PHYS-C0252 - Quantum Mechanics Part 2
Sections 5.1-5.2

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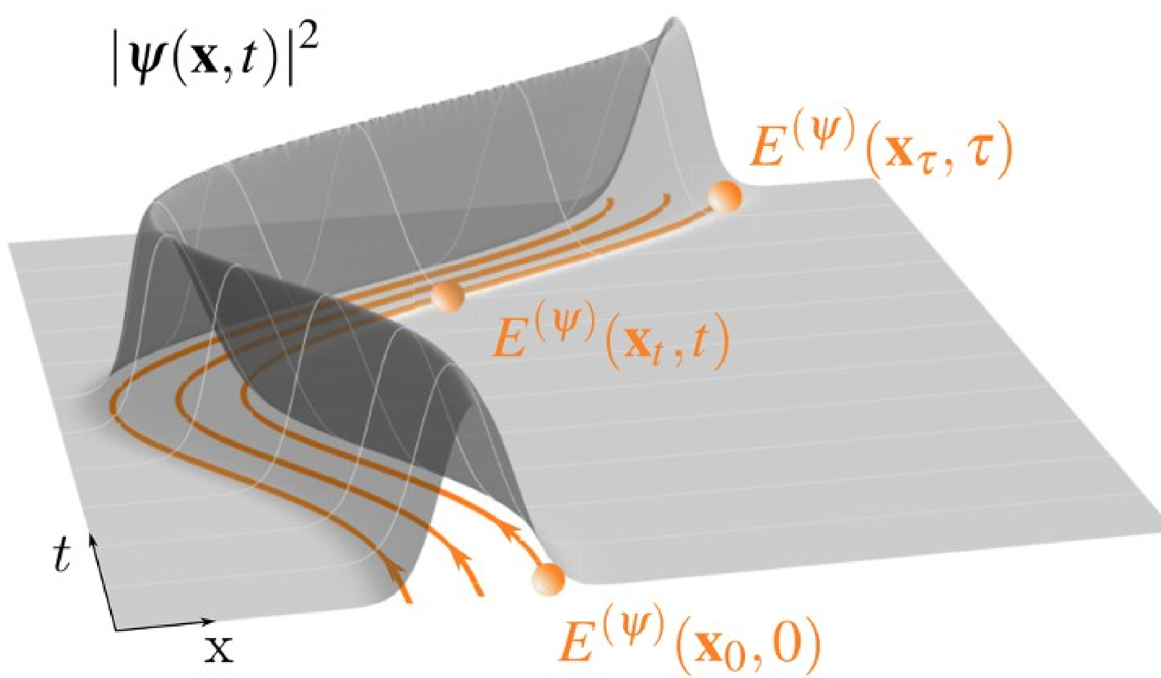


5. Bosons and Fermions

- Consider first a two-particle wave function for identical particles $\Psi(x_1, x_2, t)$. The probability for finding particle 1 at dx_1 and particle 2 at dx_2 is given by

$$|\Psi(x_1, x_2, t)|^2 dx_1 dx_2$$

If the particles are identical, they can be interchanged and thus

$$|\Psi(x_1, x_2, t)|^2 = |\Psi(x_2, x_1, t)|^2$$

which means that

$$\Psi(x_1, x_2, t) = \Psi(x_2, x_1, t) e^{\imath\delta}$$

where the phase factor $e^{\imath\delta} = \pm 1$

- If we have a Fock space of identical single-particle wave functions, the symmetric and antisymmetric (entangled) wave functions can be represented as

$$\Psi^S(x) \propto \psi_n(x_1)\psi'_n(x_2) + \psi_n(x_2)\psi'_n(x_1)$$
$$\Psi^A(x) \propto \psi_n(x_1)\psi'_n(x_2) - \psi_n(x_2)\psi'_n(x_1)$$

Qualitatively, particles with antisymmetric (entangled) wave function avoid each other – case of 1D QHO can be explicitly demonstrated:

