Problem Set 8

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Exercise 1 - PS7

Solve the following initial value problems:

(a) $\dot{y} - 2y = e^{-2t}$, $y(0) = 5$.

(b) $\dot{y} - 4y = 4 - t$, $y(0) = 2$.

(c) $\ddot{y} - 3y = 0$, $y(0) = 2$, $\dot{y}(0) = 0$.

Exercise 1 - Solution

Linear first order differential equations

- Suppose we want to solve the linear equation

$$\dot{y} = ay + b(t), \tag{11}$$

with $a \neq 0$

- Notice that $b(t)$ is a **variable coefficient** and that (11) is not separable

- Like we did at pp. 27-28, we can use the integrating factor e^{-at} to obtain the general solution

$$y(t) = Ce^{at} + e^{at} \int b(t) e^{-at} dt \tag{12}$$

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Figure 1: Lecture 19

Exercise 1 - Solution

LIATE rule [edit]

A rule of thumb has been proposed, consisting of choosing as *u* the function that comes first in the following list:[4]

- **L** – logarithmic functions: $\ln(x),\ \log_b(x)$, etc.
- **I** – inverse trigonometric functions (including hyperbolic analogues): $\arctan(x),\ \operatorname{arcsec}(x),\ \operatorname{arsinh}(x)$, etc.
- **A** – algebraic functions: $x^2,\ 3x^{50}$, etc.
- **T** – trigonometric functions (including hyperbolic analogues): $\sin(x),\ \tan(x),\ \operatorname{sech}(x)$, etc.
- **E** – exponential functions: $e^x,\ 19^x$, etc.

The function which is to be *dv* is whichever comes last in the list. The reason is that functions lower on the list generally have easier antiderivatives than the functions above them. The rule is sometimes written as "DETAIL" where *D* stands for *dv* and the top of the list is the function chosen to be *dv*.

To demonstrate the LIATE rule, consider the integral

$$\int x \cdot \cos(x)\,dx.$$

Following the LIATE rule, $u = x$, and $dv = \cos(x)\,dx$, hence $du = dx$, and $v = \sin(x)$, which makes the integral become

$$x \cdot \sin(x) - \int 1 \sin(x)\,dx,$$

which equals

$$x \cdot \sin(x) + \cos(x) + C.$$

Figure 2: How to do integration by parts in part b

Exercise 1 - Solution

Linear second order differential equations

- **First case.** $b^2 - 4ac > 0$ and the characteristic equation has two distinct real roots r_1 and r_2
- The general solution of the differential equation (2) is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- Note: C_1 and C_2 are two *distinct* unknown constants

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Figure 3: Lecture 20

Exercise 1 - Solution

(a) The general solution is

$$y(t) = Ce^{2t} + e^{2t}\int e^{-4t}dt = Ce^{2t} - \frac{1}{4}e^{-2t}.$$

In the particular solution, $C = \frac{21}{4}$.

(b) The general solution is

$$\begin{aligned} y(t) &= Ce^{4t} + e^{4t}\int (4-t)e^{-4t}dt \\ &= Ce^{4t} + e^{4t}\left(-\frac{15}{16}e^{-4t} + \frac{t}{4}e^{-4t} + D\right) \\ &= ke^{4t} - \frac{15}{16} + \frac{t}{4}. \end{aligned}$$

In the particular solution, $k = \frac{47}{16}$.

(c) The characteristic equation is $r^2 - 3 = 0$, with roots $r_1 = -\sqrt{3}$ and $r_2 = \sqrt{3}$. The general solution is

$$y(t) = C_1 e^{-\sqrt{3}t} + C_2 e^{\sqrt{3}t}.$$

In the particular solution, $C_1 = C_2 = 1$.

Exercise 2

Compute the equilibrium points of the following differential equations and determine the stability of each:

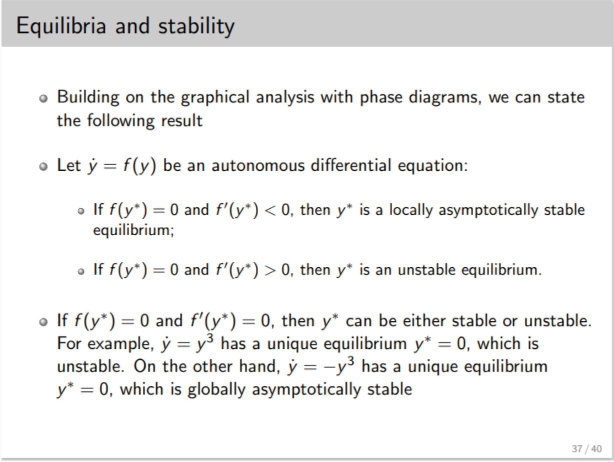
(a) $\dot{y} = \frac{y^2 - y}{y^2 + 1}$;

(b) $\dot{y} = e^y \sin y$;

(c) $\dot{y} = \frac{y}{y^2 + 1}$;

(d) $\dot{y} = y^2 - y^3$.

