PHYS-C0252 - Quantum Mechanics Part 2
Section 7

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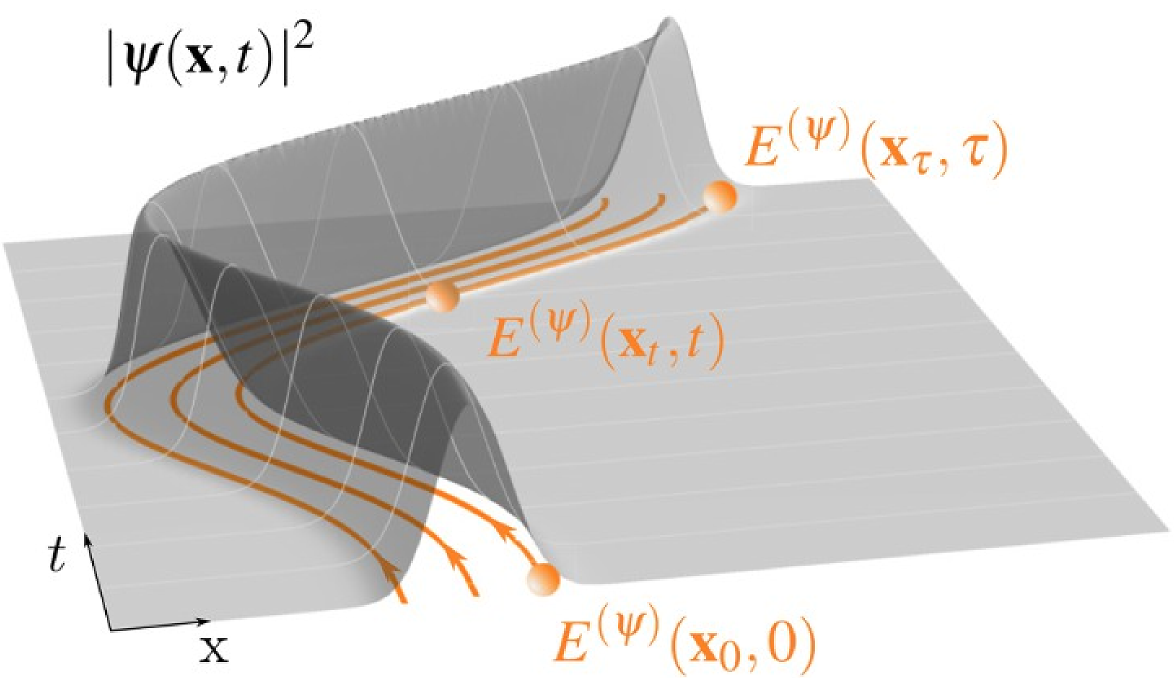

7. Time Dependence of Operators

- The *formulation of quantum dynamics is not unique for the states and operators*. Consider the expectation value of some (Hermitian) operator

$$
\begin{aligned}
\langle \hat{A}(t) \rangle &= \langle \psi(t) | \hat{A} | \psi(t) \rangle = \langle \hat{U}(t) \psi(0) | \hat{A} | \hat{U}(t) \psi(0) \rangle \\
&= \langle \psi(0) | \hat{U}^\dagger(t) \hat{A} \hat{U}(t) | \psi(0) \rangle \\
&= \left(\langle \psi(0) | \hat{U}^\dagger(t) \right) \hat{A} \left(\hat{U}(t) | \psi(0) \rangle \right) \\
&= \langle \psi(0) | \left(\hat{U}^\dagger(t) \hat{A} \hat{U}(t) \right) | \psi(0) \rangle
\end{aligned}
$$

where the time-evolution operator from the Schrödinger equation propagates the wave function

$$\Psi_n(x,t) = \hat{U}(t)\psi_n(x,t)$$

and follows the equation of motion

$$\imath\hbar \frac{\partial \hat{U}(t,t_0)}{\partial t} = \hat{H}(t)\hat{U}(t,t_0)$$

giving

$$\hat{U}(t,t_0) = e^{-\imath \int_{t_0}^{t} dt' \, \hat{H}(t')/\hbar}$$

The form where the states evolve in time is the *Schrödinger picture*

$$\langle \hat{A}(t) \rangle = \left(\langle \psi(0) | \hat{U}^\dagger(t) \right) \hat{A} \left(\hat{U}(t) | \psi(0) \rangle \right)$$

and where the operator(s) evolve in time but states stay the same is the *Heisenberg picture*:

$$\langle \hat{A}(t) \rangle = \langle \psi(0) | \left(\hat{U}^\dagger(t) \hat{A} \hat{U}(t) \right) | \psi(0) \rangle$$

