma \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}, \quad \text{thin slab part}$$

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \partial\delta\theta/\partial x \\ -\partial\delta\phi/\partial y \\ \partial\delta\theta/\partial y - \partial\delta\phi/\partial x \end{Bmatrix}^{\mathrm{T}} z^2[E]_\sigma \begin{Bmatrix} \partial\theta/\partial x \\ -\partial\phi/\partial y \\ \partial\theta/\partial y - \partial\phi/\partial x \end{Bmatrix}, \quad \text{bending part}$$

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \partial\delta w/\partial y - \delta\phi \\ \partial\delta w/\partial x + \delta\theta \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} \partial w/\partial y - \phi \\ \partial w/\partial x + \theta \end{Bmatrix}. \quad \text{shear part}$$

- The generic expressions of δw_V^{ext} and δw_A^{ext} simplify to

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = (\begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} + z \begin{Bmatrix} \delta\theta \\ -\delta\phi \\ \delta 0 \end{Bmatrix}^{\mathrm{T}}) \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix},$$

$$\delta w_A^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = (\begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} + z \begin{Bmatrix} \delta\theta \\ -\delta\phi \\ \delta 0 \end{Bmatrix}^{\mathrm{T}}) \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

- The virtual work of internal forces is obtained as integral over the domain occupied by the body (here the volume element $dV = dzd\Omega$). If $z \in [-t/2, t/2]$

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta\theta / \partial x \\ -\partial\delta\phi / \partial y \\ \partial\delta\theta / \partial y - \partial\delta\phi / \partial x \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_{\sigma} \begin{Bmatrix} \partial\theta / \partial x \\ -\partial\phi / \partial y \\ \partial\theta / \partial y - \partial\phi / \partial x \end{Bmatrix},$$

$$\delta w_{V}^{\text{int}} = -\begin{Bmatrix} \partial\delta w / \partial y - \delta\phi \\ \partial\delta w / \partial x + \delta\theta \end{Bmatrix}^{\mathrm{T}} tG \begin{Bmatrix} \partial w / \partial y - \phi \\ \partial w / \partial x + \theta \end{Bmatrix}.$$

- The contributions coming from the external forces follow in the same manner. Assuming that the volume force is constant (in an element), the surface forces do not act on the top and bottom surfaces, the expression simplifies to

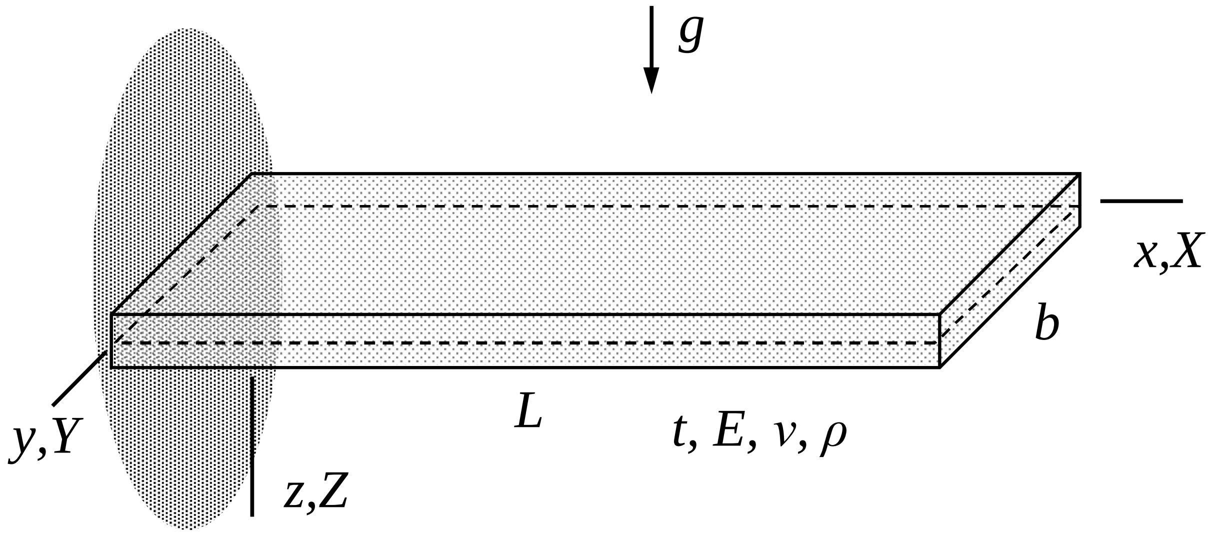
$$\delta W^{\text{ext}} = \int_{\Omega} \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} d\Omega + \int_{\partial\Omega} \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} d\Gamma. \quad \Leftarrow$$

- As the virtual work expression contains only first derivatives, the approximation should be continuous. Continuity requirement does not introduce any problems here and one may choose e.g.

$$w = \begin{Bmatrix} (1-\xi)(1-\eta) \\ \xi(1-\eta) \\ (1-\xi)\eta \\ \xi\eta \end{Bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{Bmatrix}, \ \phi = \begin{Bmatrix} (1-\xi)(1-\eta) \\ \xi(1-\eta) \\ (1-\xi)\eta \\ \xi\eta \end{Bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix}, \text{ and } \ \theta = \begin{Bmatrix} (1-\xi)(1-\eta) \\ \xi(1-\eta) \\ (1-\xi)\eta \\ \xi\eta \end{Bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix}.$$

IMPORTANT. Reissner-Mindlin plate model shares the numerical difficulties of the Timoshenko beam model and, in practice, finite element methods using low order approximations, e.g. linear or quadratic approximations on a triangle, suffer from severe shear locking that can be avoided only with carefully designed tricks.

EXAMPLE 6.9 A rectangular plate is loaded by its own weight. Determine the deflection of the plate at the free end by using the Reissner-Mindlin plate model with bi-linear, bi-quadratic and bi-cubic approximations. Thickness, width, and length of the plate are t, b and L, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν of the material are constants. Consider finally the limit $G \to \infty$.



Answer: $w(L) = u_{Z2} = u_{Z4} = \dfrac{3}{2}\dfrac{g\rho L^4}{Et^2}(1-\nu^2)$ (Bernoulli beam $w(L) = \dfrac{3}{2}\dfrac{g\rho L^4}{Et^2}$)

- The solutions given by the Mathematica code of the course are

Bi-linear: $w(L) = 0$

Bi-quadratic: $w(L) = \dfrac{g\rho L^4}{Et^2}(1-\nu^2)$

Bi-cubic: $w(L) = \dfrac{3}{2}\dfrac{g\rho L^4}{Et^2}(1-\nu^2)$

Therefore, approximations to the unknown functions should be cubic for a precise prediction.