Exercise 2 - Solution



Equilibria and stability

- Building on the graphical analysis with phase diagrams, we can state the following result
- Let $\dot{y} = f(y)$ be an autonomous differential equation:
  - If $f(y^*) = 0$ and $f'(y^*) < 0$, then y^* is a locally asymptotically stable equilibrium;
  - If $f(y^*) = 0$ and $f'(y^*) > 0$, then y^* is an unstable equilibrium.
- If $f(y^*) = 0$ and $f'(y^*) = 0$, then y^* can be either stable or unstable. For example, $\dot{y} = y^3$ has a unique equilibrium $y^* = 0$, which is unstable. On the other hand, $\dot{y} = -y^3$ has a unique equilibrium $y^* = 0$, which is globally asymptotically stable

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Figure 4: Lecture 20

Exercise 2 - Solution

(a) There are two equilibrium points: $y_1^* = 0$ and $y_2^* = 1$.
$f'(y) = \frac{(2y-1)(y^2+1)-(y^2-y)(2y)}{(y^2+1)^2} = \frac{y^2+2y-1}{(y^2+1)^2}$
The equilibrium at $y_1^* = 0$ is locally asymptotically stable because $f'(0) = -1 < 0$.
The equilibrium at $y_2^* = 1$ is unstable $f'(1) = 1/2 > 0$

(b) There are infinitely many equilibrium points:
$y^* = ..., -2\pi, \pi, 0, \pi, 2\pi, ...$
$f'(y) = e^y(sin(y) + cos(y))$
The equilibria at $y^* = 2n\pi$ are unstable because $f'(2n\pi) = e^c os(y) = e^{2n\pi} cos(2n\pi) > 0$.
The equilibria at $y^* = (2n+1)\pi$ are locally asymptotically stable because $f'((2n+1)\pi) = e^c os(y) = e^{(2n+1)\pi} cos((2n+1)\pi) < 0$.

Exercise 2 - Solution

(c) The only equilibrium point is $y^* = 0$.
$f'(y) = \frac{-y^2+1}{y^2+1}$
The equilibrium at $y^* = 0$ is unstable because $f'(0) = 1 > 0$.

(d) There are two equilibrium points, $y_1^* = 0$ and $y_2^* = 1$.
$f'(y) = 2y - 3y^2$
At $y_1^* = 0$ is $f'(0) = 0$. However, when y is sufficiently close to zero, $f(y)$ is always positive. Hence we can conclude that the equilibrium at $y_1^* = 0$ is unstable.
The equilibrium at $y_2^* = 1$ is locally asymptotically stable because $f'(1) = -1 < 0$.

Exercise 3

Find the general solution of the following differential equations:

(a) $\ddot{y} + 4\dot{y} + 8y = 0$;

(b) $3\ddot{y} + 8\dot{y} = 0$;

(c) $4\ddot{y} + 4\dot{y} + y = 0$.

Exercise 3 - Solution

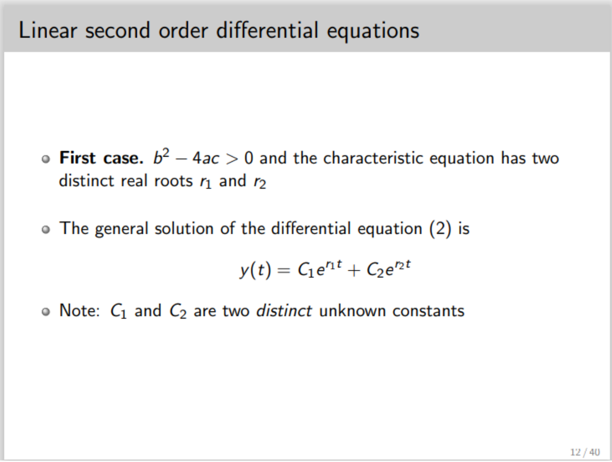

Figure 5: Lecture 20

Exercise 3 - Solution

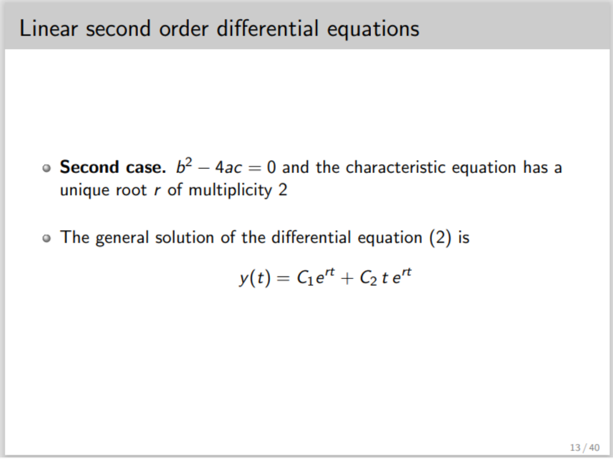

Figure 6: Lecture 20

Exercise 3 - Solution

Linear second order differential equations

- **Third case.** $b^2 - 4ac < 0$ and the characteristic equation has two distinct complex roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where i is the imaginary unit
- The general solution of the differential equation (2) is