7.1 Schrödinger Picture

- The time evolution of the state is governed by

$$\imath\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_S = \hat{H}|\psi(t)\rangle_S$$

or equivalently $|\psi(t)\rangle_S = \hat{U}(t,t_0)|\psi(t_0)\rangle_S$

For operators (expectation values)

$$\imath\hbar\frac{\partial}{\partial t}\langle\psi(t)|\hat{A}|\psi(t)\rangle_S$$

$$= \imath\hbar\left(\langle\psi(t)|\hat{A}|\dot{\psi}(t)\rangle_S + \langle\dot{\psi}(t)|\hat{A}|\psi(t)\rangle_S\right)$$

$$= \langle \psi(t)|\hat{A}\hat{H}|\psi(t)\rangle_S - \langle \psi(t)|\hat{H}\hat{A}|\psi(t)\rangle_S = \langle [\hat{A}, \hat{H}]\rangle_S$$

If the commutator is zero, the expectation value of *A* is *a constant of motion*

7.2 Heisenberg Picture

- In the HP the states do not evolve in time but the operators (expectation values) do, and we can write

$$\hat{A}_H(t) = \hat{U}^\dagger(t, t_0)\hat{A}_S\hat{U}(t, t_0)$$

which agree at time t_0

The wave functions are related by

$$|\psi_S(t)\rangle = \hat{U}(t, t_0)|\psi_H\rangle$$

The operators depend on time now and their equation of motion is given by

$$
\begin{aligned}
\frac{\partial \hat{A}_H}{\partial t} &= \frac{\partial(\hat{U}^\dagger \hat{A}_S \hat{U})}{\partial t} \\
&= \frac{\imath}{\hbar}\left(\hat{U}^\dagger \hat{H} \hat{A}_S \hat{U} - \hat{U}^\dagger \hat{A}_S \hat{H} \hat{U}\right) \\
&= \frac{\imath}{\hbar}\left(\hat{H}_H \hat{A}_H - \hat{A}_H \hat{H}_H\right) \\
&= -\frac{\imath}{\hbar}[\hat{A}, \hat{H}]_H
\end{aligned}
$$

Note that for time-independent Hamiltonian

$$
\hat{H}_H = \hat{U}^\dagger \hat{H} \hat{U} = \hat{H}
$$

There is also an *interaction (Dirac) picture*, which is used when the system Hamiltonian can be divided into two parts as (in Schrödinger picture)

$$\hat{H}_S = \hat{H}_S^0 + \hat{H}_S^I$$

where the first part is “easy” (usually solvable). Then a state vector in the IP is given by

$$|\psi_I(t)\rangle = e^{\imath \hat{H}_S^0 t/\hbar}|\psi_S(t)\rangle$$

An operator in the IP is defined by

$$\hat{A}_I(t) = e^{\imath \hat{H}_S^0 t/\hbar}\hat{A}_S e^{-\imath \hat{H}_S^0 t/\hbar}$$

Recently a *correlation picture* (transformation) has been introduced for open quantum systems

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Correlation-Picture Approach to Open-Quantum-System Dynamics

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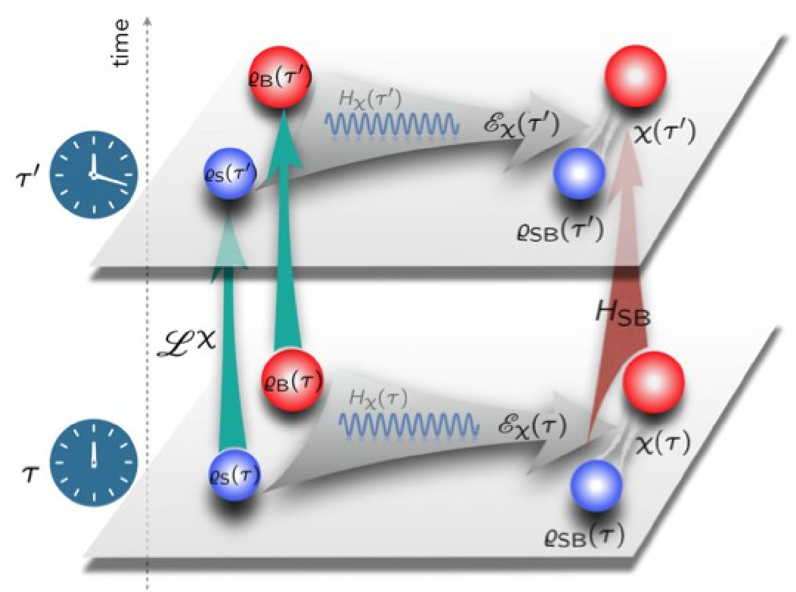

FIG. 1. Description of the correlation picture. At any time τ (or τ'), a correlating transformation $\mathscr{E}_\chi$ transforms an uncorrelated state $\varrho_S \otimes \varrho_B$ to a correlated state $\varrho_{SB} = \varrho_S \otimes \varrho_B + \chi$, at the same instant of time, due to an abstract correlation-dependent parent operator given by H_χ. Using this transformation, we obtain the temporal evolution of the uncorrelated system with a universal Lindblad-like generator $\mathscr{L}^\chi$ [see Eq. (9)] constructed from H_{SB}, the generator of the total system dynamics in the Schrödinger picture.