Consider two particles in two different single-particle states in a 1D QHO, first one with *n* and the other one with *n'*. The energy is

$$E = E_n + E_{n'} = (n + n' + 1)\hbar\omega.$$

- For two *distinguishable* particles p and q the total wave function can be of *unentangled* form:

$$\Psi_1^{(\mathrm{D})}(x_\mathrm{p}, x_\mathrm{q}, t) = \psi_n(x_\mathrm{p})\psi_{n'}(x_\mathrm{q})\,\mathrm{e}^{-i(E_n+E_{n'})t/\hbar}$$

$$\Psi_2^{(\mathrm{D})}(x_\mathrm{p}, x_\mathrm{q}, t) = \psi_n(x_\mathrm{q})\psi_{n'}(x_\mathrm{p})\,\mathrm{e}^{-i(E_n+E_{n'})t/\hbar},$$

or a linear combination as

$$\Psi^{(\mathrm{D})}(x_\mathrm{p}, x_\mathrm{q}, t) = c_1 \Psi_1^{(\mathrm{D})}(x_\mathrm{p}, x_\mathrm{q}, t) + c_2 \Psi_2^{(\mathrm{D})}(x_\mathrm{p}, x_\mathrm{q}, t)$$

- This WF is *entangled* because it associates both particles with both single-particle states

- For two identical particles there are two possible WFs as

$$\Psi^{(\mathrm{S})}(x_\mathrm{p}, x_\mathrm{q}, t) = \frac{1}{\sqrt{2}}[\psi_n(x_\mathrm{p})\psi_{n'}(x_\mathrm{q}) + \psi_n(x_\mathrm{q})\psi_{n'}(x_\mathrm{p})]\,\mathrm{e}^{-i(E_n+E_{n'})t/\hbar}$$

$$\Psi^{(\mathrm{A})}(x_\mathrm{p}, x_\mathrm{q}, t) = \frac{1}{\sqrt{2}}[\psi_n(x_\mathrm{p})\psi_{n'}(x_\mathrm{q}) - \psi_n(x_\mathrm{q})\psi_{n'}(x_\mathrm{p})]\,\mathrm{e}^{-i(E_n+E_{n'})t/\hbar}$$

- Next set particles to have identical positions $x_\mathrm{p} = x_\mathrm{q} = x_0$

- The *entangled* WF for distinguishable particles is

$$\Psi^{(\mathrm{D})}_{1,2}(x_0, x_0, t) = \psi_n(x_0)\psi_{n'}(x_0)\,\mathrm{e}^{-i(E_n+E_{n'})t/\hbar}.$$

- For two identical particles the symmetrical *entangled* WF is

$$\Psi^{(\mathrm{S})}(x_0, x_0, t) = \sqrt{2}\psi_n(x_0)\psi_{n'}(x_0)\,\mathrm{e}^{-i(E_n+E_n')t/\hbar}$$