$$y(t) = e^{\alpha t}\left(C_1 \cos \beta t + C_2 \sin \beta t\right)$$

- Note: The two complex roots of the characteristic equation are always *conjugates* of each other

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Figure 7: Lecture 20

Exercise 3 - Solution

(a) The characteristic equation is $r^2 + 4r + 8 = 0$, with roots $r_1 = -2 + 2i$ and $r_2 = -2 - 2i$. The general solution is

$$y(t) = e^{-2t}\left(C_1 \cos 2t + C_2 \sin 2t\right).$$

(b) The characteristic equation is $3r^2 + 8r = 0$, with roots $r_1 = 0$ and $r_2 = -\frac{8}{3}$. The general solution is

$$y(t) = C_1 + C_2 e^{-\frac{8}{3}t}.$$

(c) The characteristic equation is $4r^2 + 4r + 1 = 0$, whose only root $r = -\frac{1}{2}$ has multiplicity 2. The general solution is

$$y(t) = (C_1 + C_2 t)e^{-\frac{1}{2}t}.$$

Exercise 4

Consider the following system of differential equations:

$$\dot{x} = 5x - \frac{1}{2}y - 12, \dot{y} = -2x + 5y - 24$$

(a) Find the steady state.

(b) Is the steady state unique? Why or why not?

(c) Find the general solution.

(d) Find the particular solution for the initial conditions $x(0) = 12$ and $y(0) = 4$.

(e) Is the steady state globally asymptotically stable? Why or why not?

Exercise 4 - Solution

Systems of ordinary differential equations

- Consider the linear system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$
- Any vector $\mathbf{x}^*$ such that $A\mathbf{x}^* + \mathbf{b} = \mathbf{0}$ is an *equilibrium* or *steady state* of the system
- Given a steady state $\mathbf{x}^*$, the constant function $\mathbf{x}(t) = \mathbf{x}^*$ is clearly a solution of $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$
- The steady state $\mathbf{x}^* = -A^{-1}\mathbf{b}$ is unique if and only if A is invertible

Proposition

Suppose the $n \times n$ coefficient matrix A has n distinct real eigenvalues $r_1, \dots, r_n$, with corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. Let $\mathbf{x}^*$ be a steady state of the linear system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$. Then, the general solution of $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$ is

$$\mathbf{x}(t) = C_1 e^{r_1 t}\mathbf{v}_1 + C_2 e^{r_2 t}\mathbf{v}_2 + \cdots + C_n e^{r_n t}\mathbf{v}_n + \mathbf{x}^*.$$

Figure 8: Lecture 21

Exercise 4 - Solution

Equilibria and stability

- As we did for first order equations, we want to examine the **stability** of systems of differential equations
- Let's consider the linear system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$
- Let $\mathbf{x}^*$ be a steady state. We say that $\mathbf{x}^*$ is *globally asymptotically stable* if every solution $\mathbf{x}(t)$ of $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$ converges to $\mathbf{x}^*$ as $t \to \infty$. Otherwise, we say that $\mathbf{x}^*$ is unstable

Proposition (Stability of linear systems)

Consider the linear system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$ *and suppose* $\det A \neq 0$.

1. *If every real eigenvalue of A is negative and every complex eigenvalue of A has negative real part, then the steady state* $\mathbf{x}^*$ *is globally asymptotically stable.*
2. *If A has a positive real eigenvalue or a complex eigenvalue with positive real part, then* $\mathbf{x}^*$ *is an unstable equilibrium.*

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Figure 9: Lecture 21

Exercise 4 - Solution

The steady state (x^*, y^*) can be found by solving

$$5x^* - \frac{1}{2}y^* - 12 = 0, -2x^* + 5y^* - 24 = 0$$

The unique solution is $(x^*, y^*) = (3, 6)$. The steady state is unique because the system's coefficient matrix $A = \begin{pmatrix} 5 & -\frac{1}{2} \\ -2 & 5 \end{pmatrix}$ is invertible.

