HW1 problem 2

$$\mathcal{E} = \{x \mid (x - x_c)^T A^{-1} (x - x_c) \leq 1\}, \quad A^T = A \succ 0, \quad x_c \in R^n$$

$$\Rightarrow \mathcal{E} = \left\{x \mid x^T A^{-1} x - x^T A^{-1} x_c - x_c^T A^{-1} x + x_c^T A^{-1} x_c - 1 \leq 0\right\}$$

$(A^{-1})^T = A^{-1}$

$$\Longrightarrow \mathcal{E} = \left\{x \mid x^T A^{-1} x - 2 x_c^T A^{-1} x + x_c^T A^{-1} x_c - 1 \leq 0\right\}$$

$$\Rightarrow \mathcal{E} = \left\{x \mid x^T C x + 2 d^T x + e\right\}$$

where $\quad C = A^{-1}, \quad d = -A^{-1} x_c, \quad e = x_c^T A^{-1} x_c - 1$

$$\mathcal{E} = \left\{x \mid (x - x_c)^T A^{-1} (x - x_c) \leq 1\right\}$$

$$A = A^T \succ 0, \quad x_c \in R^n$$

$$\Rightarrow \mathcal{E} = \left\{x + x_c \mid x^T A^{-1} x \leq 1\right\}$$

$$= \left\{x + x_c \mid \|A^{-1/2} x\| \leq 1\right\}$$

$$= \left\{B u + x_c \mid \|u\| \leq 1\right\} \quad \text{wher} \quad B = A^{1/2}$$

HW1 problem 3

The dual of a norm cone is a cone associated with the dual of that norm.

Dual of a norm $\|x\|$ is defined as

$$\|y\|_* = \sup\{y^T x \mid \|x\| \leq 1\}$$

$$y^T x = \sum_i x_i y_i \leq \underbrace{\sum_i |x_i y_i| \leq \|y\|_p \|x\|_q \,, \frac{1}{p}+\frac{1}{q}=1}_{\text{Hölder's inequality}}$$

if $\|x\|_2 \leq 1 \Longrightarrow y^T x \leq \|y\|_2$

$\left(p=2, q=2, \frac{1}{p}, \frac{1}{q}=1\right)$

$$\|y\|_* = \|y\|_2$$

---

problem 4

similar to problem 3

if $y^T x \leq \|y\|_\infty \Longrightarrow \|y\|_* = \|y\|_\infty$

$\left(p=1, q=\infty, \frac{1}{p}+\frac{1}{q}=1\right)$

HW1 problem 6

a) $\preceq_K$ is preserved under addition

if $x \preceq_K y$ and $u \preceq_K v$

then $y - x \in K$ and $v - u \in K$

K is convex $\Rightarrow$ $y - x + v - u \in K$

$\Rightarrow x + u \preceq_K y + v$

---

b) $\preceq_K$ is transitive

if $x \preceq_K y$ and $\cancel{y} \preceq_K z$

$\Rightarrow y - x \in K$, $z - y \in K$

$\Rightarrow$ K is convex $\Rightarrow$ $z - y + y - x \in K$

$\Rightarrow z - x \in K \Rightarrow x \preceq_K z$

---

c) $\preceq_K$ is preserved under nonnegative scaling

$x \preceq_K y \Rightarrow y - x \in K$

$\Rightarrow \alpha(y - x) \in K$ $\quad \alpha \geq 0$

$\Rightarrow \alpha y - \alpha x \in K$

$\Rightarrow \alpha x \preceq_K \alpha y$

Scanned with CamScanner

HW1 Problem 6 cont.

d) $\preceq_K$ is reflexive

$0 \in K$ (cone) $\Rightarrow x - x \in K$

$\Rightarrow x \preceq_K x$

---

e) $\preceq_K$ is antisymmetric

$x \preceq_K y$ and $y \preceq_K x$

$\Rightarrow y - x \in K$, $x - y \in K$

K is pointed $\Rightarrow y - x = 0 \Rightarrow y = x$

---

f) $\preceq_K$ is preserved under limits

$\lim_{i\to\infty} x_i \preceq_K \lim_{i\to\infty} y_i$

K is closed $\rightarrow x \preceq_K y$

---

g) $x \prec_K y$ $\quad y - x \in \operatorname{int} K \subseteq K$

$\Rightarrow y - x \in K \Rightarrow x \preceq_K y$

HW2 problem 6 cont

h) $y - x \in \text{int} K$, $v - u \in K$

$\Rightarrow u + v - (x + u) \in \text{int} K$

$\Rightarrow x + u \prec_K y + u$

---

i) $y - x \in \text{int} K$

$\text{int} K$ is a cone $\rightarrow$ $ay - ax \in \text{int} K$

$\Rightarrow ax \prec_K ay$

---

j) $0 \notin \text{int} K$ (0 is on the boundary of K)

$\Rightarrow x - x = 0 \notin \text{int}(K)$

---

k) $y - x \in \text{int} K$

$y - x + (v - u) \in \text{int} K$

for small enough v, u

$\Rightarrow x + u \prec_K y + v$

HW1 problem 7

$K = \{ x = (r\cos\theta \;\; r\sin\theta) \mid r \geq 0 ,\; \theta_1 \leq \theta \leq \theta_2 \}$

where $\theta_2 - \theta_1 \leq 180°$

The dual is

$K^* = \{ y = (r\cos\theta \;\; r\sin\theta) \mid r \geq 0,$

$\theta_2 - 90 \leq \theta \leq \theta_1 + 90 \}$

(Directly from definition of dual cone)