  and the antisymmetrical one

$$\Psi^{(\mathrm{A})}(x_0, x_0, t) = 0$$

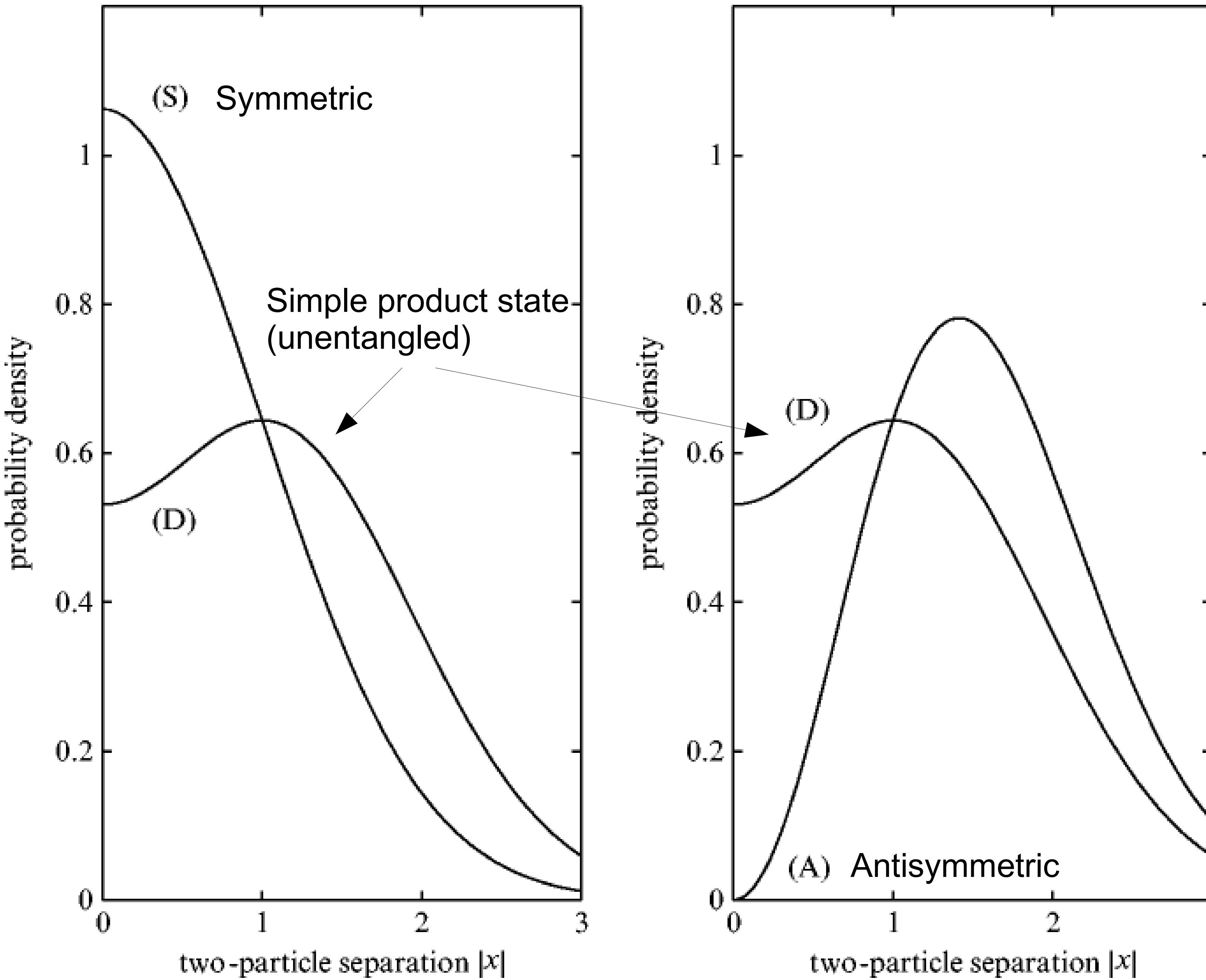
- The physical reason for these differences is *constructive or destructive interference* of the WFs

- The physical differences become even more clear if we consider two identical or distinguishable (D) particles occupying 1D QHO states with n = 0 and n' = 1, using reduced coordinates

$$x = x_{\mathrm{p}} - x_{\mathrm{q}} \quad \text{and} \quad X = \frac{x_{\mathrm{p}} + x_{\mathrm{q}}}{2}$$

- The corresponding WFs can be easily constructed (homework) and the PDFs are plotted on the next page for the symmetrical (S) and antisymmetrical (A) WFs of two identical particles and (unentangled) distinguishable particles (D)

1D QHO



(S) Symmetric
(D)
Simple product state
(unentangled)
(D)
(A) Antisymmetric
probability density
two-particle separation |x|
0
0.2
0.4
0.6
0.8
1
0
1
2
3

The *spin-statistics theorem* states that there are two fundamental classes of particles: *fermions* with half-integer spin and *bosons* with integer spin

- *Fermions:* quarks and composite particles made of them, and leptons such as the electron and neutrinos
- *Bosons:* Often force-mediating particles (photons, gluons, W and Z bosons, Higgs boson etc.), and composite particles (mesons)

5.1 Symmetrized Eigenstates for Bosons

- For bosons the total wave function must be symmetric under the interchange of any degrees of freedom (coordinates) and any number of them can have the same quantum numbers.