The matrix A has two distinct eigenvalues: $r_1 = 4$ and $r_2 = 6$. A feasible pair of corresponding eigenvectors is

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

In the particular solution, $C_1 = 4$ and $C_2 = 5$. The steady state is not stable because $r_i > 0$ for $i = 1, 2$.

Exercise 5

Find the general solution of each of the following linear systems of differential equations. For each system, also determine if the steady state $(x^*, y^*) = (0, 0)$ is globally asymptotically stable.

(a) $\dot{x} = 2x + y, \dot{y} = -12x - 5y$

(b) $\dot{x} = 6x - 3y, \dot{y} = -2x + y$

(c) $\dot{x} = x + 4y, \dot{y} = 3x + 2y$

(d) $\dot{x} = 4y, \dot{y} = -x + 4y$

(e) $\dot{x} = -2x + 5y, \dot{y} = -2x + 4y$

Exercise 5 - Solution

(a) $A = \begin{pmatrix} 2 & 1 \\ -12 & -5 \end{pmatrix}$. The matrix A has two distinct eigenvalues: $r_1 = -2$ and $r_2 = -1$. A feasible pair of corresponding eigenvectors is

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-2t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

The steady state is globally asymptotically stable because $r_i < 0$ for $i = 1, 2$.

Exercise 5 - Solution

(b) $A = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$. The matrix A has two distinct eigenvalues: $r_1 = 0$ and $r_2 = 7$. A feasible pair of corresponding eigenvectors is

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{0t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{7t} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

The steady state is unstable because $r_2 > 0$.

Exercise 5 - Solution

(c) $A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$. The matrix A has two distinct eigenvalues: $r_1 = 5$ and $r_2 = -2$. A feasible pair of corresponding eigenvectors is

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

The steady state is unstable because $r_1 > 0$.

Exercise 5 - Solution

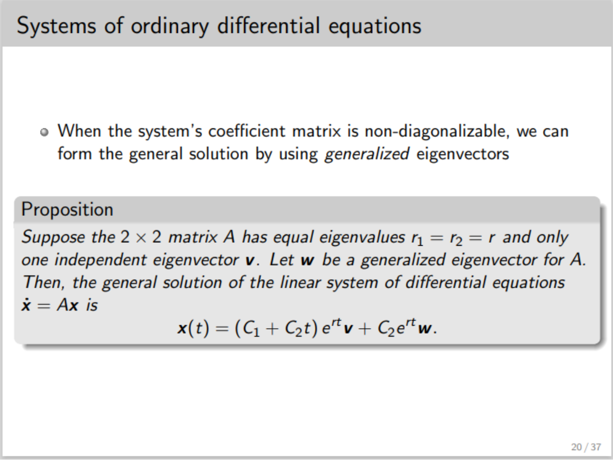

Figure 10: Lecture 21

Exercise 5 - Solution

(d) $A = \begin{pmatrix} 0 & 4 \\ -1 & 4 \end{pmatrix}$. The matrix A has one distinct eigenvalues with multiplicity 2: $r = 2$. A feasible corresponding eigenvector and a generalized eigenvector is

$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t}[C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + tC_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}]$$

The steady state is unstable because $r > 0$.

Exercise 5 - Solution

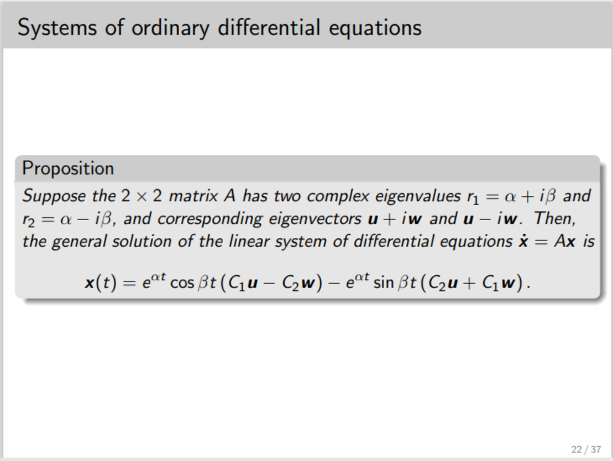

Figure 11: Lecture 21

Exercise 5 - Solution

(e) $A = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix}$. The matrix A has two complex eigenvalues:
$r_1 = 1 + i$ and $r_2 = 1 - i$. A feasible pair of corresponding eigenvectors is

$$\boldsymbol{v}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^t cos(t)(C_1 \begin{pmatrix} 5 \\ 3 \end{pmatrix} - C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}) - e^t sin(t)(C_2 \begin{pmatrix} 5 \\ 3 \end{pmatrix} + C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

The steady state is unstable because the real part of $r > 0$.

Good luck!

If you have any questions, feel free to send me an email (Hung Le - hung.h.le@aalto.fi)