HW2 Problem 1

sum of r components of x can be calculated as $a^T x$ where r entries of vector a are $\underline{1}$ and the rest are $\underline{0}$.

sum of r largest component of x can be formulated as finding the maximum (convex function) of vector $y = Ax$ (linear operation), where rows of A are all possible vectors in R^n with r entries equal to $\underline{1}$ and $n-r$ entries equal to $\underline{0}$. Thus it is a convex function.

HW2 problem 2

First order condition:

$$h'(x)(y-x) \leq h(y) - h(x) \leq h'(y)(y-x)$$

$$\Rightarrow \frac{h'(y) - h'(x)}{y-x} \geq 0$$

$y \to x$ gives $h''(x) \geq 0$

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conversely if $h''(x) \geq 0$ and $y > x$

then $\int_x^y h''(z)(y-z)\,dz \geq 0$

$$\Rightarrow -h'(x)(y-x) + h(y) - h(x) \geq 0$$

$$\Rightarrow h(y) \geq h(x) + h'(x)(y-x)$$ thus $h(x)$ is convex

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HW3 problem 3

g is positive and concave

$\Rightarrow \quad h = \frac{1}{g}$ is convex (Boyd's book ex 3.13)

$T(x,y) = x^2 y \, , \, y > 0$ is convex

$\Rightarrow T(f,h) = f^2 h = \frac{f^2}{g}$ is convex

(Boyd's book (3.10))

Problem 4

$$x = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+}}} = \frac{1}{1+x} \xRightarrow{x>0} x = 0.618$$

$$f(x^i) = \frac{1}{x_i - f(x^{i+1})} \, , \quad x^i = (x_i, x_{i+1}, \cdots)$$

$$\frac{\partial f(x^i)}{\partial x_i} = \frac{-1}{(x_i - f(x^{i+1}))^2} < 0$$

$$\frac{\partial f(x^i)}{\partial x_{i+n}} = \frac{\frac{\partial f(x^{i+1})}{\partial x_{i+1}}}{(x_i - f(x^{i+1}))^2} = c_1 \frac{\partial f(x^{i+1})}{\partial x_{i+n}}$$

$$= c_1 c_2 \cdots c_n \frac{\partial f(x^{i+n})}{\partial x_{i+n}} < 0 \, , \quad c_1, c_2, \cdots c_n > 0$$

$$\frac{\partial^2 f(x^i)}{\partial x_i^2} = \frac{2}{(x_i - f(x^i))^3} > 0$$

$$\frac{\partial^2 f(x^i)}{\partial x_{i+n}} = -2 \frac{\frac{\partial f(x^{i+1})}{\partial x_{i+n}}}{(x_i - f(x^{i+1}))^3} > 0$$

It can be shown that hessian is all positive and symmetric matrix thus it is positive definite and f is convex

HW2 problem 6

$f(v)$ is strictly convex on $\mathbb{R}^n_{++}$

$\Rightarrow f(u) > f(v) + \nabla f(v)^T(u-v)$, $u \neq v$, $u, v \in \mathbb{R}^n_{++}$

$\Rightarrow D_{KL}(u, v) > 0$ for $u \neq v$

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problem 7

a) considering $g(t) = f(Z + tV)$, $Z \succ 0$, $V \in S^n$

$$\Rightarrow g(t) = tr((Z+tV)^{-1}) = tr(Z^{-1}(I + tZ^{-1/2}VZ^{-1/2})^{-1})$$

$$= tr(Z^{-1}Q(I+t\Lambda)^{-1}Q^T) = tr(Q^TZ^{-1}Q(I+t\Lambda)^{-1})$$

$$= \sum_{i=1}^{n} (Q^TZ^{-1}Q)_{ii} (1+t\lambda_i)^{-1}$$

where $Z^{-1/2}VZ^{-1/2} = Q\Lambda Q^T$ (eigen value decomposition)

The last equality shows g as a positive weighted sum of convex functions $1/(1+t\lambda_i)$, hence it is convex

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b) considering $g(t) = f(Z + tV)$, $Z \succ 0$, $V \in S^n$

$$g(t) = (det(Z+tV))^{1/n} = (det Z^{1/2} det(I + tZ^{-1/2}VZ^{-1/2}) det Z^{1/2})^{1/n}$$

$$= (det Z)^{1/n} \left(\prod_{i=1}^{n}(1+t\lambda_i)\right)^{1/n}$$

where $\lambda_i, i = 1, \dots, n$ are the eigen values of $Z^{-1/2}VZ^{-1/2}$.

since $det Z > 0$ and $(\prod_{i=1}^{n} x_i)^{1/n}$ is concave on $\mathbb{R}^n_{++}$,

g is a concave function of t on $\{t \mid Z + tV \succ 0\}$