  Let us define a permutation operator P_{ij} by

$$P_{ij}|k_1, k_2, ..., k_i, k_j, ..., k_N\rangle = |k_1, k_2, ..., k_j, k_i, ..., k_N\rangle$$

  Sum over all the permutations includes all possible combinations of the *k*'s

$$\sum_P P|k_1, k_2, \ldots, k_N\rangle \equiv \sum (\text{all } N! \text{ permutations of momenta in } |k_1, k_2, \ldots, k_N\rangle)$$

For example

$$\sum_P P|k_1,k_2,k_3\rangle = \{|k_1,k_2,k_3\rangle + |k_2,k_1,k_3\rangle + |k_1,k_3,k_2\rangle + |k_3,k_2,k_1\rangle + |k_3,k_1,k_2\rangle + |k_2,k_3,k_1\rangle\}$$

Since there can be any number of particles with the same *k*, we must count all possible combinations of different ways of organizing the ket:

$$n_i = \text{number of particles with momentum } k_i$$

$$N = \sum_{i=1}^{N} n_i = \text{total number of particles}$$

Thus there are exactly

$$\frac{N!}{\prod_{\alpha=1}^{N} n_\alpha!}$$

different kets in $\sum_P P|k_1, k_2, ..., k_N\rangle$

Using orthonormality of the basis functions

$$\langle k_a, k_b, \ldots, k_l | k'_a, k'_b, \ldots, k'_l \rangle = \delta_{k_a,k'_a} \delta_{k_b,k'_b} \times \cdots \times \delta_{k_l,k'_l}$$

we can write the symmetrized, orthonormal *N*-body momentum eigenstate as

$$|k_1, k_2, \ldots, k_N\rangle^{(S)} = \left(\frac{N!}{\prod_{\alpha=1}^{N} n_\alpha!} \right) \sum_P P|k_1, k_2, \ldots, k_N\rangle$$

which also form a complete, orthonormal set, with identity operator

$$\hat{\mathrm{I}}^{(S)} = \frac{1}{N!} \sum_{k_1, k_2, \ldots, k_N} \left(\prod_{\alpha=1}^{N} n_\alpha! \right) |k_1, k_2, \ldots, k_N\rangle^{(S)} \ {}^{(S)}\langle k_1, k_2, \ldots, k_N|$$

5.2 Symmetrized Eigenstates for Fermions

- For fermions the total wave function must be antisymmetric under the interchange of any degrees of freedom (coordinates) and none of them can have the same quantum numbers.

  Let us again define a permutation operator P_{ij} by

$$P_{ij}|k_1, k_2, ..., k_i, k_j, ..., k_N\rangle = |k_1, k_2, ..., k_j, k_i, ..., k_N\rangle$$

  Sum over all the permutations includes all possible combinations of the *k*'s

$$\sum_P P|k_1, k_2, \ldots, k_N\rangle \equiv \sum \text{(all } N! \text{ permutations of momenta in } |k_1, k_2, \ldots, k_N\rangle)$$

The antisymmetric momentum eigenstates can be written as

$$|k_1, k_2, \ldots, k_N\rangle^{(A)} = \frac{1}{\sqrt{N!}} \sum_P (-1)^P P|k_1, \ldots, k_N\rangle$$

where *P* is the number of permutations (changes)

- For example,

$$\sum_P (-1)^P P|k_1, k_2, k_3\rangle = \{|k_1, k_2, k_3\rangle - |k_2, k_1, k_3\rangle - |k_1, k_3, k_2\rangle - |k_3, k_2, k_1\rangle + |k_3, k_1, k_2\rangle + |k_2, k_3, k_1\rangle\}.$$

The antisymmetric fermion wave function $\langle r_1, r_2, ..., r_N | k_1, k_2, ..., k_N \rangle^{(A)}$ can be written as the *Slater determinant*

$$\langle r_1, r_2, \ldots, r_N | k_1, k_2, \ldots, k_N \rangle^{(A)} = \frac{1}{\sqrt{N}} \begin{pmatrix} \langle r_1 | k_1 \rangle & \langle r_1 | k_2 \rangle & \cdots & \langle r_1 | k_N \rangle \\ \langle r_2 | k_1 \rangle & \langle r_2 | k_2 \rangle & \cdots & \langle r_2 | k_N \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle r_N | k_1 \rangle & \langle r_N | k_2 \rangle & \cdots & \langle r_N | k_N \rangle \end{pmatrix}$$

which naturally gives zero for any pair of equal quantum numbers